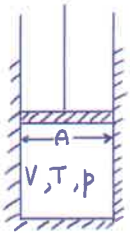
SHORT 1 Part IA Thermofluids 2024

(a)



$F = 1000$ N is added to the piston $\Rightarrow$ the pressure of the gas must increase by:

$$p_2 - p_1 = \frac{F}{A} = \frac{1000\,\text{N}}{0.01\,\text{m}^2} = 10^5\ \text{Nm}^{-2} \quad \Rightarrow \quad p_2 = 2\times10^5\ \text{Nm}^{-2}$$

Assume the compression is adiabatic and reversible, and therefore isentropic $\Rightarrow$ $pV^\gamma = \text{const}$.

$$\Rightarrow p_1 V_1^\gamma = p_2 V_2^\gamma \quad \Rightarrow \quad V_2/V_1 = \left(p_1/p_2\right)^{1/\gamma} \quad \Rightarrow \quad V_2 = 0.01\times\left(\tfrac{1}{2}\right)^{3/5} = 6.598\times10^{-3}\ \text{m}^3$$

[2 marks]

Most students answered this well. A few assumed the compression was isothermal.

(b) There are two equally good methods and either can be given.

Method 1:
$$U_2 - U_1 = mc_v(T_2 - T_1) = mc_vT_1\left(\frac{T_2}{T_1} - 1\right) \text{ where } \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \text{ and } p_1V_1 = mRT_1 \Rightarrow mT_1 = \frac{p_1V_1}{R}$$

$$= p_1V_1\frac{c_v}{R}\left(\frac{T_2}{T_1} - 1\right) = \frac{p_1V_1}{\gamma-1}\left(\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) = \frac{10^5\times10^{-2}}{2/3}\left(2^{2/5} - 1\right) = 479.3\ \text{J}$$

Method 2:
$$U_2 - U_1 = -W = -\int_{V_1}^{V_2} p\,dV = -\int_{V_1}^{V_2} p_1V_1^\gamma\frac{dV}{V^\gamma} = -p_1V_1^\gamma\left[\frac{V^{1-\gamma}}{1-\gamma}\right]_{V_1}^{V_2} = \frac{p_1V_1^\gamma}{\gamma-1}\left\{V_2^{1-\gamma} - V_1^{1-\gamma}\right\}$$

$$\boxed{V_1^\gamma = V_1V_1^{\gamma-1}}$$

$$= \frac{p_1V_1}{\gamma-1}\left\{\left(\frac{V_1}{V_2}\right)^{\gamma-1} - 1\right\} = \frac{p_1V_1}{\gamma-1}\left\{\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right\} = 479.3\ \text{J as above.}$$

[3 marks]

Only around 25% of students answered this well, and many used c_v and R from the databook, which was OK but required more effort. Although many students wrote down $\Delta U = mc_v\Delta T$, most couldn't calculate ΔT.

(c) The mass of argon remains unchanged so, by the ideal gas equation of state:

$$p_1V_1 = mRT_1 = mRT_3 = p_3V_3 \text{ where } p_3 = p_2 \quad \Rightarrow \quad V_3 = V_1\,p_1/p_3 = 0.01\times\tfrac{1}{2} = 5.00\times10^{-3}\ \text{m}^3$$

[1 mark]

Most students answered this well.

(d) Method 1: Heat is transferred at constant pressure so $Q = mc_p(T_2 - T_3)$ where $T_3 = T_1$ and $\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$

$$Q = mc_pT_1\left(\frac{T_2}{T_1} - 1\right) = p_1V_1\frac{c_p}{R}\left(\frac{T_2}{T_1} - 1\right) = p_1V_1\frac{\gamma}{\gamma-1}\left(\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) = 10^3\times\frac{5}{2}\times\left(2^{2/5} - 1\right) = 798.8\ \text{J}$$

Method 2: The internal energy returns to its original value because $T_3 = T_1$ so the heat lost to the surroundings, Q, equals the work done on the system in both stages:

$$Q = \underbrace{479.3\ \text{J}}_{\text{from (b)}} + p_2(V_2 - V_3) = 479.3\ \text{J} + 2\times10^5(6.598 - 5.00)\times10^{-3}\ \text{J} = 798.8\ \text{J}$$

[3 marks]

Those who answered (b) also answered (d).

(e) When the gas cools from T_2 to T_3 it contracts so the weight does additional work on the gas, which is dissipated as heat to the environment.

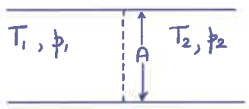
[1 mark]

Few students answered this well, showing a lack of physical understanding.

M. Juniper 2024

SHORT 2

(a)



By the ideal gas equation of state, $\rho_1 = p/RT_1 = \dfrac{2.87 \times 10^5}{287 \times 10^3} = 1.00\ \text{kg m}^{-3}$

The mass flowrate $\dot{m} = \rho_1 A V_1 \quad \Rightarrow \quad V_1 = \dfrac{\dot{m}}{\rho_1 A} = \dfrac{2.00}{2.00 \times 0.01} = 100\ \text{ms}^{-1}$

[2 marks]

Almost all students answered this well.

(b) The flow is adiabatic and no shaft work is done on the gas so, by the S.F.E.E:

$$h_1 + \tfrac{1}{2}V_1^2 = h_2 + \tfrac{1}{2}V_2^2 \quad \Rightarrow \quad V_2^2 = V_1^2 + 2c_p(T_1 - T_2) = 100^2 + 2 \times 1005 \times 28.66 = 67607$$

$$\Rightarrow \quad V_2 = 260\ \text{ms}^{-1}$$

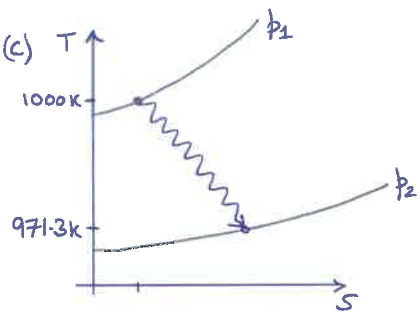
The mass flowrate is unchanged so $\dot{m} = \rho_1 A V_1 = \rho_2 A V_2 \Rightarrow \rho_2 = \rho_1 V_1/V_2 = 1.00 \times \dfrac{100}{260} = 0.385\ \text{kg m}^{-3}$

By the ideal gas equation of state, $p_2 = \rho_2 R T_2 = 0.385 \times 287 \times (1000 - 28.66) = 1.072$ bar

[4 marks]

This was answered well by around half the students. A few tried to use Bernoulli's equation, which only considers mechanical energy and is therefore not appropriate.

(c)



Most students drew the diagram well but many drew a solid line between the two states, which is incorrect because the intermediate states are not in quasi-equilibrium.

[2 marks]

(d) From the perfect gas relationships in the databook, the specific entropy change is:

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = 1005 \ln\left(\frac{1000 - 28.66}{1000}\right) - 287 \ln\left(\frac{1.072}{2.87}\right) = 253.4\ \text{J kg}^{-1}\text{K}^{-1}$$

The mass flowrate is $2.00\ \text{kg s}^{-1}$ so the rate of entropy increase is $506.8\ \text{J kg}^{-1}\text{K}^{-1}\text{s}^{-1}$. [2 marks]

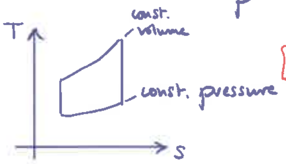
This was well answered by many students, even if they had not answered the earlier parts well.

