4C8 2026

Q1

(a) (i) 10 Marks

- m_s represents a quarter of the total sprung mass of a car
- Suspension is modelled as a parallel linear spring-damper system, without friction
- Unsprung mass (i.e. mass of axle or tyre) is ignored
- Tyre is assumed to be a point, always contacting the road (no loss of contact)
- Displacement z_s is deviations from static equilibrium positions.

(a) (ii) 10 Marks

- The equation of motion of the one degree-of-freedom linear model can be expressed as:

$$m_s \ddot{z}_s = -k(z_s - z_r) - c(\dot{z}_s - \dot{z}_r) \tag{1}$$

- Taking Laplace transform to (1) and assuming zero initial conditions result in:

$$s^2 m_s z_s(s) = -k\left[z_s(s) - z_r(s)\right] - sc\left[z_s(s) - z_r(s)\right] \tag{2}$$

- Replacing Laplace operator s with $j\omega$ in (2) yields:

$$\frac{z_s(j\omega)}{z_r(j\omega)} = \frac{j\omega c + k}{-\omega^2 m_s + j\omega c + k} \tag{3}$$

(a) (iii) 35 Marks

- The response quantity $\mathrm{E}\left[\left(z_s - z_r\right)^2\right]$ represents suspension working space (WS). It is a measure of the vertical space required in a suspension system. There is usually limited space available for relative displacements between sprung mass and unsprung masses in road vehicles.
- Define the working space transfer function $H_{WS}(j\omega)$ as:

$$H_{WS}(j\omega) = \frac{z_s(j\omega) - z_r(j\omega)}{\dot{z}_r(j\omega)} \tag{4}$$

- (4) can be written as:

$$\begin{aligned} H_{WS}(j\omega) &= \frac{z_s(j\omega) - z_r(j\omega)}{\dot{z}_r(j\omega)} \\ &= \frac{z_s(j\omega)}{\dot{z}_r(j\omega)} - \frac{z_r(j\omega)}{\dot{z}_r(j\omega)} \\ &= \frac{1}{j\omega} \cdot \frac{z_s(j\omega)}{z_r(j\omega)} - \frac{1}{j\omega} \cdot \frac{z_r(j\omega)}{z_r(j\omega)} \\ &= \frac{1}{j\omega} \cdot \frac{z_s(j\omega)}{z_r(j\omega)} - \frac{1}{j\omega} \end{aligned} \tag{5}$$

- Substituting (3) into (5) to eliminate $z_s(j\omega)/z_r(j\omega)$ then gives

$$
\begin{aligned}
H_{WS}(j\omega) &= \frac{1}{j\omega}\cdot\frac{j\omega c+k}{-\omega^2 m_s + j\omega c + k} - \frac{1}{j\omega} \\
&= \frac{-(j\omega)m_s}{(j\omega)^2 m_s + (j\omega)c + k}
\end{aligned} \tag{6}
$$

- Following the lecture notes, the $\mathrm{E}\left[(z_s - z_r)^2\right]$ required can be obtained using the expression below:

$$
\mathrm{E}\left[(z_s - z_r)^2\right] = \int_{-\infty}^{\infty} |H_{WS}(j\omega)|^2 S_{\dot{z}_r}(\omega)d\omega \tag{7}
$$

- As the question suggests, $S_{\dot{z}_r}(\omega) = S_0$, which is a white noise (i.e. independent of angular frequency ω). Consequently, (7) becomes:

$$
\mathrm{E}\left[(z_s - z_r)^2\right] = S_0 \int_{-\infty}^{\infty} |H_{WS}(j\omega)|^2 d\omega \tag{8}
$$

