Version XN/3

EGT3
ENGINEERING TRIPOS PART IIB

Thursday 30 April 2026   9.30 to 11.10

Module 4C8

VEHICLE DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C8 Vehicle Dynamics data sheet (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Figure 1(a) shows a linear, one-degree-of-freedom model with vertical displacement z_s. The model may be used to study the vertical vibration of a vehicle travelling over a road surface with random vertical displacement z_r. The sprung mass is denoted by m_s, the suspension stiffness by k, and the suspension damping by c. The vertical velocity of the road surface is characterised by a double-sided mean square spectral density S_0, which can be treated as white noise.

(i) State the key assumptions implicit in the model. [10%]

(ii) Show that the transfer function $z_s(j\omega)/z_r(j\omega)$ can be expressed in the following form, under the assumption of zero initial conditions:

$$\frac{z_s(j\omega)}{z_r(j\omega)} = \frac{j\omega c + k}{-\omega^2 m_s + j\omega c + k} \,,$$

where ω is the angular frequency. [10%]

(iii) A response quantity of the model is defined as $\mathrm{E}[(z_s - z_r)^2]$. State the relevance of this response quantity to vehicle performance. Derive an expression for this quantity. [35%]

You may use the following result:

If a second-order stable transfer function $H_2(j\omega)$ is expressed as

$$H_2(j\omega) = \frac{(j\omega)B_1 + B_0}{(j\omega)^2 A_2 + j\omega A_1 + A_0} \,,$$

the mean square integral of $H_2(j\omega)$ can be expressed as

$$\int_{-\infty}^{\infty} |H_2(j\omega)|^2 \, d\omega = \frac{\pi(A_0 {B_1}^2 + A_2 {B_0}^2)}{A_0 A_1 A_2} \,.$$

(b) The model shown in Fig. 1(a) is extended to include an unsprung mass m_u, supported by a tyre with vertical stiffness k_t, as illustrated in Fig. 1(b). The extended model has two degrees of freedom, z_s and z_u. A response quantity of the model is defined as $\mathrm{E}\left[\left(k_t(z_r - z_u)\right)^2\right]$, which may be expressed as:

$$\mathrm{E}\left[\left(k_t(z_r - z_u)\right)^2\right] = \frac{\pi S_0[(m_u + m_s)^3 k^2 - 2(m_u + m_s) m_u m_s k_t k + (m_u + m_s)^2 k_t c^2 + m_u {m_s}^2 {k_t}^2]}{{m_s}^2 c} \,.$$

(i) State the relevance of this response quantity to vehicle performance. Find the value of suspension stiffness k that minimises this response quantity. [25%]

(ii) Using the value of k determined from (b)(i), and considering the fact that $m_s \gg m_u$, find the suspension damping c that minimises the response quantity given in (b). [20%]

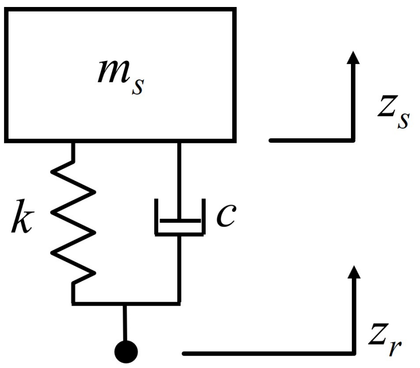

Fig. 1(a)

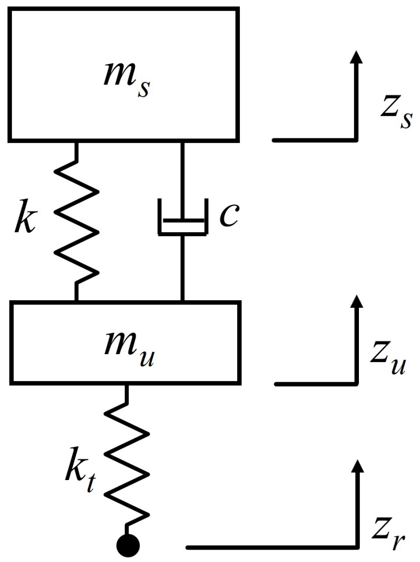

Fig. 1(b)

2 Figure 2 shows a pitch-plane model of a vehicle with mass m and pitch inertia I. The front suspension has stiffness k_1 and damping c_1, and the rear suspension has stiffness k_2 and damping c_2. The distances from the vehicle body centre of mass to the front and rear axles are a and b, respectively. The vehicle travels at a constant speed V over a randomly rough road surface. The vertical displacements of the road surface at the front and rear axles are z_{r1} and z_{r2}, respectively. The vertical displacement and pitch angle of the vehicle body at the centre of mass are represented using z and θ, respectively.