M. Juniper 2024

LONG

(a) Process (i) is isentropic and the gas is perfect. Therefore $pv^{\gamma} = \text{const.}$ and $\dfrac{T}{p^{\frac{\gamma-1}{\gamma}}}$ is const.

$$\Rightarrow \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = r^{2/7} \quad \text{and} \quad \frac{V_2}{V_1} = \left(\frac{p_1}{p_2}\right)^{1/\gamma} = r^{-5/7} \quad [2]$$



[2]

Many students left γ in the expression. This was not intended but was not penalised. Only a simple T-s diagram was required here

(b) Process (ii) is at constant volume. Therefore $Tds = du = c_v dT$

$$\Rightarrow \frac{1}{T}\frac{dT}{ds} = \frac{1}{c_v} \; ; \text{ now, } \frac{d\ln T}{ds} = \frac{d\ln T}{dT}\frac{dT}{ds} \text{ by chain rule } = \frac{1}{T}\frac{dT}{ds} \Rightarrow \frac{d(\ln T)}{ds} = \frac{1}{c_v} \quad [2]$$

If heat is added at constant volume then $W=0$ and, by 1st law, $q = \Delta u = c_v \Delta T$

$$\Rightarrow T_3 - T_2 = q/c_v \; ; \text{ for next part, } \frac{T_3-T_2}{T_2} = \frac{T_3}{T_2} - 1 = \frac{q}{c_v T_2} \Rightarrow \frac{T_3}{T_2} = 1 + \frac{q}{c_v \Delta T} \quad [2]$$

Using the perfect gas relationship in the databook or the expression above:

$$s_3 - s_2 = c_v \ln\left(\frac{T_3}{T_2}\right) = c_v \ln\left(1 + \frac{q}{c_v \Delta T}\right)$$

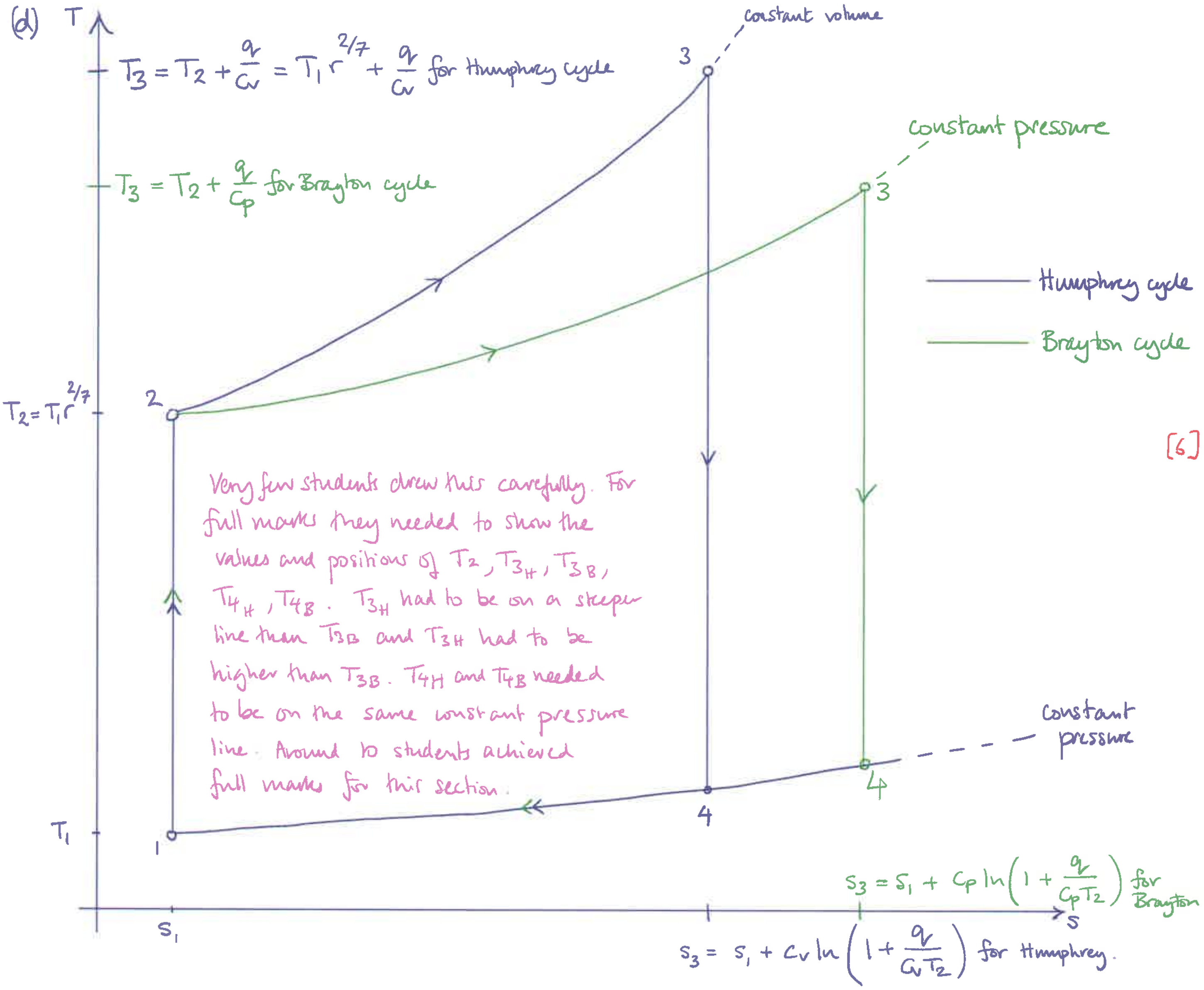
Almost all students achieved this [2]

(c) Process (iv) is at constant pressure. Therefore $Tds = dh = c_p dT$

$$\Rightarrow \frac{1}{T}\frac{dT}{ds} = \frac{1}{c_p} \Rightarrow \frac{d(\ln T)}{ds} = \frac{1}{c_p} \quad \text{using same argument as in (b)}$$

Almost all students achieved this [2]

(d)



[6]

Very few students drew this carefully. For full marks they needed to show the values and positions of $T_2, T_{3H}, T_{3B}, T_{4H}, T_{4B}$. T_{3H} had to be on a steeper line than T_{3B} and T_{3H} had to be higher than T_{3B}. T_{4H} and T_{4B} needed to be on the same constant pressure line. Around 10 students achieved full marks for this section.

(e) In process (ii) heat is added at constant pressure $\Rightarrow T_3 - T_2 = q/c_p$

Using the perfect gas relationships, $s_3 - s_2 = c_p \ln\left(\frac{T_3}{T_2}\right) = c_p \ln\left(1 + \frac{q}{c_p T_2}\right)$

Consider $\frac{d(s_3 - s_2)}{dq} = \frac{c_p}{1 + \frac{q}{c_p T_2}} \times \frac{1}{c_p T_2} = \frac{c_p}{c_p T_2 + q}$ for Brayton and $\frac{c_v}{c_v T_2 + q}$ for Humphrey.

For all q, this is always larger for Brayton than for Humphrey, so integrating w.r.t. q shows that $s_3 - s_2$ of the Brayton cycle is always greater than that of the Humphrey cycle, [6]

Most students answered this satisfactorily, using a variety of methods. Few, however, drew the Brayton cycle carefully on the diagram, as described previously.

(f) The Humphrey cycle is more efficient than the Brayton cycle because:

(i) for the same heat in, q, the Humphrey cycle has a larger temperature drop during the isentropic expansion, which is when the work is extracted. By the first law, this will lead to larger work being extracted from the same q

or (ii) for the same heat in, q, the Brayton cycle has a larger entropy increase $s_3 - s_2$. Consequently, during the heat rejection stage, the Brayton cycle has to reject more heat to the environment, resulting in less work being extracted.

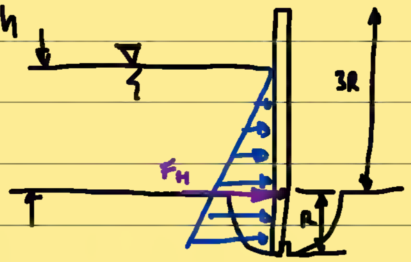
or (iii) The temperature of heat addition is always greater for the Humphrey cycle, while the temperature of heat rejection is always less or equal to that of the Brayton cycle.

Any of these explanations is sufficient, and there are others too. [6]

Only around a quarter of students answered this correctly, even if they had completed the question well this far. A very common error was to try to compare the areas swept out by each cycle on badly-drawn graphs. Bad drawings led to bad conclusions. All the above arguments are safer.

