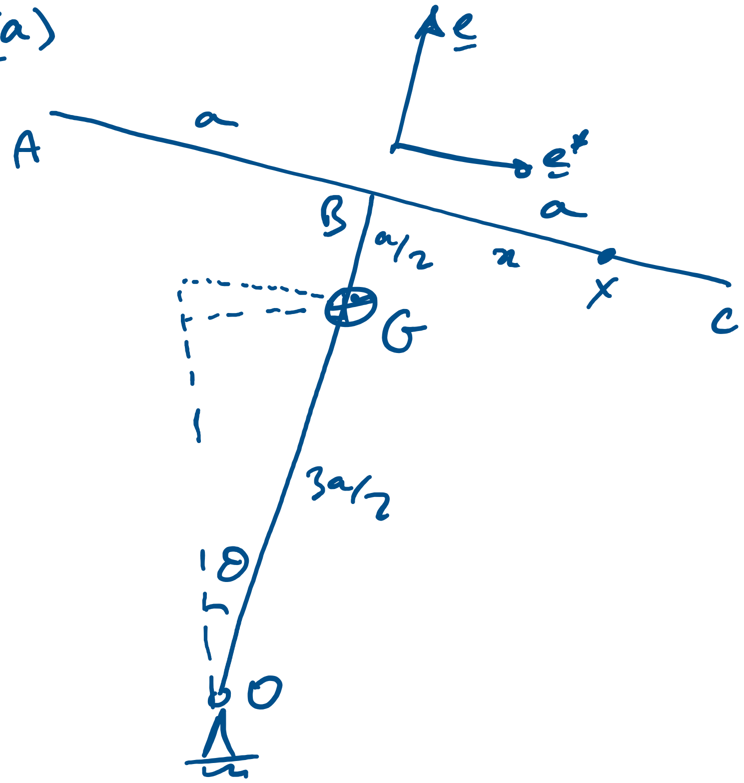
Paper 1

MECHANICAL ENGINEERING

SOLUTIONS

Q1 (a)



(i) CoM G @ midpoint between CoM of each bar,

so $\underline{r}_G = \frac{3a}{2}\underline{e}$

(ii) $$I_G = I_{G_1} + m\left(\tfrac{a}{2}\right)^2 + I_{G_2} + m\left(\tfrac{a}{2}\right)^2$$

$$= 2\left\{\frac{m.(2a)^2}{12} + \frac{ma^2}{4}\right\} = \frac{7}{6}ma^2$$

(b) $$\underline{r}_X = 2a\,\underline{e} + x\,\underline{e}^*$$

$$\dot{\underline{r}}_X = 2a\dot{\theta}\,\underline{e}^* - x\dot{\theta}\,\underline{e}$$

$$\ddot{\underline{r}}_X = 2a\ddot{\theta}\,\underline{e}^* - 2a\dot{\theta}^2\,\underline{e} - x\ddot{\theta}\,\underline{e} - x\dot{\theta}^2\,\underline{e}^*$$

$$= -\left(2a\dot{\theta}^2 + x\ddot{\theta}\right)\underline{e} + \left(2a\ddot{\theta} - x\dot{\theta}^2\right)\underline{e}^*$$

(c) $V = 2mg\frac{3a}{2}(1-\cos\theta)$ taking $V(0)=0$.

$= 3mga(1-\cos\theta)$

$T = \frac{1}{2}2m v_G^2 + \frac{1}{2}I_G\dot{\theta}^2$

$= m\left(\frac{3}{2}a\dot{\theta}\right)^2 + \frac{1}{2}\cdot\frac{7}{6}ma^2\dot{\theta}^2$

$= \frac{17}{6}ma^2\dot{\theta}^2$

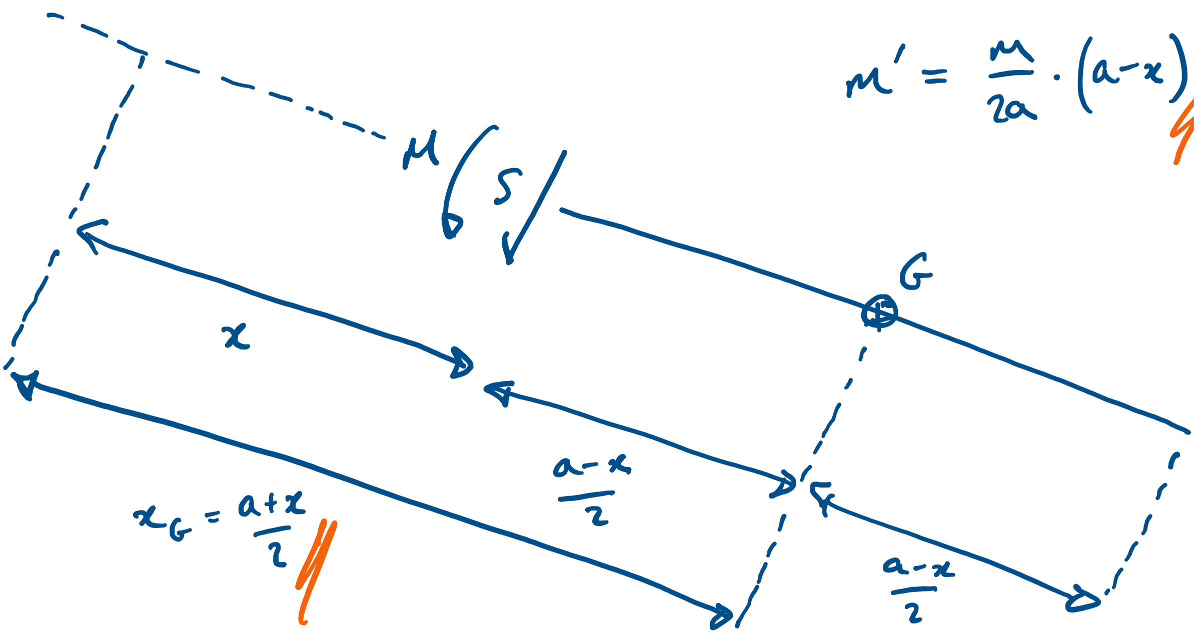
$$3\cancel{m}g\cancel{a}(1-\cos\theta) = \frac{17}{6}\cancel{m}a^{\cancel{2}}\dot{\theta}^2$$