- Comparing the $H_{WS}(j\omega)$ described by (6) with the $H_2(j\omega)$ given in the question, it can be seen that: $B_0 = 0$, $B_1 = -m_s$, $A_0 = k$, $A_1 = c$, and $A_2 = m_s$.
- Using the mean square integral of $H_2(j\omega)$ provided in the question, (8) can be calculated:

$$
\begin{aligned}
\mathrm{E}[(z_s - z_r)^2] &= S_0 \int_{-\infty}^{\infty} |H_{WS}(j\omega)|^2 d\omega \\
&= S_0 \frac{\pi\left(A_0 B_1^2 + A_2 B_0^2\right)}{A_0 A_1 A_2} \\
&= S_0 \frac{\pi k m_s^2}{k c m_s} \\
&= \frac{\pi S_0 m_s}{c}
\end{aligned} \tag{9}
$$

(b) (i) $\boxed{\text{25 Marks}}$

- The response quantity $\mathrm{E}\left[k_t (z_r - z_u)^2\right]$ represents tyre dynamic force (TF). It is a measure of the ability of the tyres to generate accelerating, braking or cornering force. This ability is reduced when tyre force oscillates, or in other words, when the quantity increases.
- The suspension stiffness k that minimises this response quantity satisfies:

$$
\frac{\partial\left\{\mathrm{E}\left[k_t (z_r - z_u)^2\right]\right\}}{\partial k} = 0 \tag{10}
$$

Substituting the expression of $\mathrm{E}\left[k_t (z_r - z_u)^2\right]$ given into (10) yields:

$$
\frac{\pi S_0 \left\{2(m_u + m_s)^3 k - 2(m_u + m_s) m_u m_s k_t\right\}}{m_s^2 c} = 0,
$$

which in turn gives

$$
k = \frac{m_u m_s k_t}{(m_u + m_s)^2} \tag{11}
$$

(b) (ii) 20 Marks

- The suspension damping c that minimises this response quantity satisfies:

$$\frac{\partial\left\{\mathrm{E}\left[k_t\left(z_r - z_u\right)^2\right]\right\}}{\partial c} = 0 \tag{12}$$

Substituting the expression of $\mathrm{E}\left[k_t\left(z_r - z_u\right)^2\right]$ given into (12) yields:

$$\pi S_0\left\{-\frac{\left(m_u + m_s\right)^3 k^2}{m_s^2 c^2} + \frac{2\left(m_u + m_s\right) m_u m_s k_t k}{m_s^2 c^2} + \frac{\left(m_u + m_s\right)^2 k_t}{m_s^2} - \frac{m_u m_s^2 k_t^2}{m_s^2 c^2}\right\} = 0,$$

which in turn gives

$$c^2 = \frac{m_u m_s^2 k_t^2 + \left(m_u + m_s\right)^3 k^2 - 2\left(m_u + m_s\right) m_u m_s k_t k}{\left(m_u + m_s\right)^2 k_t} \tag{13}$$

- Substituting the value of k described in (11) into (13) gives:

$$c^2 = \frac{m_u m_s^2 k_t^2 - \dfrac{m_u^2 m_s^2 k_t^2}{m_u + m_s}}{\left(m_u + m_s\right)^2 k_t} \tag{14}$$

Dividing both the numerator and the denominator of the right-hand side of (14) by m_s^2 gives:

$$c^2 = \frac{m_u k_t\left(-\dfrac{1}{m_u + m_s}\right)}{\left(\dfrac{m_u^2}{m_s^2} + 2\dfrac{m_u}{m_s} + 1\right)} \tag{15}$$

Considering $m_s \gg m_u \gg 1$, (15) reduces to $c^2 = m_u k_t$, or equivalently $c = \sqrt{m_u k_t}$.

Q2

(a) 20 Marks

- The equations of motion can be expressed as:

$$m\ddot{z} = -k_1\left(z - a\theta - z_{r1}\right) - c_1\left(\dot{z} - a\dot{\theta} - \dot{z}_{r1}\right) - k_2\left(z + b\theta - z_{r2}\right) - c_2\left(\dot{z} + b\dot{\theta} - \dot{z}_{r2}\right) \tag{1.1}$$

$$I\ddot{\theta} = ak_1\left(z - a\theta - z_{r1}\right) + ac_1\left(\dot{z} - a\dot{\theta} - \dot{z}_{r1}\right) - bk_2\left(z + b\theta - z_{r2}\right) - bc_2\left(\dot{z} + b\dot{\theta} - \dot{z}_{r2}\right) \tag{1.2}$$