M. Juniper 2024

4) a)



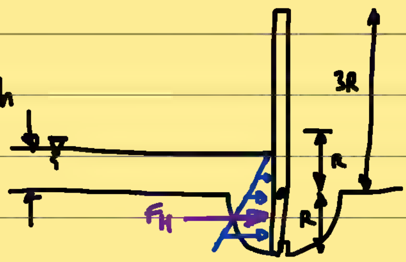
When force acts above hinge gate will begin to open.

$$\frac{h+R}{3} > R$$

$$\Rightarrow h > 3R - R > 2R$$

Scop always $\frac{h+R}{3}$ from base so unstable.

b)



$F_H = \rho g \frac{(R+h)^2}{2}$ @ $\frac{(h+R)}{3}$ from bottom of gate

5) VERSION2

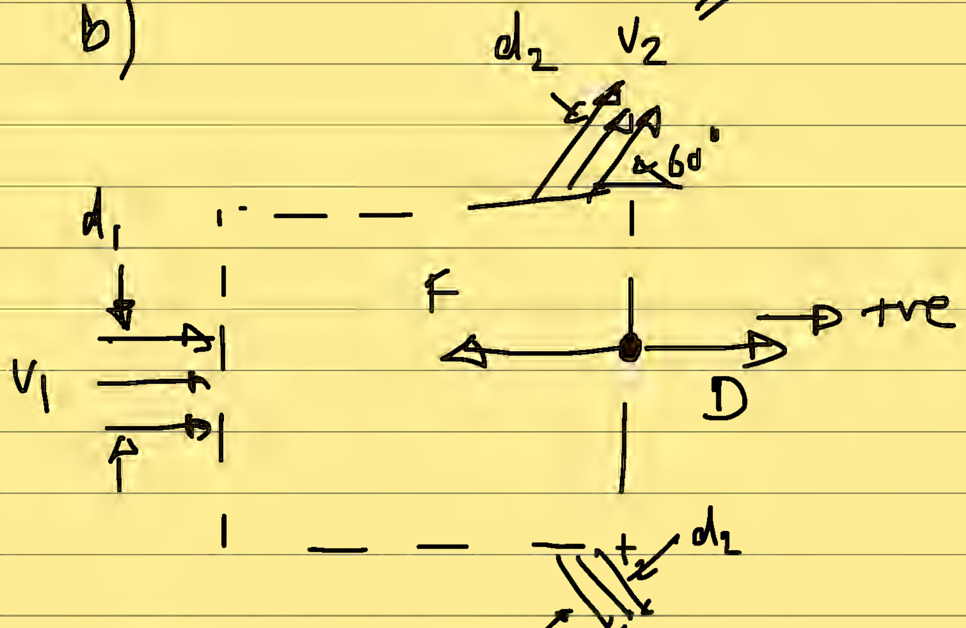
Bernoulli $p_a + \frac{1}{2}\rho V_1^2 = p_a + \frac{1}{2}\rho V_2^2$

a) $\Rightarrow V_1 = V_2$

$$d_1 \cancel{V_1} = 2 \times d_2 \cancel{V_2}$$

$\Rightarrow d_2 = d_1/2$ BY SYMMETRY

b)



FORCE ON FLUID

$$\sum F_x + \cancel{\sum (pA)_x} = \sum(\dot{m} V_{out})_x - \sum(\dot{m} V_{in})_x$$

$$-F = \cancel{2} \times \frac{\dot{m}}{\cancel{2}} V_2 \cos 60° - \dot{m} V_1$$

$$= \dot{m} V_1 (\cos 60° - 1)$$

$$+F = +\dot{m} V_1/2$$

$F = \rho V_1^2 d_1/2$ FORCE ON FLUID

FORCE ON OBSTACLE $D = \rho V_1^2 d_1/2$ TO THE RIGHT

6)

a) $V_2 = V_1 \frac{\pi D_1^2}{4} \bigg/ \frac{\pi D_2^2}{4} = V_1 \frac{D_1^2}{D_2^2}$ //

b) $p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$

$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_1^2 \left(\frac{D_1}{D_2}\right)^4$

$\Rightarrow p_2 = p_1 + \frac{1}{2}\rho V_1^2 \left\{ 1 - \left(\frac{D_1}{D_2}\right)^4 \right\}$

c)

		LHS RHS		
12	$p_1 - p_{pitot}$	→	LHS lower	$\boxed{\text{SAME A 14}}$
23	$p_{pitot} - p_2$	→	$\boxed{\text{LHS lower}}$	OPPOSITE
34	$p_2 - p_{pitot}$	→	$\boxed{\text{RHS lower}}$	OPPOSITE
24		→	$\boxed{\text{ZERO}}$	(pitot on both sides)
14	$p_1 - p_{pitot}$	→	$\boxed{\text{SAME AS 12}}$	
13	$p_1 - p_2$	→	LHS lower	

d) i) $p_1 - (p_1 + \frac{1}{2}\rho V_1^2) = \rho_L g h_{12}$

$\Rightarrow h_{12} = -\frac{\rho V_1^2}{2\rho_L g}$ //

ii) $p_1 - p_2 = -\frac{1}{2}\rho V_1^2 \left\{ 1 - \left(\frac{D_1}{D_2}\right)^4 \right\} = \rho_L g h_{13}$

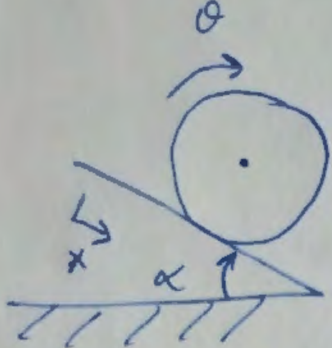
$h_{13} = \frac{-\frac{1}{2}\rho V_1^2 \left\{ 1 - (D_1/D_2)^4 \right\}}{\rho_L g} = \frac{\rho V_1^2}{2\rho_L}\left[\left(\frac{D_1}{D_2}\right)^4 - 1\right]$

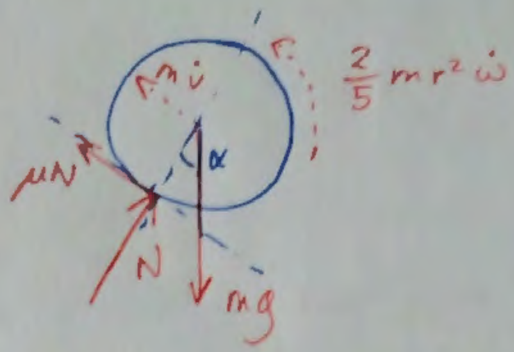
iii) $p_{pitot} - p_2 = \rho_l g h_{23}$

$$\cancel{p_1} + \cancel{\tfrac{1}{2}\rho v_1^2} - \left[\cancel{p_1} + \cancel{\tfrac{1}{2}\rho v_1^2}\left\{1 - \left(\frac{D_1}{D_2}\right)^4\right\}\right] = \rho_l g h_{23} \Rightarrow \frac{1}{2}\rho v_1^2 \left(\frac{D_1}{D_2}\right)^4 = \rho_l g h_{23}$$

$$\Rightarrow h_{23} = \frac{\rho}{2g\rho_l} v_1^2 \left(\frac{D_1}{D_2}\right)^4$$

Q 7





a) rolling without slip: $x = r\theta \rightarrow \ddot{x} = r\ddot{\theta}$

$$\Longrightarrow \dot{v} = r\dot{\omega}$$

b) Resolving forces normal to the slope

$$N = mg\cos(\alpha)$$

c) · At the onset of sliding, we can assume that:

$$\begin{cases} \dot{v} = r\dot{\omega} & \text{: just about to slide, but not yet sliding} \\ \text{Friction force} = \mu N & \text{: Friction force has reached its limit.} \end{cases}$$

· Resolving forces along the slope

$$\mu N + m\dot{v} = mg\sin\alpha$$

· Moments about the center:

$$\mu N r = \frac{2}{5} m r^2 \dot{\omega} = \frac{2}{5} m r \dot{v} \Longrightarrow \mu N = \frac{2}{5} m\dot{v}$$

Substituting in the previous equation, we get:

$$mg\sin\alpha = \mu N + \frac{5}{2}\mu N = 7/2\,\mu N$$

But $N = mg\cos\alpha \quad \Longrightarrow \quad \mu = \frac{2}{7}\tan\alpha$

d) For $\alpha = \frac{\pi}{4}$ and friction coef $= \frac{1}{7}$.

Is it sliding ?

$$\frac{1}{7} < \frac{2}{7} \tan \frac{\pi}{4}$$

There is not enough friction and there is indeed sliding.

Therefore :

- The friction force is μN
- But $\dot{v} \neq r\dot{\omega}$

To find $\dot{v}$ $\rightarrow$ resolve force along the slope.

$$\mu N + \dot{v} m = mg \sin \alpha$$

$$\dot{v} = g \sin \alpha - \frac{1}{7} g \cos \alpha = \frac{6}{7} \frac{\sqrt{2}}{2} g$$

To find $\dot{\omega}$ $\rightarrow$ moments on center.

$$\mu N r = \frac{2}{5} m r^2 \dot{\omega}$$

$$\dot{\omega} = \frac{5}{2mr^2} \frac{1}{7} mg \cos(\alpha) r$$

$$\dot{\omega} = \frac{5}{14} \frac{\sqrt{2}}{2} \frac{g}{r}$$

$$v_1 \cdot r = v_2 \cdot 2r \text{ , or equivalently } v_1 = 2v_2 \qquad (3).$$

▶ Then using energy conservation $E_1 = E_2$, where

$$E_1 = \frac{1}{2} m v_1^2 \quad \text{and} \quad E_2 = \frac{1}{2} m v_2^2 + \underline{\underline{2mg \cdot r}}$$

At time instant #2, the mass 2m is at a height of r relatively to its position at time instant #