$$\dot{\theta}^2 = \frac{3g(1-\cos\theta)}{17/6\cdot a} = \frac{18g(1-\cos\theta)}{17a}$$ //

$$\ddot{\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = \frac{1}{2}\cdot\frac{18g\sin\theta}{17a} = \frac{9g\sin\theta}{17a}$$ //

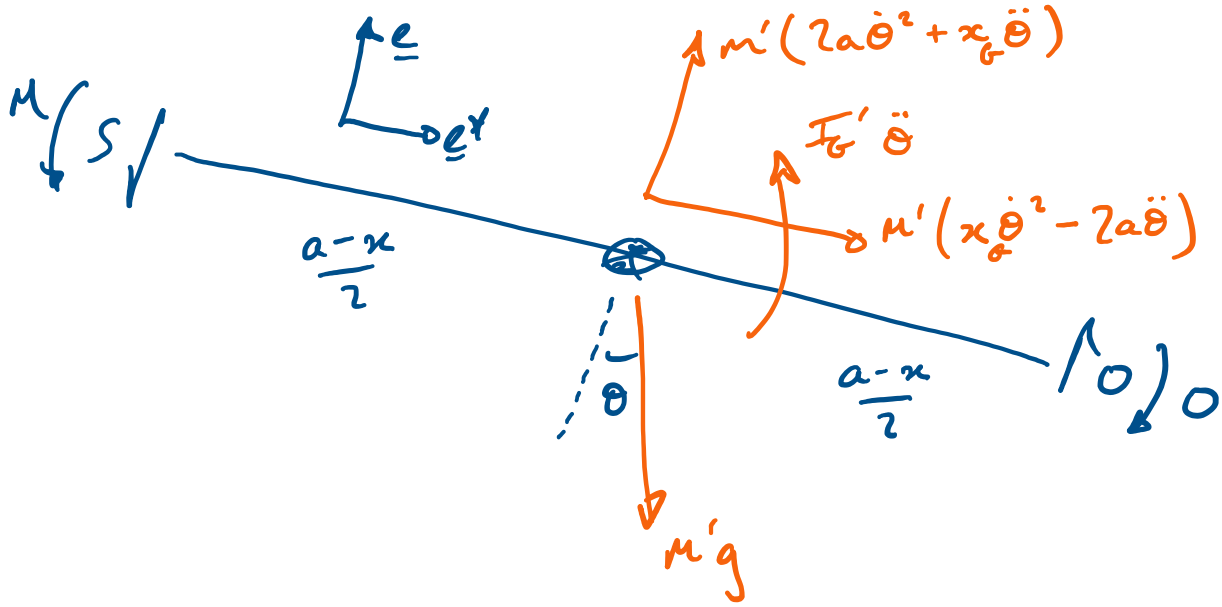
(d)(i) Cut @ distance x from B:

$m' = \frac{m}{2a}\cdot(a-x)$ //



$x_G = \frac{a+x}{2}$ //

(ii)



$$\sum F /\!/ \underline{e} \Rightarrow S = m'(2a\dot\theta^2 + x_G\ddot\theta) - m'g\cos\theta$$

$$= \frac{M}{2a}(a-x)\left[\left(2a\dot\theta^2 + \frac{a+x}{2}\ddot\theta\right) - g\cos\theta\right]$$

$$S = \frac{M}{2a}(a-x)\left[2\cancel{a}\cdot\frac{18g(1-\cos\theta)}{17\cancel{a}} + \frac{a+x}{2}\cdot\frac{9g\sin\theta}{17a} - g\cos\theta\right]$$

$$S = \frac{M}{2a}(a-x)g\left[\underbrace{\frac{36}{17}}_{2.12} - \underbrace{\frac{53}{17}}_{3.12}\cos\theta + \underbrace{\frac{9}{34}}_{0.26}\left(1+\frac{x}{a}\right)\sin\theta\right]$$

(e) $\theta = \pi/2 \Rightarrow \cos\theta = 0,\ \sin\theta = 1.$

$M = \frac{dS}{dx}$, max when $S = 0$.

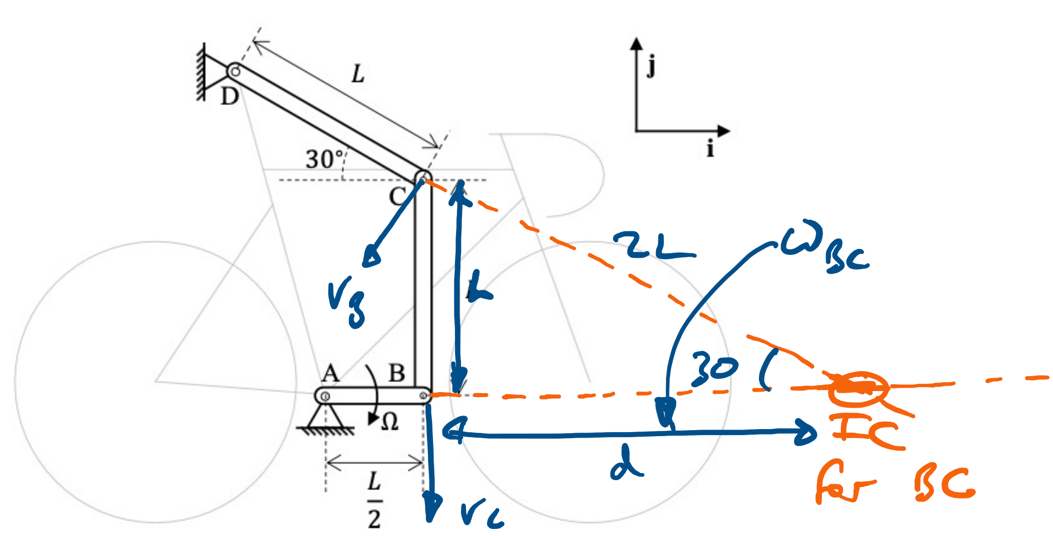
ie @ $x = a$ (free end ✓)