(a) Show that the equations of motion can be expressed in the following matrix form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{P}\dot{\mathbf{u}} + \mathbf{R}\mathbf{u}\ ,$$

where $\mathbf{x} = \{z \quad \theta\}^\top$ and $\mathbf{x} = \{z_{r1} \quad z_{r2}\}^\top$. State the expressions for matrices **M**, **C**, **K**, **P** and **R**. [20%]

(b) It is proposed to estimate the undamped natural frequency associated with the bounce mode, ω_1 using

$$\omega_1 = \sqrt{\frac{k_1 + k_2}{m}}\ .$$

Using the results from (a) to state the condition under which the proposed estimate is accurate. Under the same condition, find the undamped natural frequency associated with the pitch mode ω_2. [20%]

(c) When $a = b$, $k_1 = k_2$, and $c_1 = c_2$, the transfer functions relating road inputs z_{r1} and z_{r2} to vehicle responses z and θ can be expressed as

$$\begin{Bmatrix} z(j\omega) \\ \theta(j\omega) \end{Bmatrix} = \begin{bmatrix} H_z(j\omega) & H_z(j\omega) \\ -H_\theta(j\omega) & H_\theta(j\omega) \end{bmatrix} \begin{Bmatrix} z_{r1}(j\omega) \\ z_{r2}(j\omega) \end{Bmatrix}\ .$$

By considering the relationship between z_{r1} and z_{r2}, derive expressions for

$$\left|\frac{z(j\omega)}{z_{r1}(j\omega)}\right| \quad \text{and} \quad \left|\frac{\theta(j\omega)}{z_{r1}(j\omega)}\right|$$

in terms of ω, a, V, $H_z(j\omega)$ and $H_\theta(j\omega)$. [40%]

(d) Using the results obtained from (c), explain the mechanism of ‘wheelbase filtering’. [20%]

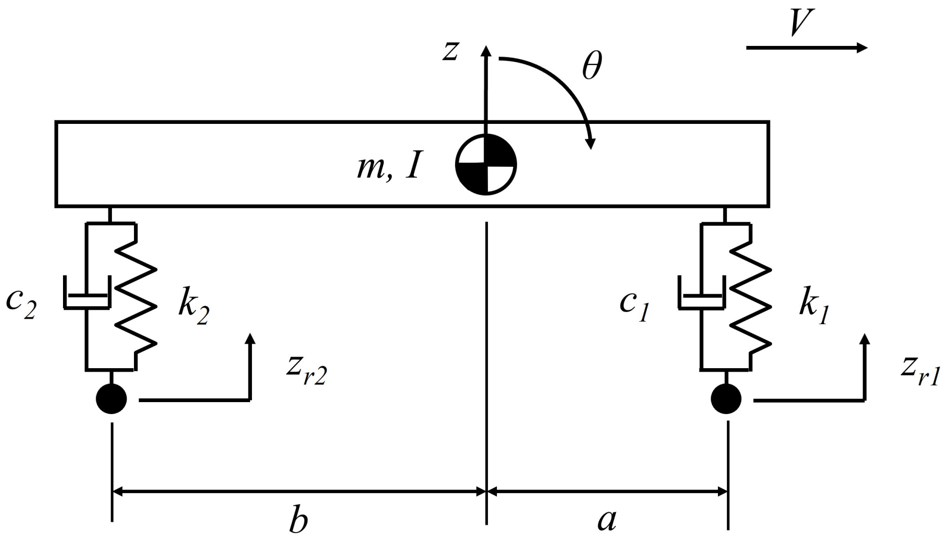

Fig. 2

3 (a) Use the 'brush' model with lateral bristle stiffnesses K_y to derive an expression for the realigning moment generated by a wheel with rectangular contact area having dimensions $2l \times 2h$, a normal load Z, and uniform contact pressure, rolling with small yaw angle δ. Define the pneumatic trail and calculate it in this case. Explain scenarios in vehicle dynamics when the realigning moment can be ignored. [30%]

(b) The castered wheel of a baggage trolley is idealised as shown in plan view in Fig. 3. A wheel of mass m and diametral moment of inertia I, is free to rotate about a horizontal axis (with stiff bearings) through its centre of mass D. The lateral creep coefficient of the wheel is C. The axle is attached to a light, rigid yoke BD, of length a which is free to rotate about a vertical axis at B, with yaw angle θ. The leg of the trolley is flexible and allows lateral movement x of point B. This is modelled as a linear spring of stiffness k joining B to the body of the trolley at A as shown. The body of the trolley moves in a straight line at constant speed u, and the floor is smooth and level.