- Arranging (1.1) and (1.2) into matrix form gives:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{P}\dot{\mathbf{u}} + \mathbf{R}\mathbf{u} \tag{2}$$

where

$$\mathbf{x} = \begin{Bmatrix} z \\ \theta \end{Bmatrix},\ \mathbf{u} = \begin{Bmatrix} z_{r1} \\ z_{r2} \end{Bmatrix},\ \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_1a + c_2b \\ -c_1a + c_2b & c_1a^2 + c_2b^2 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_1a + k_2b \\ -k_1a + k_2b & k_1a^2 + k_2b^2 \end{bmatrix},$$

$$\mathbf{P} = \begin{bmatrix} c_1 & c_2 \\ -ac_1 & bc_2 \end{bmatrix},\ \text{and}\ \mathbf{R} = \begin{bmatrix} k_1 & k_2 \\ -ak_1 & bk_2 \end{bmatrix}.$$

(b) 20 Marks

- The undamped natural frequencies of the system can be obtained by resolving the following equation:

$$\left|-\omega^2\mathbf{M} + \mathbf{K}\right| = \mathbf{0} \tag{3}$$

- When the matrix **K** involved in (3) becomes a diagonal matrix, that is, when the term $-k_1a + k_2b$ in matrix **K** becomes zero, the two natural frequencies can be obtained as

$$\omega_1 = \sqrt{\frac{k_1 + k_2}{m}} \tag{4.1}$$

$$\omega_2 = \sqrt{\frac{k_1a^2 + k_2b^2}{I}} \tag{4.2}$$

- Note that when $-k_1a + k_2b = 0$, the two equations involved in $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$ become decoupled. Hence, the bounce mode is independent of the pitch mode, and its natural frequency can be obtained by simplifying the system into a single mass-spring model, where the two springs k_1 and k_2 are in parallel, which results in an equivalent spring stiffness of $k_1 + k_2$.

(c) 40 Marks

- Following lecture notes, z_{r2} can be related to z_{r1} in the time domain using the following equation:

$$z_{r2}(t) = z_{r1}\left(t - \frac{a+b}{V}\right) \tag{5}$$

- Taking Laplace transform, using $a = b$, and setting $s = j\omega$ gives:

$$z_{r2}(j\omega) = z_{r1}(j\omega)e^{-j\omega\frac{2a}{V}} \tag{6}$$

- The transfer function provided in the question tells:

$$z(j\omega)=H_z(j\omega)z_{r1}(j\omega)+H_z(j\omega)z_{r2}(j\omega) \tag{7}$$

- Substituting (6) into (7) to eliminate $z_{r2}(j\omega)$ gives:

$$\frac{z(j\omega)}{z_{r1}(j\omega)}=H_z(j\omega)\left[1+e^{-j\omega\frac{2a}{V}}\right] \tag{8}$$

- Using Euler's equation to simplify (8):

$$\frac{z(j\omega)}{z_{r1}(j\omega)}=H_z(j\omega)\left[1+\cos\left(\frac{2a\omega}{V}\right)-j\sin\left(\frac{2a\omega}{V}\right)\right] \tag{9}$$

- Taking module of (9) then gives:

$$\begin{aligned}\left|\frac{z(j\omega)}{z_{r1}(j\omega)}\right| &=|H_z(j\omega)|\sqrt{\left[1+\cos\left(\frac{2a\omega}{V}\right)\right]^2+\sin\left(\omega\frac{2a\omega}{V}\right)^2}\\ &=|H_z(j\omega)|\sqrt{1+\cos\left(\frac{2a\omega}{V}\right)^2+2\cos\left(\frac{2a\omega}{V}\right)+\sin\left(\omega\frac{2a\omega}{V}\right)^2}\\ &=|H_z(j\omega)|\sqrt{2+2\cos\left(\frac{2a\omega}{V}\right)}\\ &=\sqrt{2}|H_z(j\omega)|\sqrt{1+\cos\left(\frac{2a\omega}{V}\right)}\end{aligned} \tag{10}$$