& $\frac{36}{17} + \frac{9}{34}\left(1+\frac{x}{a}\right) = 0$

$$\frac{x}{a} = \frac{-36/17}{9/34} - 1 \Rightarrow x = -9a$$

BUT analysis valid for $0 < x < a$, so this is outside of bar, & max must therefore be at joint B

Q2 // (a)



IC for AB : A //

IC for CD : D //

IC for BC : $r_{IC/B}$ $\Rightarrow$ $\tan 30 = L/d$

so $d = L/\tan 30 = \underline{\sqrt{3}L}$

$\sqrt{3}L$ to the right of B //.

(b) $\omega_{AB} = \Omega$ ↻

$v_B = \omega_{BC} \cdot \sqrt{3}L = \Omega L/2$

$\Rightarrow$ $\omega_{BC} = \Omega/2\sqrt{3} = \frac{\sqrt{3}}{6}\Omega$ ↺ anticlockwise //

$v_C = \omega_{BC} \cdot 2L = \omega_{CD} \cdot L$

$\omega_{CD} = 2\omega_{BC} = \frac{\sqrt{3}}{3}\Omega$ ↻ clockwise //

(c) relative ω

Joint A : $\omega_{AB} = \Omega$

B : $\omega_{AB} + \omega_{BC} = \left(\sqrt{3}/6 + 1\right)\Omega$

C : $\omega_{BC} + \omega_{CD} = \left(\sqrt{3}/3 + \sqrt{3}/6\right)\Omega = \frac{\sqrt{3}}{2}\Omega$

D : $\omega_{CD} = \sqrt{3}/3\,\Omega$

$$Q_C \cdot \sqrt{3}/2\, \cancel{\Omega} = Q_A \cdot \cancel{\Omega} + Q_B\left(1 + \sqrt{3}/6\right)\cancel{\Omega}$$

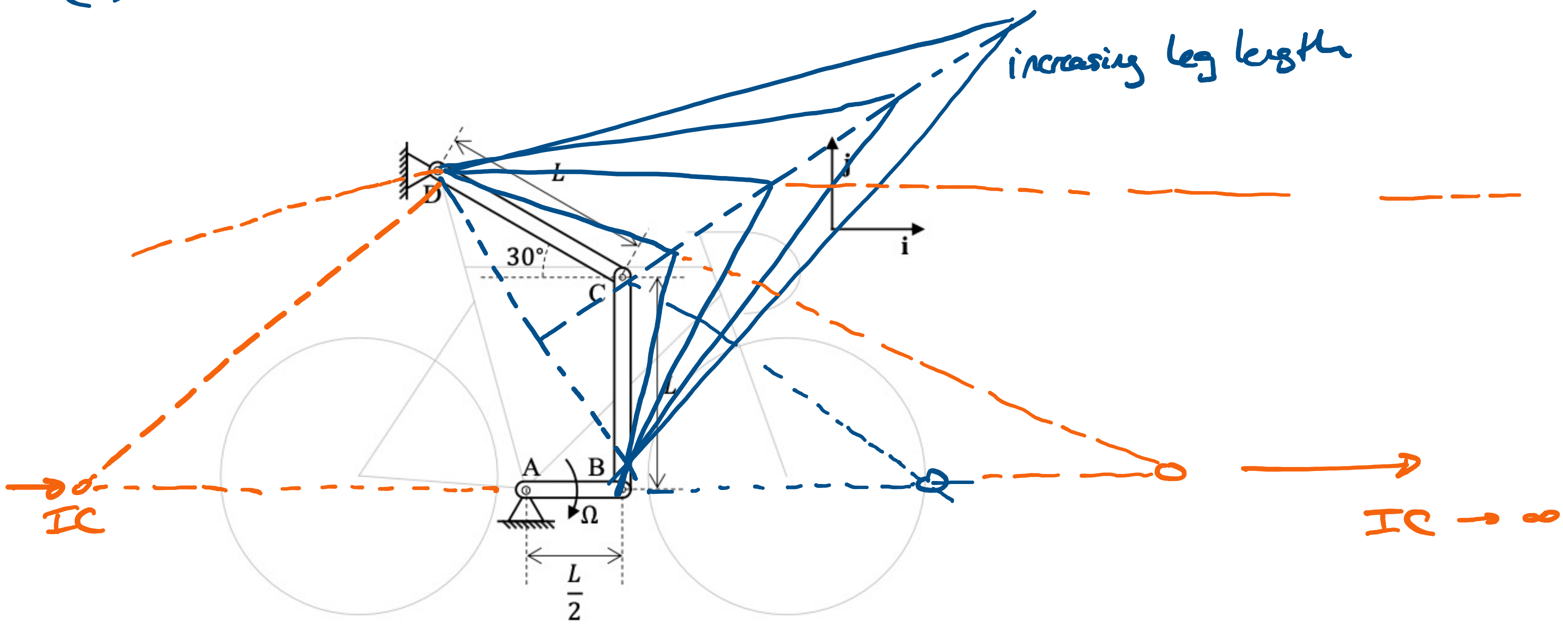
$$Q_C = \frac{2\sqrt{3}}{3} Q_A + \left(\frac{2\sqrt{3}}{3} + \frac{1}{3}\right) Q_B$$