(i) Derive equations for small lateral and yaw oscillations of the system in the horizontal plane (neglecting gyroscopic effects). [35%]

(ii) Use the Routh-Hurwitz criterion to determine the conditions for which the motion of the system would be stable. [35%]

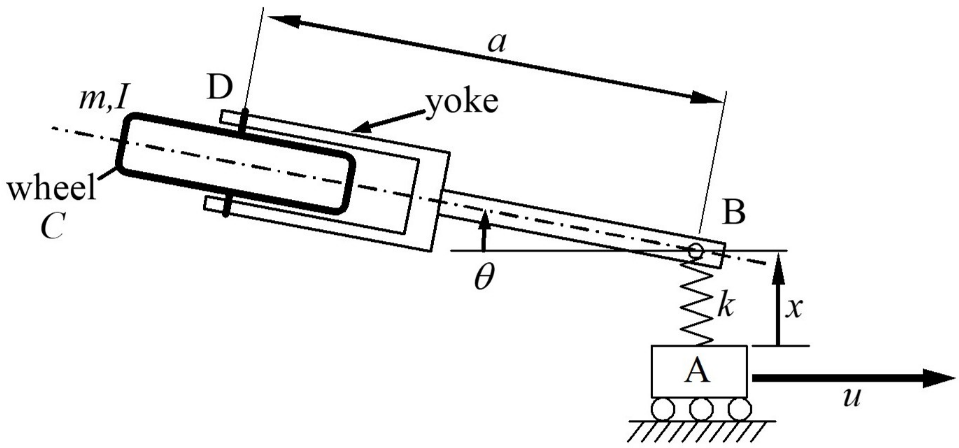

Fig. 3

4 Figure 4 shows a schematic plan view of an idealised railway bogie which consists of two light, rigid wheelsets, connected to a light frame AB of length $2a$ by suspensions with yaw stiffness k. The wheels all have lateral and longitudinal creep coefficients C, effective conicity ε, and average rolling radii r when running in the central position. The bogie runs on a straight track of gauge $2d$ at steady forward speed u, with a constant lateral force P applied at its mid point C. The wheelsets take up steady yaw angles θ_1 and θ_2 with constant lateral tracking errors y_1 and y_2 as shown (exaggerated) on the figure.

(a) Show that the net yawing moment acting on the front wheelset as a result of the wheel/rail contact forces is

$$N = \frac{2dC\varepsilon y_1}{r} ,$$

and derive an expression for the net lateral force acting on the front wheelset. State your assumptions. [40%]

(b) Derive a set of equations that could be used to find the steady state configuration of the bogie $(y_1,\ y_2,\ \theta_1,\ \theta_2)$ and explain how you would solve these equations to find the yaw angle of the bogie frame relative to the track centre-line. [30%]

(c) Use your equations from part (b) to determine the lateral tracking error at C and the yaw angle of the bogie, for the case $k \to \infty$. Sketch the bogie configuration in this case. [30%]

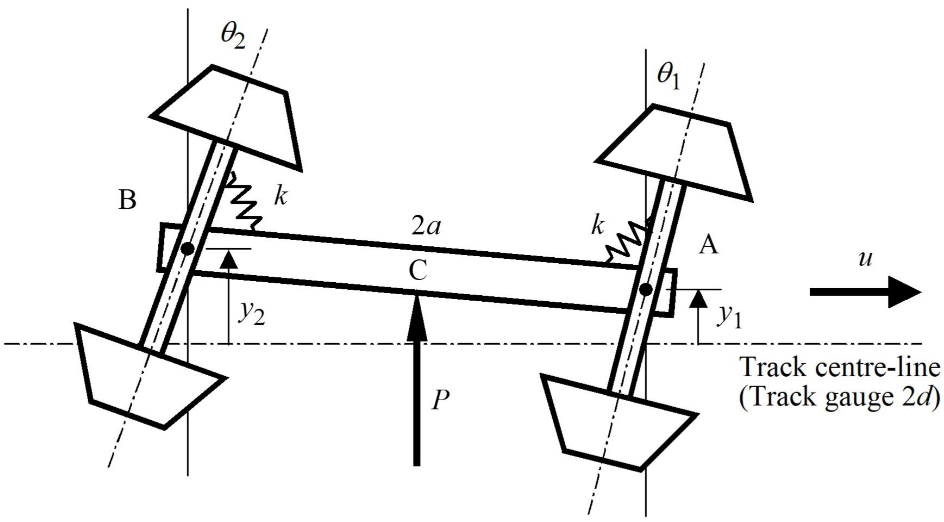

Fig. 4

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Engineering Tripos Part IIB

Data sheet for Module 4C8: Vehicle Dynamics

1 Creep Forces in Rolling Contact

1.1 Surface Tractions

Longitudinal force:

$$X = \iint_A \sigma_x \, dA$$

Lateral force:

$$Y = \iint_A \sigma_y \, dA$$

Realigning moment:

$$N = \iint_A (x\sigma_y - y\sigma_x) \, dA$$

where

$\sigma_x, \ \sigma_y$ = longitudinal, lateral surface tractions

$x, \ y$ = coordinates along, across contact patch

A = area of contact patch

1.2 Brush Model

$$\sigma_x = K_x q_x, \quad \sigma_y = K_y q_y \quad \text{for} \quad \sqrt{\sigma_x^2 + \sigma_y^2} \leq \mu p$$

where

$q_x, \ q_y$ = longitudinal, lateral displacements of 'bristles' relative to wheel rim

$K_x, \ K_y$ = longitudinal, lateral stiffness per unit area

μ = coefficient of friction

p = local contact pressure

1.3 Linear Creep Equations

$$\begin{aligned} X &= -C_{11}\xi \\ Y &= -C_{22}\alpha + C_{23}\psi \\ N &= C_{32}\alpha + C_{33}\psi \end{aligned}$$

where X, Y, N are defined as in 1.1 above, and

C_{ij} = coefficients of linear creep

ξ = longitudinal creep ratio = longitudinal creep speed / forward speed

α = lateral creep ratio = (lateral speed / forward speed) - steer angle

ψ = spin creep ratio = spin angular velocity / forward speed

2 Plane Motion in a Moving Coordinate Frame

$$\ddot{\mathbf{R}}_{O_1} = (\dot{u} - v\Omega)\mathbf{i} + (\dot{v} + u\Omega)\mathbf{j}$$

with $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ axis system fixed to body at point O_1

where

u = speed of point O_1 in $\mathbf{i}$ direction

v = speed of point O_1 in $\mathbf{j}$ direction

Ω = angular rate of body in $\mathbf{k}$ direction

3 Routh-Hurwitz Stability Criteria

$$(a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0)\ y = x(t) \qquad \text{is stable if all } a_i > 0$$

$$(a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0)\ y = x(t) \qquad \text{is stable if (i) all } a_i > 0 \text{ and (ii) } a_1a_2 > a_0a_3$$

$$(a_4\frac{d^4}{dt^4} + a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0)\ y = x(t) \qquad \text{is stable if (i) all } a_i > 0 \text{ and (ii) } a_1a_2a_3 > a_0a_3^2 + a_4a_1^2$$

4 Vehicle Vibrations

4.1 Random Vibration

$$E[x^2(t)] = \frac{1}{T}\int_{t=0}^{t=T} x^2(t)\,dt = \int_{\omega=-\infty}^{\omega=\infty} S_x(\omega)\,d\omega \qquad \left(\text{or } \int_{\omega=0}^{\omega=\infty} S_x(\omega)\,d\omega \;\text{ if } S_x(\omega) \text{ is single sided}\right)$$

$$S_{\dot{x}}(\omega) = \omega^2 S_x(\omega) \qquad (\text{Spectrum of time derivative of } x \text{ is } \omega^2 \text{ times spectrum of } x)$$

4.2 Single Input – Single Output System

$$S_y(\omega) = |H_{yx}(\omega)|^2 S_x(\omega)$$
$$y(\omega) = H_{yx}(\omega)\; x(\omega)$$

4.3 Two Input – Two Output System

$$\begin{bmatrix} y_1(\omega) \\ y_2(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} \begin{bmatrix} x_1(\omega) \\ x_2(\omega) \end{bmatrix}$$

$$\begin{bmatrix} S^y_{11}(\omega) & S^y_{12}(\omega) \\ S^y_{21}(\omega) & S^y_{22}(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^* \begin{bmatrix} S^x_{11}(\omega) & S^x_{12}(\omega) \\ S^x_{21}(\omega) & S^x_{22}(\omega) \end{bmatrix} \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^{\mathbf{T}}$$

where * means complex conjugate and T means transpose.

If x_1 and x_2 are uncorrelated, the following equations hold:

$$S_{(x_1+x_2)}(\omega) = S_{x_1}(\omega) + S_{x_2}(\omega)$$

$$S^x_{12}(\omega) = S^x_{21}(\omega) = 0$$
$$E[(x_1(t) + x_2(t))^2] = E[x_1^2(t)] + E[x_2^2(t)]$$