- The transfer function provided in the question also tells:

$$\theta(j\omega)=-H_\theta(j\omega)z_{r1}(j\omega)+H_\theta(j\omega)z_{r2}(j\omega) \tag{11}$$

- Substituting (6) into (11) to eliminate $z_{r2}(j\omega)$ gives:

$$\frac{\theta(j\omega)}{z_{r1}(j\omega)}=H_\theta(j\omega)\left[-1+e^{-j\omega\frac{2a}{V}}\right] \tag{12}$$

- Using Euler's equation and then taking module gives:

$$\begin{aligned}\left|\frac{\theta(j\omega)}{z_{r1}(j\omega)}\right| &=|H_\theta(j\omega)|\sqrt{\left[-1+\cos\left(\frac{2a\omega}{V}\right)\right]^2+\sin\left(\omega\frac{2a\omega}{V}\right)^2}\\ &=|H_\theta(j\omega)|\sqrt{1+\cos\left(\frac{2a\omega}{V}\right)^2-2\cos\left(\frac{2a\omega}{V}\right)+\sin\left(\omega\frac{2a\omega}{V}\right)^2}\\ &=|H_\theta(j\omega)|\sqrt{2-2\cos\left(\frac{2a\omega}{V}\right)}\\ &=\sqrt{2}|H_\theta(j\omega)|\sqrt{1-\cos\left(\frac{2a\omega}{V}\right)}\end{aligned} \tag{13}$$

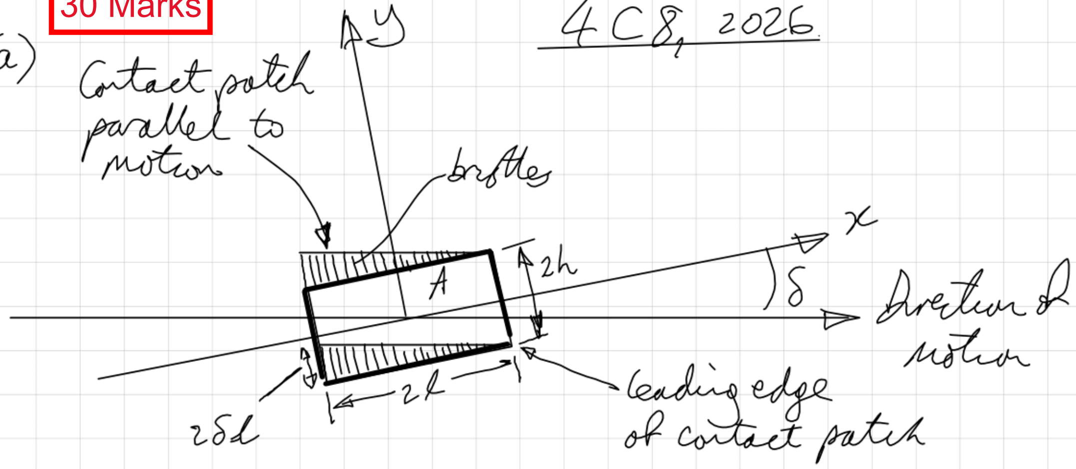
(d) $\boxed{\text{20 Marks}}$

- The term $\sqrt{1+\cos\left(\frac{2a\omega}{V}\right)}$ in Equation (10) acts essentially as a 'filter', parameterised by wheelbase $2a$ and vehicle speed V. When the road excitation frequency ω is of such values enabling $\frac{2a\omega}{V}=\pi+2k\pi$, for $k=0,1,2\ldots$ the bounce response vanishes.
- The term $\sqrt{1-\cos\left(\frac{2a\omega}{V}\right)}$ in Equation (13) also acts essentially as a 'filter', parameterised by wheelbase $2a$ and vehicle speed V. When the road excitation frequency ω is of such values enabling $\frac{2a\omega}{V}=0+2k\pi$, for $k=0,1,2\ldots$ the pitch response vanishes.
- For a particular vehicle of a specific wheelbase, wheelbase filtering leads to the vibration response of the vehicle body being strongly dependent on vehicle speed.