(d) Power: $Q_C' = Q_C + \Delta Q_C$

$$\Delta Q_C \sqrt{3}/2\, \Omega = R v$$
$$= \alpha v^3$$

$$\Delta Q_C = \frac{2\alpha}{\sqrt{3}} \frac{v^3}{\Omega} = \frac{2\sqrt{3}\alpha v^3}{3\Omega}$$

(e)

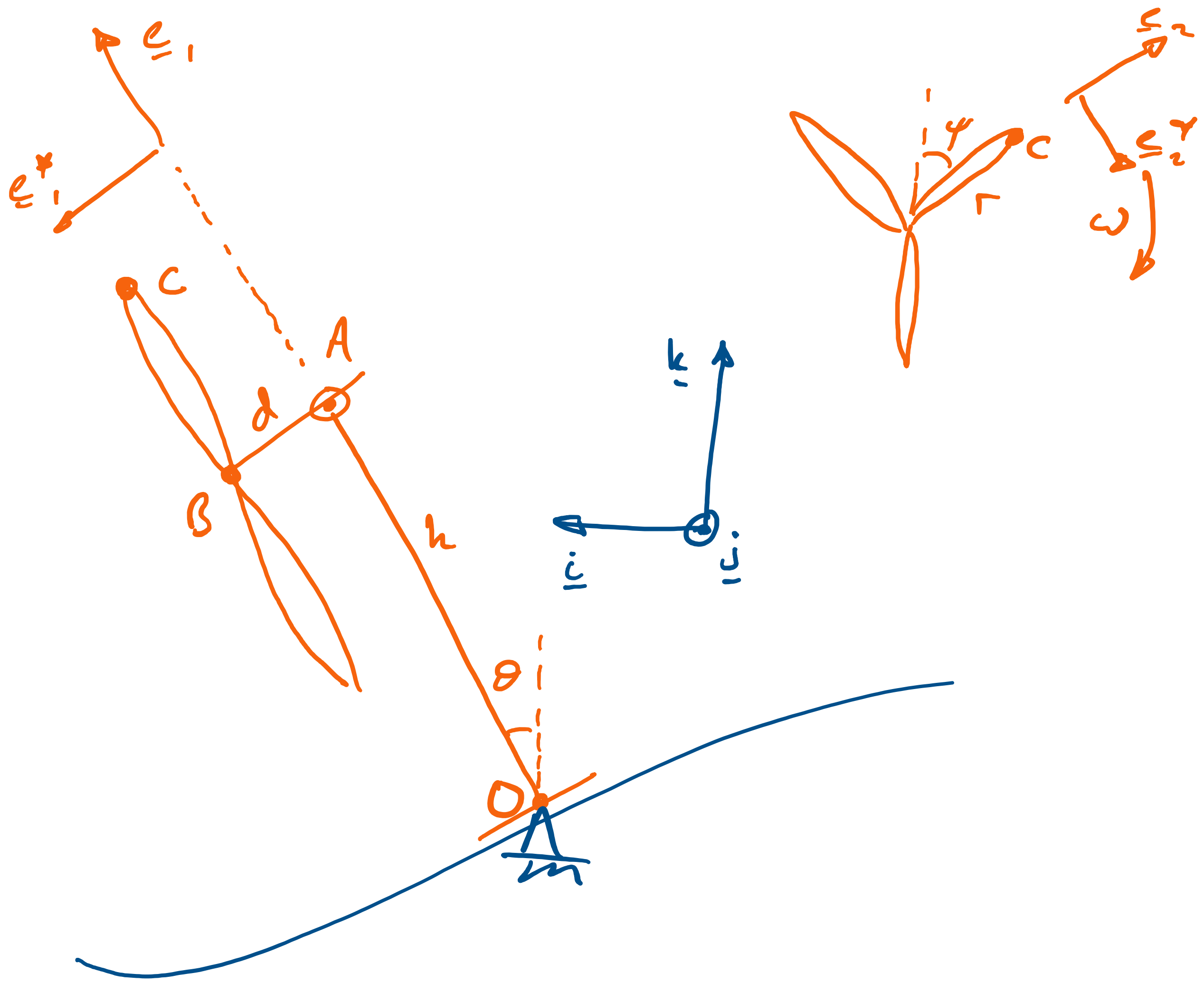


As leg length L increases, IC moves to right, tending to ∞ when AB is horizontal. Then switches to ∞ on left side, and approaches a final position in line with DC & orthogonal to AC.

So L increasing ⇒
- longer distance from IC to B
- ω_{BC} & ω_{CD} lower
- joint ω_c lower ⇒ Q_B higher //.