So, $\frac{1}{2} m v_1^2 = \frac{1}{2} m v_2^2 + 2mgr \qquad (4).$

▶ Substituting (3) into (4) to eliminate v_1 finally gives $\underline{\underline{v_2 = \sqrt{\frac{4gr}{3}}}}$. ✓✓✓

2024 1P1

1

9 (short) A rigid circular disc of mass m and radius r is shown in Fig.6. A point A is located on the edge of the disc at $\{x, y\} = r\{1,1\}/\sqrt{2}$.
(a) Find the moment of inertia I_{zz} at A. [5]
(b) A shaft parallel to the z axis is fixed to the disc at point A. Find the angular acceleration of the disc when a torque Q is applied along the axis of this shaft. [5]

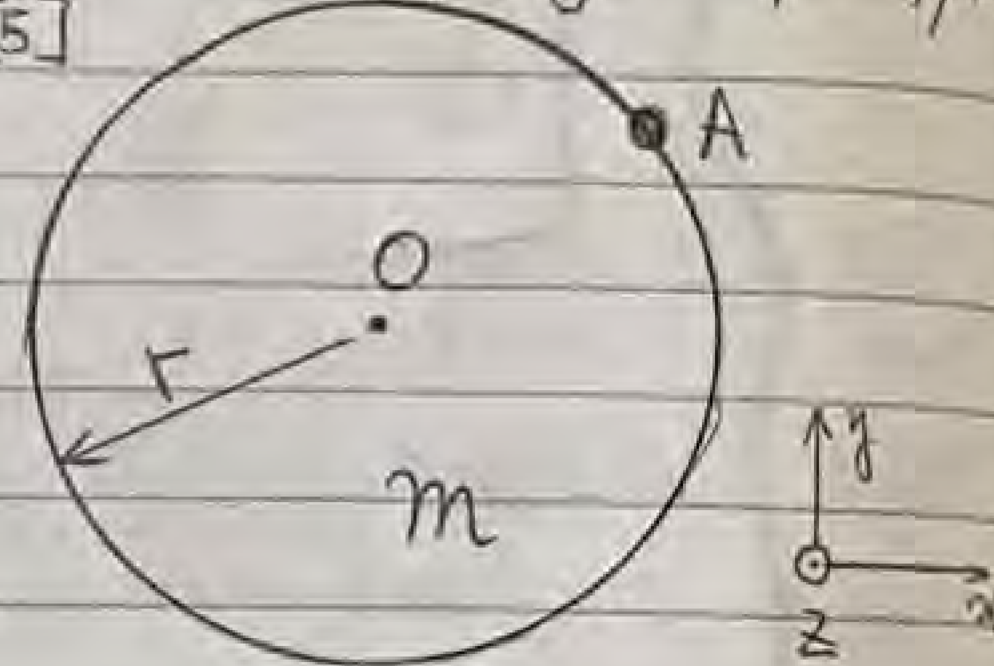


Fig.6

(a)

▶ I_{zz}/A can be found following two steps.

▶ STEP1. Find I_{zz}/O using the data book and Perpendicular Axis Theorem.
- According to the data book, $I_{xx} = I_{yy} = m\frac{r^2}{4}$.
- So, $I_{zz}/O = I_{xx} + I_{yy} = \frac{1}{2}mr^2$ (1)

▶ STEP2. Find I_{zz}/A using Parallel Axis Theorem.
- So, $I_{zz}/A = I_{zz}/O + mr^2$ (2)
- Substituting (1) into (2) results in: $I_{zz}/A = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$ ✓✓✓

(b)

▶ This part is essentially concerned with a rigid body rotates about a fixed axis. Relevant analysis is covered by Section 11 in the 1P1 Mechanics lecture notes. The key equation is

$$\underbrace{Q}_{\text{Total torque applied about the fixed axis}} = \underbrace{J}_{\text{Mass moment of inertia of the body about the fixed axis}}\ \underbrace{\dot{\omega}}_{\text{Angular acceleration of the body about the fixed axis at the instant when } Q \text{ is the total torque applied.}} \quad (3)$$

▶ By applying equation (3) to the question, the angular acceleration of the disc $\dot{\omega}$ resulted from the torque Q is:

$$Q = I_{zz}/A\ \dot{\omega} \quad (4)$$

So, $\dot{\omega} = Q / I_{zz}/A = \dfrac{Q}{\frac{3}{2}mr^2} = \dfrac{2Q}{3mr^2}$. ✓✓✓

10 (short) A flexible chain of length L and mass m is resting on a set of scales. At time $t=0$ the chain is lifted from one end at a steady speed v as shown in Fig. 7. The time-varying force required to lift the chain is $f(t)$.

(a) Find an expression for the force $f(t)$ at $t=0$, the time at which the chain first lifts off the scales. [4]

(b) Sketch a graph of $f(t)$ vs time for the entire duration of the motion. Show also the reading on the scales for the same period. [6]

(a)

- This question is almost the same as the example covered in Section 7 of the lecture notes.

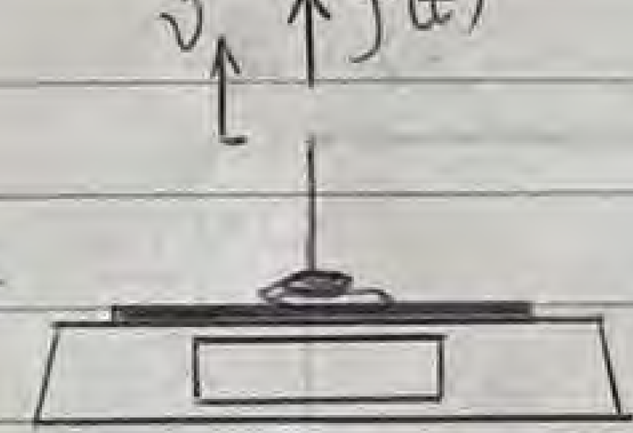
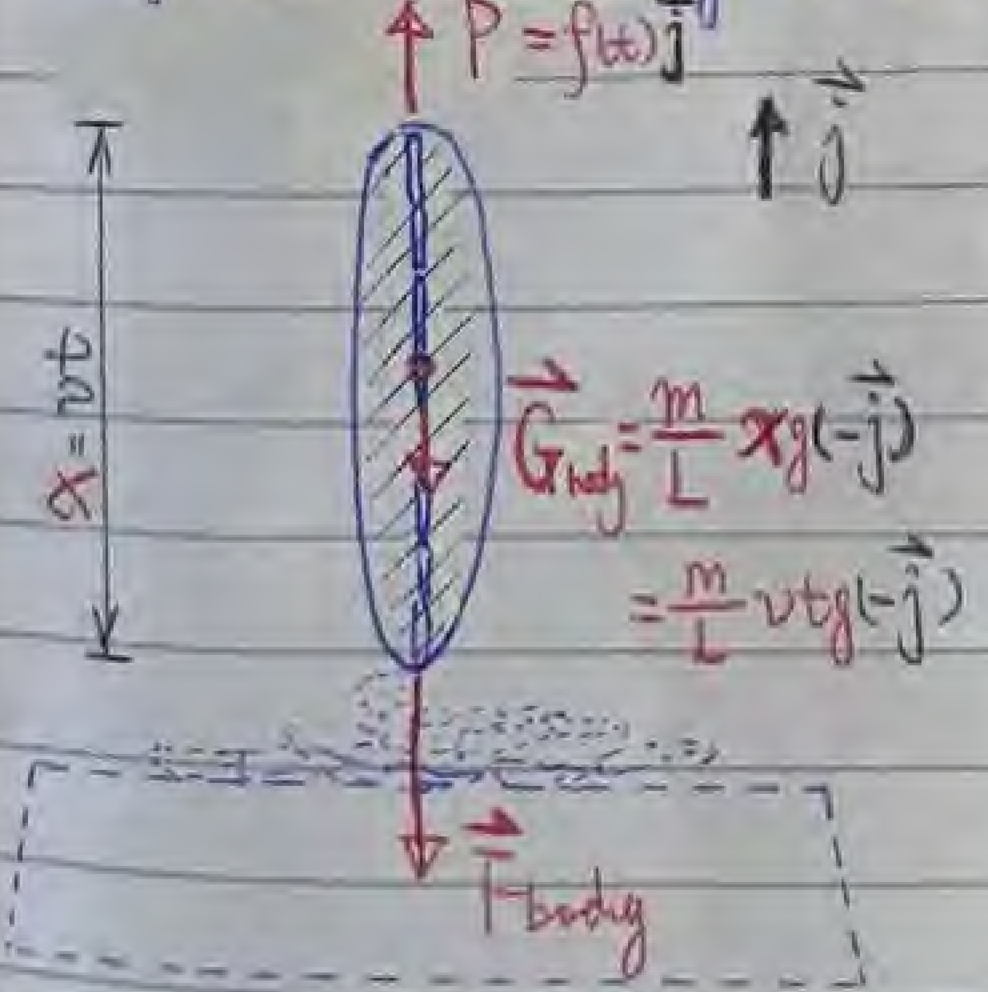


Fig. 7.