30 Marks

4C8, 2026

Q3 (a)



The lateral deflections of the tips of the bristles from their unstressed position is

$$q_y(x,y) = \delta(l-x) \qquad \text{①}$$

Provided there is no longitudinal traction or braking force

$$q_x(x,y) = 0 \qquad \text{②}$$

The realigning moment is given by (datasheet)

$$N = \iint_A (x\sigma_y - y\sigma_x)\, dA \qquad \text{③}$$

The contact traction generated by the bristles is:

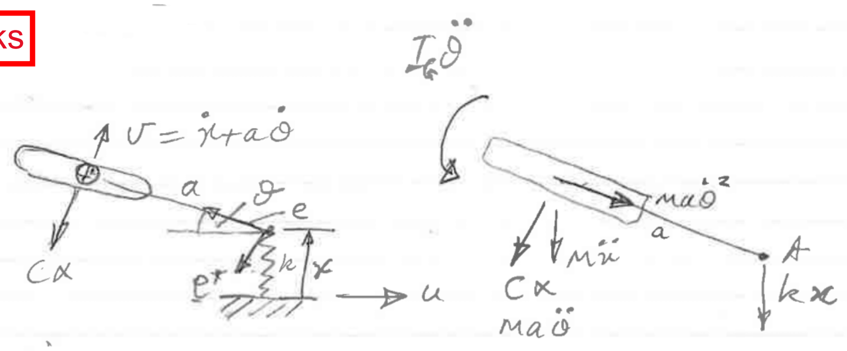
$$\sigma_y = K_y q_y \quad \& \quad \sigma_x = 0 \qquad \text{④}$$

①, ② & ④ into ③ gives

$$N = \int_{-h}^{h} dy \int_{-l}^{l} \left(x\, K_y\, \delta(l-x) - 0\right) dx = -\left(\tfrac{4}{3} l^2 h K_y\right)\delta$$

Pneumatic trail = distance of centre of pressure of lateral tractions behind the geometric centre of the contact patch. For a triangular pressure distribution, centre of pressure is the centroid, $2l/3$ from base of triangle so the pneumatic trail is $l/3$ (i.e. 1/6 length of contact patch)

Realigning moment can be ignored when it is much smaller than ½ wheelbase — so that moments of the lateral forces about G are >> realigning moments (b)(i) 35 Marks



$$\underline{R}_G = x\underline{j} + a\underline{e}$$

$$\dot{\underline{R}}_G = \dot{x}\underline{j} + a\dot{\theta}e^*$$

$$\ddot{\underline{R}}_G = \ddot{x}\underline{j} + a\ddot{\theta}\underline{e}^* - a\dot{\theta}^2\underline{e}$$

Slip angle $\quad \alpha = \dfrac{\dot{x} + a\dot{\theta}}{u} + \theta$

$$\downarrow \Sigma F_V: \quad m(\ddot{x} + a\ddot{\theta}) + c\left(\frac{\dot{x} + a\dot{\theta}}{u} + \theta\right) + kx = 0$$

$$\circlearrowleft \Sigma M_A: \quad I_G\ddot{\theta} + a\left[m(\ddot{x} + a\ddot{\theta}) + c\left(\frac{\dot{x} + a\dot{\theta}}{u} + \theta\right)\right] = 0$$

in matrix form

$$\begin{bmatrix} m & ma \\ ma & I_G + ma^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c/u & ca/u \\ ca/u & ca^2/u \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & c \\ 0 & ca \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = 0$$