So longer leg length, or smaller frame ⇒ increased torque needed (at least for position analysed) //

Q3



(a) $\theta(t) = 0, \quad \dot{\psi} = \omega$

(i) $\dot{\underline{e}}_2 = \dot{\psi}\,\underline{e}_2^* = \omega\,\underline{e}_2^*$

$\dot{\underline{e}}_2^* = -\dot{\psi}\,\underline{e}_2 = -\omega\,\underline{e}_2$

(ii) $\underline{r}_c = h\underline{e}_1 + d\underline{e}_1^* + r\underline{e}_2$

$\dot{\underline{r}}_c = \underbrace{\cancel{h\dot{\underline{e}}_1}}_{0} + \underbrace{\cancel{d\dot{\underline{e}}_1^*}}_{0} + r\dot{\underline{e}}_2 = r\dot{\psi}\,\underline{e}_2^* = r\omega\,\underline{e}_2^*$

$$\ddot{\underline{r}}_c = r\cancelto{0}{\dot{\omega}}\, \underline{e}_2^\varphi + r\omega\, \dot{\underline{e}}_2^\varphi$$

$$= -r\omega^2 \underline{e}_2$$

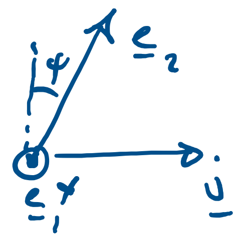
(b) (i) Angular velocity of $\underline{e}_1, \underline{e}_1^\varphi$: $\underline{\omega}_1 = \dot{\theta}\,\underline{j}$

Angular velocity of $\underline{e}_2, \underline{e}_2^\varphi$: $\underline{\omega}_2 = \dot{\theta}\,\underline{j} - \omega\, \underline{e}_1^\varphi$

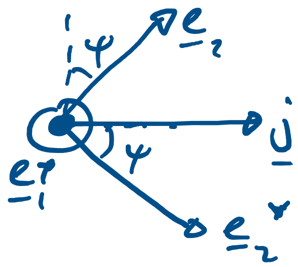
$$\dot{\underline{e}}_1 = \dot{\theta}\, \underline{e}_1^\varphi \qquad \& \qquad \dot{\underline{e}}_1^\varphi = -\dot{\theta}\, \underline{e}_1$$

$$\dot{\underline{e}}_2 = \underline{\omega}_2 \times \underline{e}_2 = (\dot{\theta}\,\underline{j} - \omega\, \underline{e}_1^\varphi) \times \underline{e}_2$$

Need $\underline{j} \times \underline{e}_2 = \sin(90 - \varphi)\, \underline{e}_1^\varphi$
$= \cos\varphi\, \underline{e}_1^\varphi$



Also need $\underline{j} \times \underline{e}_2^\varphi = -\sin\varphi\, \underline{e}_1^\varphi$



So $\dot{\underline{e}}_2 = \underline{\omega}_2 \times \underline{e}_2 = (\dot{\theta}\,\underline{j} - \omega\, \underline{e}_1^\varphi) \times \underline{e}_2$

$$= \dot{\theta}\,\underline{j} \times \underline{e}_2 - \omega\, \underline{e}_1^\varphi \times \underline{e}_2$$

$$= \dot{\theta}\cos\varphi\, \underline{e}_1^\varphi + \omega\, \underline{e}_2^\varphi$$

$$\dot{\underline{e}}_2^{\varphi} = \underline{\omega}_2 \times \underline{e}_2^{\varphi} = \left(\dot{\theta}\,\underline{j} - \omega\, e_1^{\varphi} \right) \times \underline{e}_2^{\varphi}$$

$$= -\dot{\theta} \sin\varphi \, \underline{e}_1^{\varphi} - \omega\, \underline{e}_2$$

(ii) $$\underline{r}_c = h\,\underline{e}_1 + d\,\underline{e}_1^{\varphi} + r\,\underline{e}_2$$

$$\dot{\underline{r}}_c = h\,\dot{\underline{e}}_1 + d\,\dot{\underline{e}}_1^{\varphi} + r\,\dot{\underline{e}}_2$$

$$= h\dot{\theta}\,\underline{e}_1^{\varphi} - d\dot{\theta}\,\underline{e}_1 + r\,\dot{\theta}\cos\varphi\,\underline{e}_1^{\varphi} + r\omega\,\underline{e}_2^{\varphi}$$

$$= -d\dot{\theta}\,\underline{e}_1 + \dot{\theta}(h + r\cos\varphi)\,\underline{e}_1^{\varphi} + r\omega\,\underline{e}_2^{\varphi}$$

$$\ddot{\underline{r}}_c = -d\ddot{\theta}\,\underline{e}_1 - d\dot{\theta}\,\dot{\underline{e}}_1 + \ddot{\theta}(h + r\cos\varphi)\,\underline{e}_1^{\varphi}$$
$$- \dot{\theta} r \dot{\varphi} \sin\varphi\,\underline{e}_1^{\varphi} + \dot{\theta}(h + r\cos\varphi)\,\dot{\underline{e}}_1^{\varphi} + \overset{0}{\cancel{r\dot{\omega}}}\,\underline{e}_2^{\varphi}$$
$$+ r\omega\,\dot{\underline{e}}_2^{\varphi}$$

$$= -d\ddot{\theta}\,\underline{e}_1 - d\dot{\theta}^2\,\underline{e}_1^{\varphi} + \ddot{\theta}(h + r\cos\varphi)\,\underline{e}_1^{\varphi}$$
$$- r\omega\dot{\theta}\sin\varphi\,\underline{e}_1^{\varphi} - \dot{\theta}^2(h + r\cos\varphi)\,\underline{e}_1$$
$$- r\omega\dot{\theta}\sin\varphi\,\underline{e}_1^{\varphi} - r\omega^2\,\underline{e}_2$$

$$\ddot{\underline{r}}_c = -\underline{e}_1\left(d\ddot{\theta} + \dot{\theta}^2(h + r\cos\varphi)\right)$$
$$+ \underline{e}_1^{\varphi}\Big(\ddot{\theta}(h + r\cos\varphi) \underbrace{- 2r\omega\dot{\theta}\sin\varphi}_{\text{Coriolis term.}} - d\dot{\theta}^2\Big) - r\omega^2\,\underline{e}_2$$

(iii) Coriolis term $\underline{a}_{corr} = -2 r \omega \dot{\theta} \sin\psi \, \underline{e}_1^*$.