- Following the lecture notes, we assume that at a time t, the chain in the air, i.e. the "body" has a length X. So, $x = vt$ (1). This indicates that at time $t=0$, $x=0$.
- Let's first work out the force applied by the rest part of the chain on the "body". This can be done, as suggested in the lecture notes, by assuming that in a time dt, the mass lifted up is dm, which represents a small segment of the chain. To facilitate our analysis, let's draw a FBD for the "body" at time t, as shown below.



- In this FBD, the time-varying force applied to lift up the chain is $\vec{P}=f(t)\vec{j}$. The weight of the "body" is $\vec{G}_{body} = \frac{m}{L}xg(-\vec{j}) = \frac{m}{L}vtg(-\vec{j})$ (2) where $\frac{m}{L}$ is the linear density of the chain and x the length of the body. $\vec{F}_{body}$ is the force applied by the rest chain on the scale to the body. Note that this $\vec{F}_{body}$ is essentially applied by the segment just below the body.

- Continuing to follow lecture notes, $\vec{F}_{body}$ can be found following a specific procedure, which a dm segment of the chain is lifted up.
  - Focusing on a very short time period dt during which a dm segment of the chain is lifted up.
  - The change of momentum of this dm is $d\vec{p} = \vec{v}dm - 0\,dm = \vec{v}dm = v\,dm\,\vec{j}$ (3.1).
  - The force applied by the body to this dm, $\vec{F}_{particle}$ is $\vec{F}_{particle} = d\vec{p}/dt$ (3.2)
  - Substituting (3.1) into (3.2) gives: $\vec{F}_{particle} = v\frac{dm}{dt}\vec{j}$ (4).

- Note that $\frac{dm}{dt} = \frac{dm}{dx}\frac{dx}{dt} = \frac{dm}{dx} v$, and $\frac{dm}{dx} = \frac{m}{L}$, i.e. linear density.
  Therefore $\frac{dm}{dt} = \frac{m}{L} v$ (5).
- Substituting (5) into (4) gives $\vec{F}_{particle} = \frac{m}{L} v^2 \vec{j}$ (6).
- Remember that $\vec{F}_{particle} = -\vec{F}_{body}$, as per the Third Law of Newton, we have

$$\vec{F}_{body} = \frac{m}{L} v^2 (-\vec{j}) \qquad (7)$$

Note that the "$-$" sign in (7) makes physical sense because the rest of the chain must pull the body downwards.

▶ As the body moves up at a constant velocity $v\vec{j}$, we must have:

$$\sum \vec{F} = \vec{P} + \vec{G}_{body} + \vec{F}_{body} = \vec{0} \qquad (8)$$

Substituting $\vec{P} = f(t)\vec{j}$ and (2) and (7) into (8) gives:

$$f(t)\vec{j} + \frac{m}{L} vtg(-\vec{j}) + \frac{m}{L} v^2 (-\vec{j}) = \vec{0}$$, ~~or~~ equivalently

$$f(t) = \frac{m}{L}(v^2 + vgt) \qquad (9)$$

(9) shows how time-varying force $f(t)$ changes as t increases from 0.

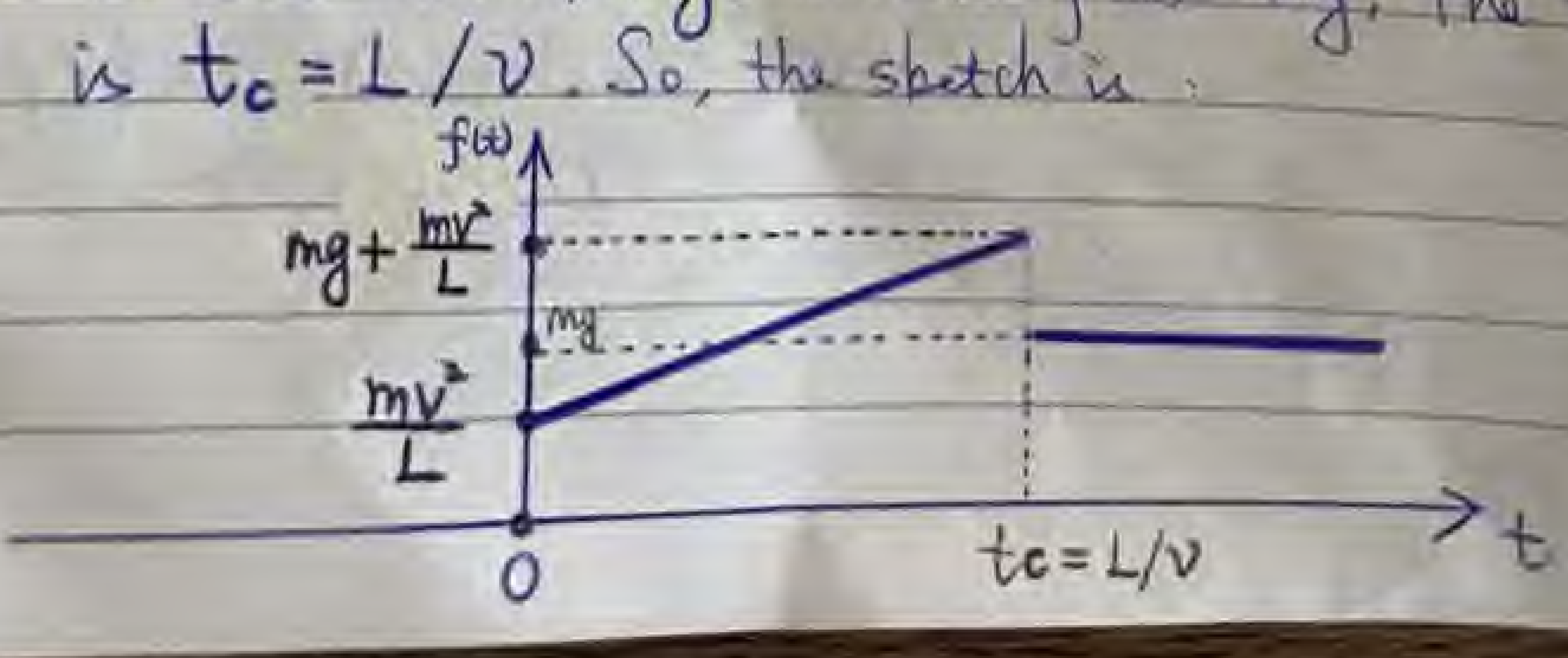
▶ So, for $f(t)$ at $t=0$, substituting $t=0$ into (9):

$$f(0) = \frac{m}{L} v^2 \checkmark \qquad (10)$$

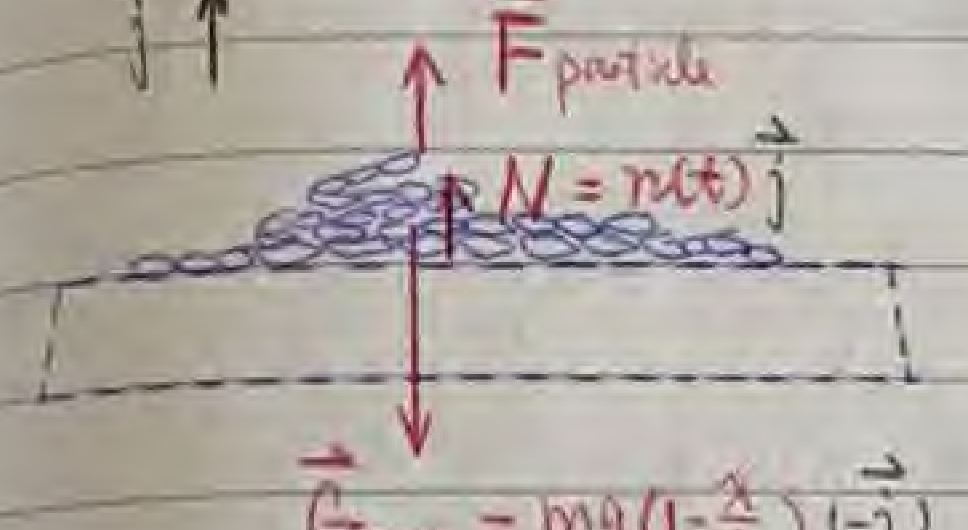
It can be seen that (10) suggests that at $t=0$, $\vec{G}_{body} = \vec{0}$.