(b)(ii)
35 Marks

Let $\underset{\sim}{q} = \begin{Bmatrix} X \\ \Theta \end{Bmatrix} e^{st}$ Then the characteristic eqn is:

$$\begin{vmatrix} ms^2 + cs/u + k & mas^2 + \frac{ca}{u}s + c \\ mas^2 + \frac{cas}{u} & (I_G + ma^2)s^2 + \frac{ca^2 s}{u} + ca \end{vmatrix} = \underline{0}$$

from which:

$$0 = \left(ms^2 + cs/u + k\right)\left[(I_G + ma^2)s^2 + \frac{ca^2 s}{u} + ca\right] - \left(mas^2 + \frac{cas}{u} + c\right)\left(mas^2 + \frac{cas}{u}\right)$$

$$\Rightarrow s^4\left[m(I_G + \cancel{ma^2}) - \cancel{m^2a^2}\right] + s^3\left[\cancel{\frac{mca^2}{u}} + \frac{c}{u}(I_G + \cancel{ma^2}) - \cancel{\frac{mca^2}{u}} - \cancel{\frac{mca^2}{u}}\right]$$

$$+ s^2\left[k(I_G + ma^2) + \cancel{\frac{c^2a^2}{u^2}} + mca - mac - \cancel{\frac{c^2a^2}{u^2}}\right]$$

2 cont

$$+ s\left[\frac{c^2 \cancel{a}}{\cancel{u}} + \frac{ca^2k}{u} - \frac{\cancel{c^2 a}}{\cancel{u}}\right] + kca.$$

ie $$s^4(\underbrace{mI_G}_{a_4}) + s^3\left(\underbrace{\frac{I_G c}{u}}_{a_3}\right) + s^2 k(\underbrace{I_G + ma^2}_{a_2})$$

$$+ s\left(\underbrace{\frac{ca^2k}{u}}_{a_1}\right) + \underbrace{kca}_{a_0}$$

Stability conditions (Routh-Hurwitz)

(i) All a's +ve $\Rightarrow u > 0$.

(ii) $a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$

$$\Rightarrow k\left(\frac{\cancel{c}a^2k}{\cancel{u}}\right)(I_G + ma^2)\left(\frac{I_G \cancel{c}}{\cancel{u}}\right) > (kca)\left(\frac{I_G \cancel{c}}{\cancel{u}}\right)^2 + (mI_G)\left(\frac{\cancel{c}a^2k}{\cancel{u}}\right)^2$$

$$\Rightarrow k\cancel{I_G} a^{\cancel{2}} \cancel{k}(I_G + ma^2) > I_G^{\cancel{2}} \cancel{k} c \cancel{a} + m\cancel{I_G} a^{\cancel{4}3} k^{\cancel{2}}$$

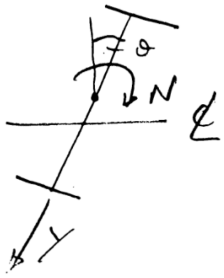
$$ka(\cancel{I_G} + \cancel{ma^2}) > \cancel{I_G} c + \cancel{ma^3}k$$

Dimensionally correct ✓

$$\Rightarrow \underline{\underline{k > c/a}}$$

40 Marks

Q4 (a) Force and moment acting on rigid wheelset see lecture notes (derivation required). In general case, net yawing moment is