$$\theta = \theta_0 \sin \Omega t.$$
$$\dot{\theta} = \Omega \theta_0 \cos \Omega t$$

$$a_{corr} = -2 r \omega . \Omega \theta_0 \cos \Omega t . \sin \psi . \underline{e}_1^*$$

Max when all sin/cos terms are 1

$$a_{max} = 2 r \omega \Omega \theta_0 \,//$$

Crib question 4 (AC) Given the system:

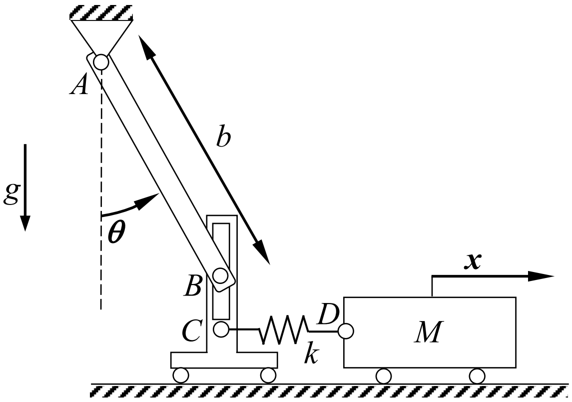


Fig. 4

(a) The kinetic energy is:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}J_A\dot{\theta}^2 \tag{1}$$

where: $J_A = \frac{mb^2}{3}$.

The potential energy is:

$$V = \frac{1}{2}k\left(x - b\sin\left(\theta\right)\right)^2 - mg\frac{b}{2}\cos(\theta) \tag{2}$$

Therefore, the Lagrangian is:

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\frac{mb^2}{3}\dot{\theta}^2 - \frac{1}{2}kx^2 - \frac{1}{2}k\left(b\sin\left(\theta\right)\right)^2 + kxb\sin\left(\theta\right) + mg\frac{b}{2}\cos(\theta) \tag{3}$$

(b) The Lagrange Equation for the generalised coordinate q_j is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{L}{\partial\dot{q}_j}\right) - \frac{\mathrm{d}Ł}{\partial q_j} = 0 \tag{4}$$

with $q_1 = x$, $q_2 = \theta$.

Taking the derivatives

$$\left(\frac{\partial L}{\partial\dot{x}}\right) = M\dot{x} \tag{5}$$

$$\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{mb^2}{3}\dot{\theta} \tag{6}$$

$$\frac{\partial L}{\partial x} = -kx + kb\sin(\theta) \tag{7}$$

$$\frac{\partial L}{\partial \theta} = -kb^2\cos(\theta)\sin(\theta) - mg\frac{b}{2}\sin(\theta) + kxb\cos(\theta) = 0 \tag{8}$$

Leading to:

$$m\ddot{x} + kx - kb\sin(\theta) = 0 \tag{9}$$

$$\frac{mb^2}{3}\ddot{\theta} + kb^2\cos(\theta)\sin(\theta) + mg\frac{b}{2}\sin(\theta) - kxb\cos(\theta) = 0 \tag{10}$$

(c) Linearise the Equations of motion about $x = 0$, $\theta = 0$, $\dot{x} = 0$, $\dot{\theta} = 0$. Write the equations of motion in matrix form.

$$M\ddot{x} + kx - kb\theta = 0 \tag{11}$$

$$\frac{mb^2}{3}\ddot{\theta} + kb^2\theta + mg\frac{b}{2}\theta - kbx = 0 \tag{12}$$

$$\begin{bmatrix} M & 0 \\ 0 & \frac{mb^2}{3} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & -kb \\ -kb & kb^2 + mg\frac{b}{2} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

(d) Find the natural frequencies of the normal modes of the system

Using the harmonic trial solutions:

$$\begin{aligned} x &= A\mathrm{e}^{i\omega t} \\ \theta &= B\mathrm{e}^{i\omega t} \end{aligned} \tag{14}$$

$$\begin{bmatrix} k - M\omega^2 & -kb \\ -kb & kb^2 + mg\frac{b}{2} - \frac{mb^2}{3}\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{15}$$

Leading to:

$$(k - M\omega^2)(kb^2 + mg\frac{b}{2} - \frac{mb^2}{3}\omega^2) - k^2b^2 = 0 \tag{16}$$

$$k^2b^2 + kmg\frac{b}{2} - k\frac{mb^2}{3}\omega^2 - Mkb^2\omega^2 - mMg\frac{b}{2}\omega^2 + M\frac{mb^2}{3}\omega^4 - k^2b^2 = 0 \tag{17}$$

$$M\frac{mb^2}{3}\omega^4 - \left(k\frac{mb^2}{3} + Mkb^2 + mMg\frac{b}{2}\right)\omega^2 + kmg\frac{b}{2} = 0 \tag{18}$$

$$M\frac{mb^2}{3}\omega^4 - \beta\omega^2 + kmg\frac{b}{2} = 0 \tag{19}$$

where

$$\beta = 3\left(k\frac{mb^2}{3} + Mkb^2 + mMg\frac{b}{2}\right) \tag{20}$$

so that:

$\omega_1^2 = \frac{\beta - \sqrt{\beta^2 - 6m^2b^3kMg}}{2Mmb^2}$

$\omega_2^2 = \frac{\beta + \sqrt{\beta^2 - 6m^2b^3kMg}}{2Mmb^2}$

(e) Describe the motion corresponding to each normal mode:

For mode 1: point b of the bar and the mass move in the same direction with relatively different displacements.

For mode 2: point b of the bar and the mass move in the opposite directions with relatively different displacements.