(b)

▶ To sketch $f(t)$ versus t, it should be noted that (9) derived above holds when the body in our FBD is pulled down by the rest of the chain on the scale. After the last segment of chain lefts the scale, $f(t)$ becomes the weight of the entire chain, mg, that is $f(t) = mg$. The time corresponding to this instant is $t_c = L/v$. So, the sketch is:



▶ To work out the reading of the scale, we will need to do a similar analysis but for the rest of the chain on the scale. Below is a FBD for the "rest" at an arbitrary time t:



$\vec{G}_{rest} = mg(1-\frac{x}{L})(-\vec{j})$
$= mg(1-\frac{vt}{L})(-\vec{j})$

▶ We've worked out $\vec{F}_{particle} = \frac{m}{L}v^2\vec{j}$ (6).
We know that $\vec{G}_{rest} = mg(-\vec{j}) - \vec{G}_{body}$. So substituting $\vec{G}_{body} = \frac{m}{L}vtg(-\vec{j})$ into gives $\vec{G}_{rest} = mg(1-\frac{vt}{L})(-\vec{j})$ (11)

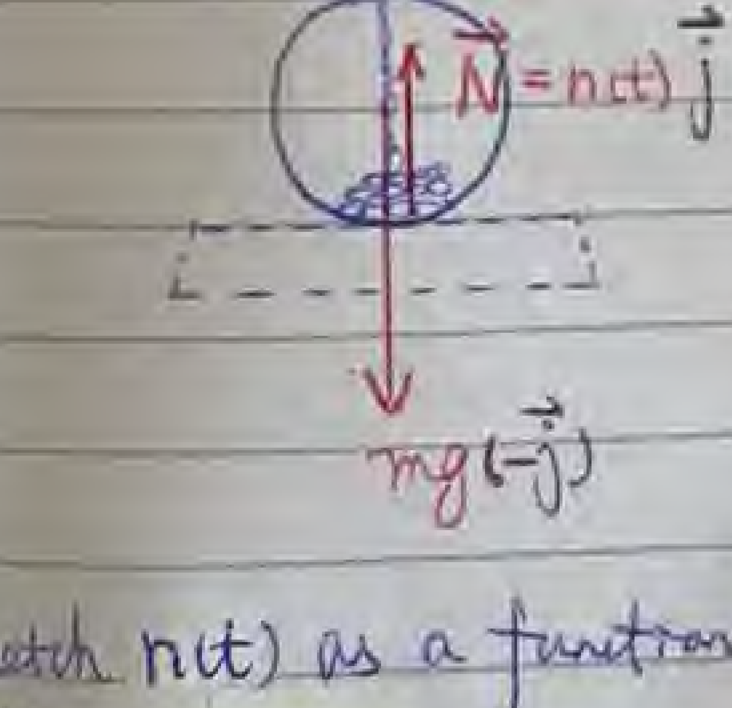
▶ We know that the "rest" is in rest, which means $\sum\vec{F} = \vec{F}_{particle} + \vec{G}_{rest} + \vec{N} = \vec{0}$ (12)

▶ Substituting (6) and (11) into (12) gives:

$$\frac{m}{L}v^2\vec{j} + mg(1-\frac{vt}{L})(-\vec{j}) + n(t)\vec{j} = \vec{0}$$, or equivalently

$$n(t) = \frac{m}{L}(gL - gvt - v^2) \quad (13)$$

▶ Note that (13) can be also obtained by viewing the chain as a whole system. As a result, the system's FBD is:

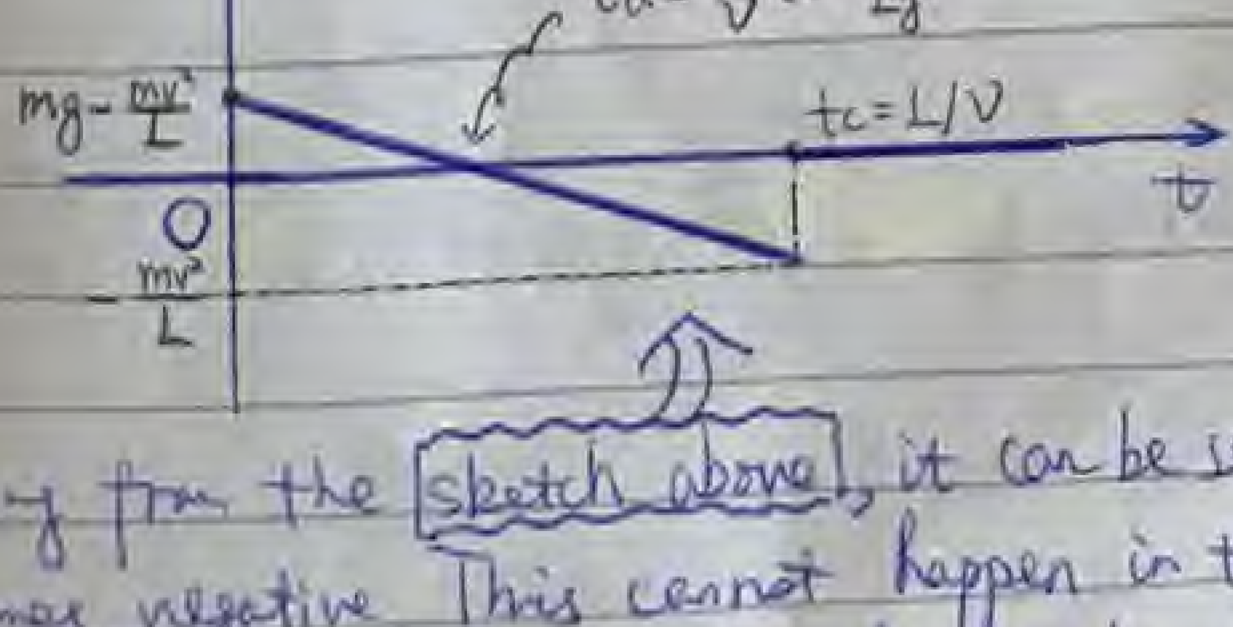


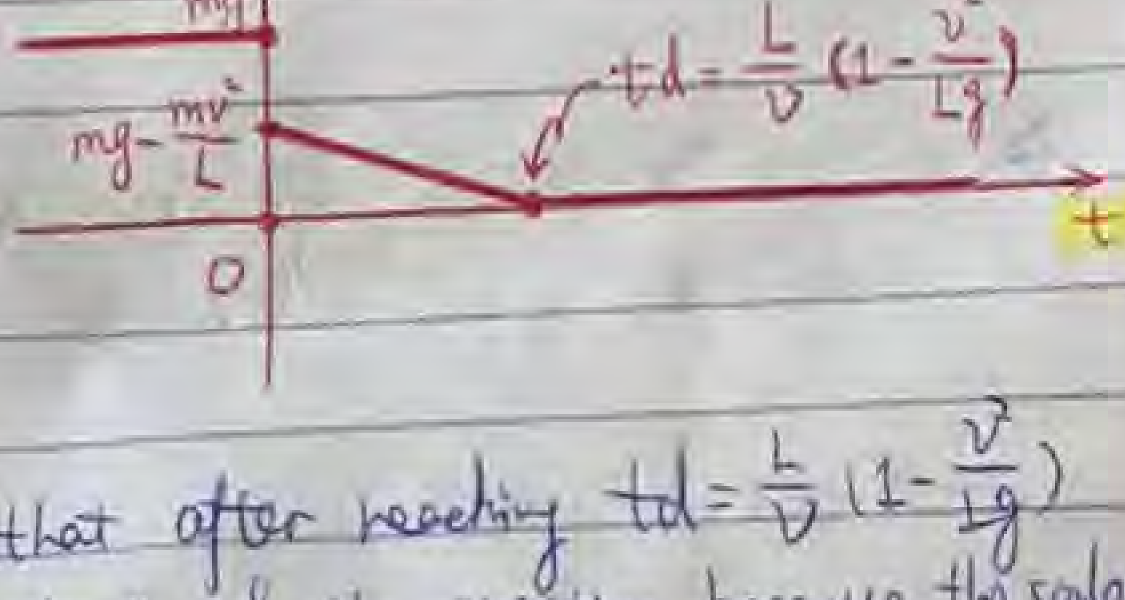
▶ As the chain's acceleration is $\vec{0}$ because either the lifting part or the rest part is holding a $\vec{0}$ acceleration, we must have:

$$\sum\vec{F} = \vec{P} + mg(-\vec{j}) + \vec{N} = \vec{0} \quad (14)$$

▶ Substituting $\vec{P} = f(t)\vec{j}$, (9) and $\vec{N} = n(t)\vec{j}$ into (14) gives (13)

▶ Sketch $n(t)$ as a function of time t (noting that at $t = t_c$ $n(t) = 0$ as chain separates from scale):





Viewing from the $\boxed{\text{sketch above}}$, it can be seen that after reaching $t_d = \frac{L}{v}(1-\frac{v^2}{Lg})$ $n(t)$ becomes negative. This cannot happen in the context of the question because the scale has no chance to pull the chain downwards, but only to support it. So, the reading will become zero at t_d. The reading vs t sketch is plotted next to $n(t)$-t plot.