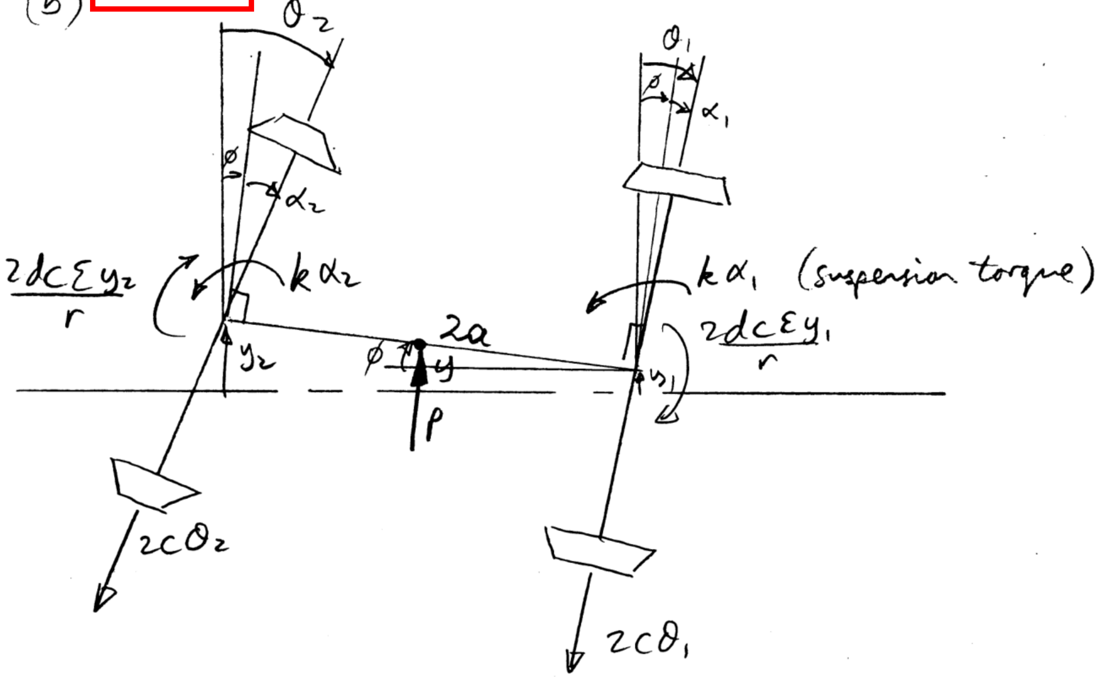
$$N = 2dC\left(\frac{\varepsilon y}{r} - \frac{d\dot{\theta}}{u} - d/R\right)$$

$$\& \quad Y = 2c(\theta + \dot{y}/u)$$

So for straight line motion $R \to \infty$ and in the steady state $\dot{\theta} = \dot{y} = 0$

$$N_1 = \frac{2dc\varepsilon y_1}{r} \quad \& \quad Y_1 = 2c\theta_1$$

(b) 30 Marks



$$\left.\begin{aligned} \phi &= \frac{y_2 - y_1}{2a} & \alpha_2 &= \theta_2 - \phi = \theta_2 - \left(\frac{y_2 - y_1}{2a}\right) \\ y &= \frac{y_1 + y_2}{2} & \alpha_1 &= \theta_1 - \left(\frac{y_2 - y_1}{2a}\right) \end{aligned}\right\} \quad (1)$$

Moment equilibrium for each axle:

$$\frac{2dC\varepsilon}{r} y_1 = k\left[\theta_1 + \left(\frac{y_1 - y_2}{2a}\right)\right] \qquad \text{——— (2)}$$

$$\frac{2dC\varepsilon}{r} y_2 = k\left[\theta_2 + \left(\frac{y_1 - y_2}{2a}\right)\right] \qquad \text{——— (3)}$$

2) cont

Force equilibrium on whole vehicle

$$2c(\theta_1 + \theta_2) = P \qquad \text{——— (4)}$$

& Moments about frame centre

$$\frac{2dc\varepsilon}{r}(y_1 + y_2) + 2ac(\theta_1 - \theta_2) = 0 \qquad \text{—— (5)}$$

4 variables $\theta_1, \theta_2, y_1, y_2$ & 4 equations (2)-(5)

30 Marks (c) If $k \to \infty$, (1) $\Rightarrow$ $\theta_1 + \frac{y_1 - y_2}{2a} = \frac{2dc\varepsilon y_1}{rk} \to 0$

Similarly (2) $\Rightarrow$ $\theta_2 + \frac{y_1 - y_2}{2a} = 0$

So $\theta_1 = \theta_2 = \frac{y_2 - y_1}{2a} = \phi$ (from (1))

$2y =$

Hence (4) $\Rightarrow$ $\phi = P/4c$

and (5) $\Rightarrow$ $y_1 = -y_2 = a\phi = \frac{aP}{4c}$

$\therefore$ lateral tracking error at the centre of the bogie is zero ie $y = \frac{y_1 + y_2}{2} = 0$