Crib question 5 (AC) Given the system:

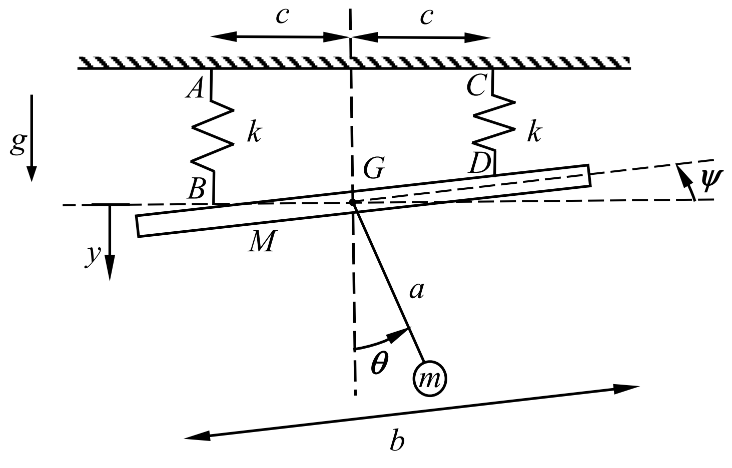


Fig. 6

(a) The kinetic energy of the bar is:

$$T_{bar} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}\frac{mb^2}{12}\dot{\psi}^2 \tag{21}$$

The kinetic energy of the bob is:

$$T_{bob} = \frac{1}{2}mv_{bob}^2 \tag{22}$$

where

$$\begin{aligned} v_{bob} &= \left(a\dot{\theta}\cos(\theta)\right)^2 + \left(\dot{y} - a\dot{\theta}\sin(\theta)\right)^2 \\ &= \dot{y}^2 + a^2\dot{\theta}^2 + 2a\dot{y}\dot{\theta}\sin(\theta) \end{aligned} \tag{23}$$

The potential energy of the bar is:

$$V = \frac{1}{2}k\left(L_0 + y - c\sin(\psi)\right)^2 + \frac{1}{2}k\left(L_0 + y + c\sin(\psi)\right)^2 - Mgy \tag{24}$$

where L_0 is the extension of the spring when in equilibrium. This becomes:

$$V = k\left((L_0 + y)^2 - c^2\sin^2(\psi)\right) - Mgy \tag{25}$$

The potential energy of the bob is:

$$V_{bob} = -mg\left(y + a\cos(\theta)\right) \tag{26}$$

Therefore, the Lagrangian is:

$$\begin{aligned} L = T - V = \frac{1}{2}M\dot{y}^2 + \frac{mb^2}{24}\dot{\psi}^2 + \frac{1}{2}m\left(\dot{y}^2 + a^2\dot{\theta}^2 - 2a\dot{y}\dot{\theta}\sin\left(\theta\right)\right) \\ -k\left((L_0 + y)^2 - c^2\sin^2\left(\psi\right)\right) + Mgy + mg\left(y + a\cos(\theta)\right) \end{aligned} \tag{27}$$

which can be rewritten as:

$$\begin{aligned} L = \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{24}mb^2\dot{\psi}^2 + \frac{1}{2}m\left(a^2\dot{\theta}^2 - 2a\dot{y}\dot{\theta}\sin\theta\right) \\ -k((L_0 + y)^2 + c^2\sin^2\psi) + (M+m)gy + mga\cos\theta \end{aligned}$$

Note that $2kL_0 = (M+m)g$ (from the equilibrium condition), giving:

$$\begin{aligned} L = \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{24}mb^2\dot{\psi}^2 + \frac{1}{2}m\left(a^2\dot{\theta}^2 - 2a\dot{y}\dot{\theta}\sin\theta\right) \\ -k(y^2 + c^2\sin^2\psi) + mga\cos\theta + kL_0^2 \end{aligned}$$

i.e. $D = kL_0^2$ for the chosen datum of V.

(b) The Lagrange Equation for the generalised coordinate q_j is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{L}{\partial\dot{q}_j}\right) - \frac{\mathrm{d}Ł}{\partial q_j} = 0 \tag{28}$$

with $q_1 = y$, $q_2 = \theta$, $q_3 = \psi$.

Leading to:

$$(m + M)\,\ddot{y} + 2ky + a\ddot{\theta}m\sin\left(\theta\right) + a\dot{\theta}^2m\cos\left(\theta\right) - (M+m)g = 0 \tag{29}$$

$$ma^2\ddot{\theta} + ma^2\ddot{y}\sin\left(\theta\right) + ma\dot{y}\cos\left(\theta\right)\dot{\theta} + mga\sin\left(\theta\right) = 0 \tag{30}$$

$$\frac{mb^2}{12}\ddot{\psi} + 2kc^2\psi\cos\left(\psi\right) = 0 \tag{31}$$

(c) Linearise the Equations of motion. Write the equations of motion in matrix form.

$$(m + M)\,\ddot{y} + am\ddot{\theta} + 2ky = 0 \tag{32}$$

$$ma^2\ddot{\theta} + ma\ddot{y} + mga\theta = 0 \tag{33}$$

$$\frac{mb^2}{12}\ddot{\psi} + 2kc^2\psi = 0 \tag{34}$$

$$\begin{bmatrix} m+M & 0 & 0 \\ 0 & ma^2 & 0 \\ 0 & 0 & \frac{mb^2}{12} \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} 2k & 0 & 0 \\ 0 & mga & 0 \\ 0 & 0 & 2kc^2 \end{bmatrix} \begin{bmatrix} y \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

(d) There are four equilibrium positions.