11 (short) An electron microscope is suited in the basement of a building. It is sensitive to horizontal vibrations and so is isolated as shown in Fig. 8. The microscope is modelled as a rigid mass m whose displacement is y. The basement motion is x. The total isolation stiffness and damping are $2k$ and 2λ respectively.

(a) Find a differential equation for the motion y of the microscope given the basement motion x. [4]

(b) The basement is moving sinusoidally according to $x(t) = X\cos\omega t$ and motion of the microscope likewise is $y(t) = Y\cos\omega t$. Use the Mechanics Data Book or otherwise to sketch a graph of the ratio Y/X versus frequency ω.

For part (b) use $m = 1000\,kg$, $k = 8\,N/m$, and $\lambda = 400\,Ns/m$.

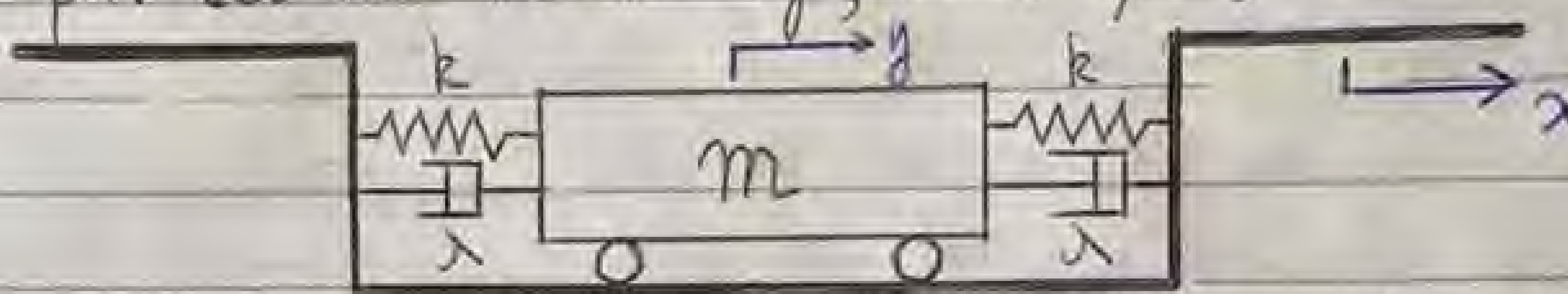
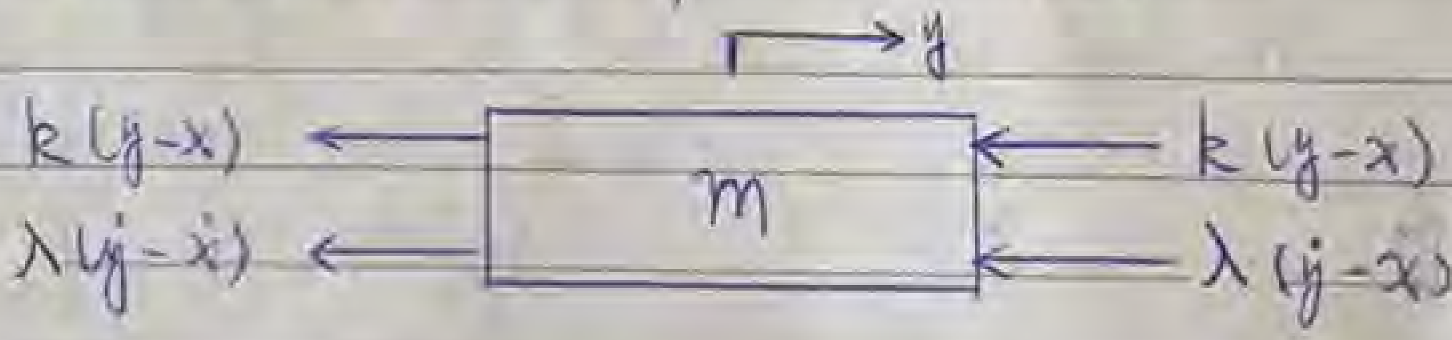


Fig. 8

(a)

▶ To find the differential equation governing the motion of the electron microscope m, we will need to do a FBD for m, as below:



(Fig. S1)

▶ Therefore, the equation of motion is $m\ddot{y} = -2k(y-x) - 2\lambda(\dot{y}-\dot{x})$. This can be rewritten as

$$m\ddot{y} + 2\lambda\dot{y} + 2ky = 2\lambda\dot{x} + 2kx \quad (1)$$

Dividing (1) by $2k$, we have:

$$\frac{m}{2k}\ddot{y} + \frac{\lambda}{k}\dot{y} + y = \frac{\lambda}{k}\dot{x} + x \quad (2)$$

▶ It can be seen that (2) represents "Case (c)" described in the lecture notes. By using the data book, we can obtain:

$$\omega_n = \sqrt{\frac{2k}{m}} \quad (3.1) \quad \text{and} \quad \zeta = \frac{\lambda}{\sqrt{2km}} \quad (3.2)$$

[Note by Xiaoxiang Na] Occasionally, students may ask how to assume the relation between x and y when deriving the differential equation of motion. My personal tip is assuming one is larger than the other and simultaneously moving faster than the other. For instance, we assume x is larger than y and $\dot{x}$ is larger than $\dot{y}$ at the same time. As a result, the forces can be directly resolved, as shown in <Fig. S1>. The "truth" behind this assumption is that we're dealing with equation of motion, which applies to any conditions in terms of motion. We just assumed a special case, i.e. $\begin{cases} x>y \\ \dot{x}>\dot{y} \end{cases}$ to derive the equation. This makes our lives EASIER.

(b)

▶ Since we've worked out that the equation of motion corresponds to "Case (c)", we will be able to compute ration of response $|Y/X|$ and sketch it, given that we know the values of ω_n and ξ.

▶ Substituting $m = 1000$ kg, $k = 8$ N/m, and $\lambda = 400$ Ns/m into (3.1) and (3.2):

$$\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2\times 8}{1000}} = \underline{\underline{0.127 \text{ rad/s}}}, \text{ and}$$

$$\xi = \frac{\lambda}{\sqrt{2km}} = \frac{400}{\sqrt{2\times 8\times 1000}} = \underline{\underline{3.162}}.$$

▶ Note that this is a heavily damped scenario. Therefore, the "Maximum response" equation or the "Half-power bandwidth" equation which requires "$\xi \ll 1$" cannot be used.

▶ To plot the response ratio versus angular frequency ω, we will need to plug specific ω into response ratio equation provided in the data book. The equation is given as:

$$\left|\frac{Y}{X}\right| = \frac{\left\{1 + (2\xi\omega/\omega_n)^2\right\}^{\frac{1}{2}}}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\xi\omega/\omega_n)^2\right\}^{\frac{1}{2}}} \quad (4)$$

▶ Looking at the curve in data book corresponding to $\xi = 1.5$, it can be seen that the maximum response is larger than 1, and it happens at around $\omega/\omega_n = 0.6$. It also can be seen from the cohort of curves given in the data sheet that as ξ increased, the frequency in which the maximum response appears reduces!!! So, for this question's $\xi = 3.162$ case, let's try a $\omega/\omega_n = 0.4$. This corresponds to $\omega = 0.4\,\omega_n = 0.4\times 0.127 = 0.0508$. Substituting $\omega = 0.0508$ into (4) gives:

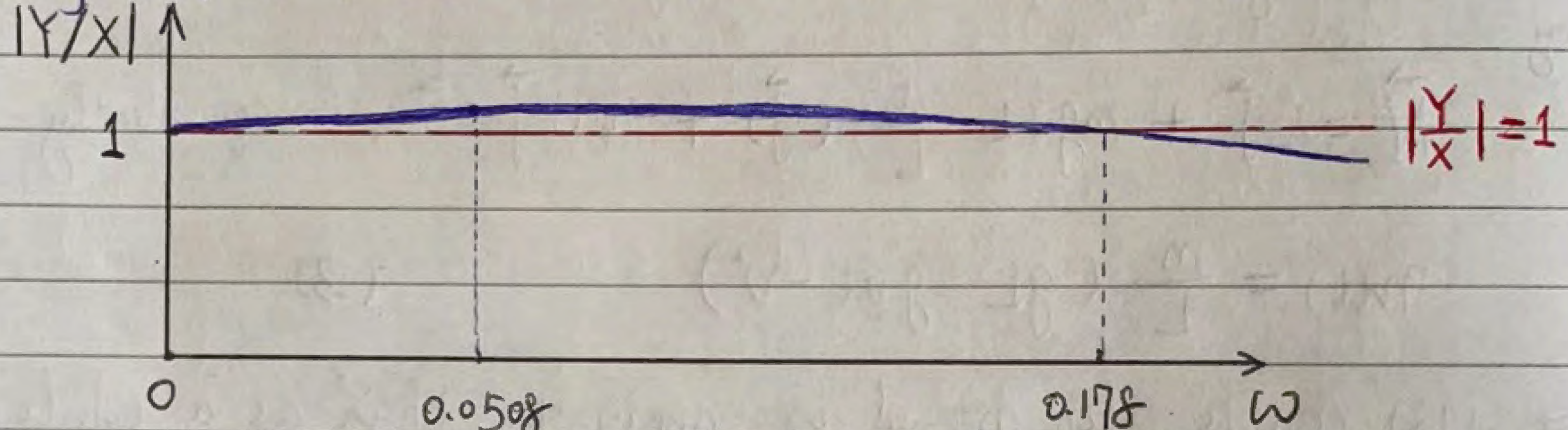
$$\left|\frac{Y}{X}\right|_{@\,\omega=0.0508} = 1.0205 \quad (5)$$

▶ (5) suggests that the response curve will for sure exceeds 1.

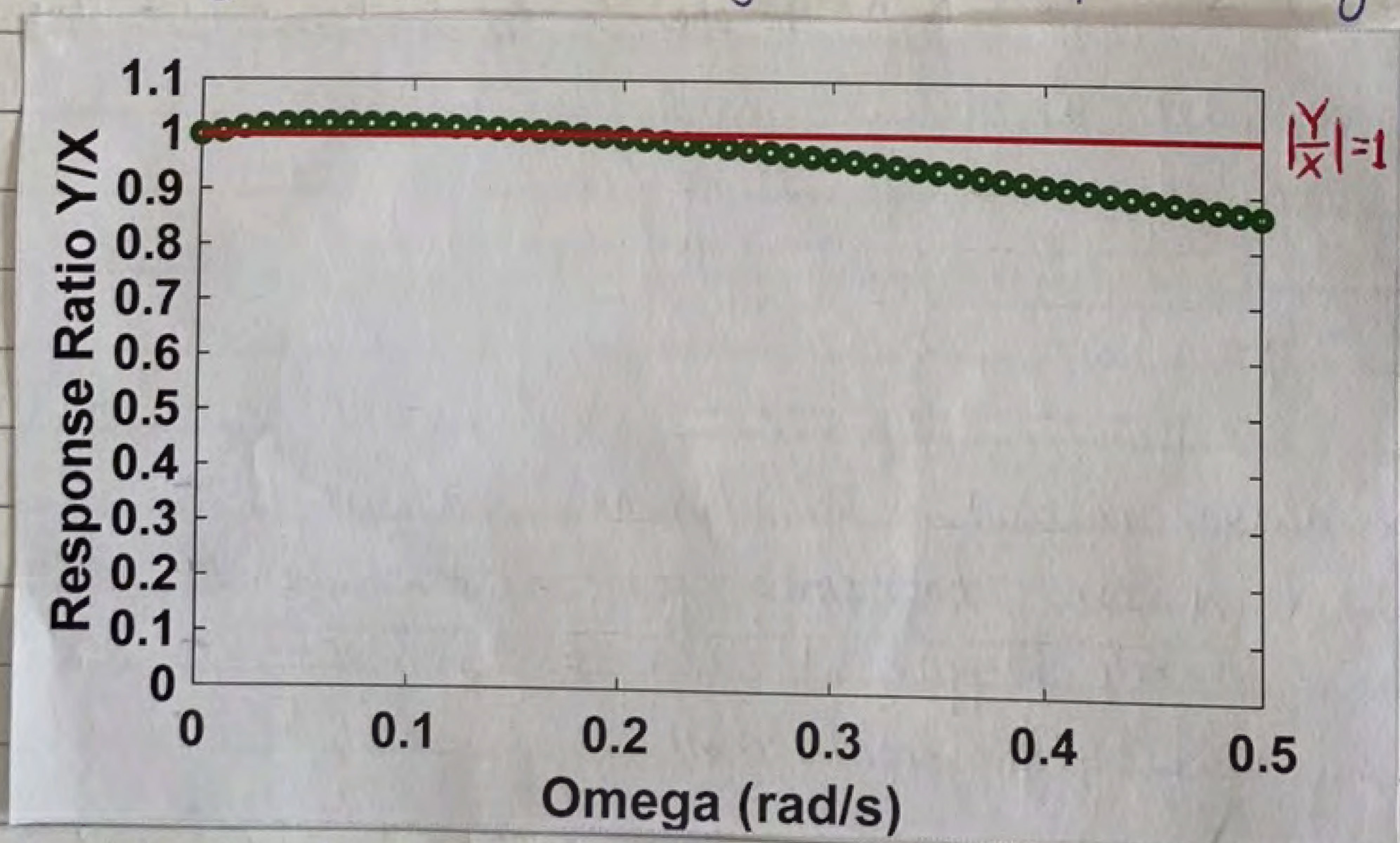
▶ Again by observing the cohort of curves given in the data book and the |Y/X| equation it can be found that all curves have zero response ratio around $\omega/\omega_n=\sqrt{2}$. So, let's do $\omega/\omega_n=\sqrt{2}$. This gives $\omega=\sqrt{2}\times\omega_n=1.414\times 0.127=0.178$. This indicates that:

$$\left|\frac{Y}{X}\right|_{@\,\omega=0.178}=1.00. \qquad (6)$$

▶ By using (5) and (6), and the fact that at $\omega=0$ $|Y/X|=1.00$, a sketch of |Y/X| versus ω can be made, as shown below.



▶ Below please find a "full" version of the sketch plotted using Matlab:

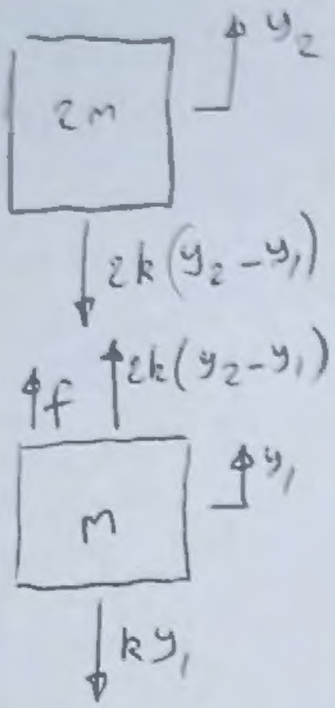


▶ It can be found from numerical simulation that the maximum response ratio |Y/X| is 1.0206. The corresponding ω is 0.056 rad/s.

Q 12

long

(a)



$$-2k(y_2-y_1) = 2m\ddot{y}_2$$

$$2k(y_2-y_1) - ky_1 + f = m\ddot{y}_1$$

$$\therefore \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Mass matrix   Stiffness matrix

(b) put $f = F\cos\omega t$, $y_1 = Y_1 \cos\omega t$
$y_2 = Y_2 \cos\omega t$

$$\begin{bmatrix} 3k - m\omega^2 & -2k \\ -2k & 2k - 2m\omega^2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 2k - 2m\omega^2 & 2k \\ 2k & 3k - m\omega^2 \end{bmatrix} \begin{bmatrix} F \\ 0 \end{bmatrix}$$

(6)

where $\Delta = 2(3k - m\omega^2)(k - m\omega^2) - 4k^2$

$= 2\left[3k^2 - 4km\omega^2 + m^2\omega^4 - 2k^2\right]$

$= 2k^2\left[\left(\frac{\omega}{\omega_n}\right)^4 - 4\left(\frac{\omega}{\omega_n}\right)^2 + 1\right]$

where $\omega_n^2 = \frac{k}{m}$

(c) Two natural frequencies when $[\quad] = 0$

$\therefore \left(\frac{\omega}{\omega_n}\right)^2 = 2 \pm \frac{1}{2}\sqrt{16-4}$

$= 2 \pm \sqrt{3}$

$$\boxed{\begin{aligned} Y_1 &= \frac{F}{\Delta}\, 2k\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right) \\ Y_2 &= \frac{F}{\Delta}\, 2k \end{aligned}}$$

(d)