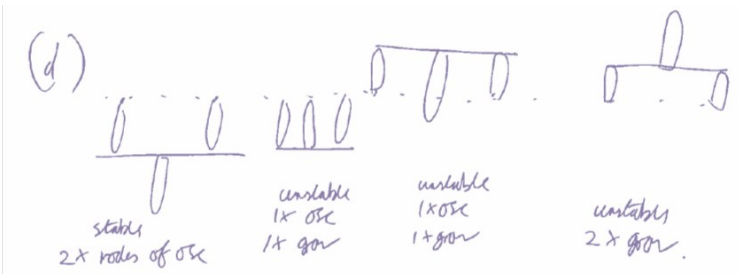


Fig. 7

While the first one is stable ($\theta = 0$ and $\dot{\theta} = 0$, $\psi = 0$ and $\dot{\psi} = 0$ that is the pendulum and the rods vertically down), the other 3 are unstable: (i) rods down, pendulum up; (ii) rods up, pendulum down; (iii) rods and pendulum up. For a small perturbation in the first configuration, the system will oscillate wrt the vertically down configuration. While for a small perturbation in the other configurations, the system will move away from the equilibrium.

Crib, question 6 - AC

Given the system:

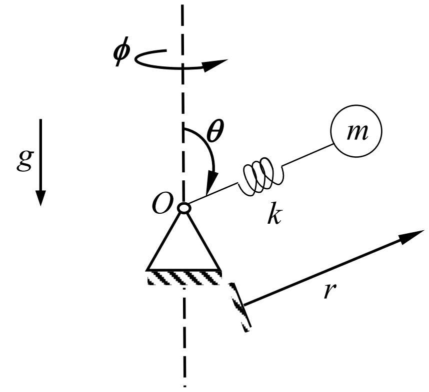


Fig. 5

a(i) Determine the Lagrangian.

The kinetic energy is

$$T = \frac{1}{2} m \left[\dot{r}^2 + r^2 \sin^2(\theta_0)\, \Omega^2 \right] \tag{36}$$

The potential energy is

$$V = mgr\cos(\theta_0) + \frac{1}{2} k r^2 \tag{37}$$

The Lagrangian is

$$L = T - V = \frac{1}{2} m \left[\dot{r}^2 + r^2 \sin^2(\theta_0)\, \Omega^2 \right] - mgr\cos(\theta_0) - \frac{1}{2} k r^2 \tag{38}$$

a(ii) Equation of motion for r. Substitute the Lagrangian derived above into:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial \text{Ł}}{\partial r} = 0 \tag{39}$$

to obtain the governing equation in r.

$$m\ddot{r} - mr\sin^2(\theta_0)\, \Omega^2 + mg\cos(\theta_0) + kr = 0$$

a(iii) For the equilibrium $\ddot{r} = \dot{r} = 0$. Using the Lagrange Equation for r_0:

$$-mr_0\sin^2(\theta_0)\, \Omega^2 + mg\cos(\theta_0) + kr_0 = 0 \tag{40}$$

It follows that:

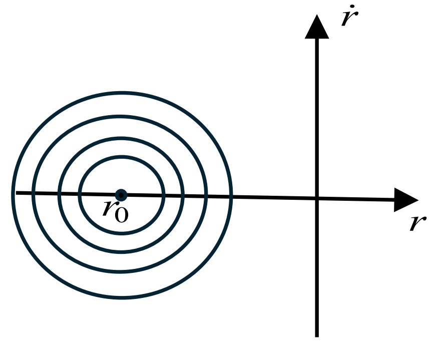
$$r_0 = \frac{mg\cos(\theta_0)}{m\sin^2(\theta_0)\,\Omega^2 - k} \tag{41}$$

The locus is obtained as

$$m\dot{r}^2 - mr^2\sin^2(\theta_0)\,\Omega^2 + kr^2 + mgr\cos(\theta_0) = const$$

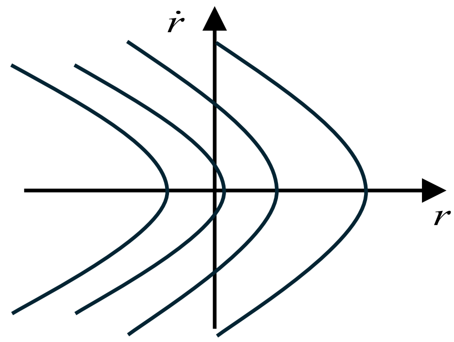
For $\Omega^2 < k/(msin^2\theta_0)$,

$$r_0 = \frac{mg\cos(\theta_0)}{m\sin^2(\theta_0)\,\Omega^2 - k} < 0$$



and the phase portrait is

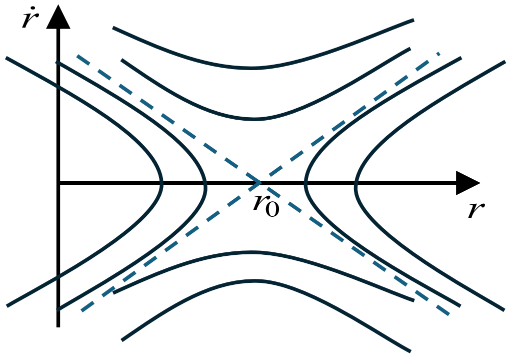
For $\Omega^2 = k/(msin^2\theta_0)$, there is no equilibrium point. and the phase



portrait is

For $\Omega^2 > k/(msin^2\theta_0)$,

$$r_0 = \frac{mg\cos(\theta_0)}{m\sin^2(\theta_0)\,\Omega^2 - k} > 0$$



b(i) The kinetic energy is

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\,\dot{\phi}^2\right] \tag{42}$$

The potential energy is

$$V = mgr\cos(\theta) + \frac{1}{2}kr^2 \tag{43}$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\,\dot{\phi}^2\right] - mgr\cos(\theta) - \frac{1}{2}kr^2 \tag{44}$$

b(ii) The momentum is defined as

$$p_\phi = \left(\frac{\partial L}{\partial \dot{\phi}}\right) = mr^2\theta^2\dot{\phi} \tag{45}$$

since $F_\phi = 0$ and

$$\frac{\partial L}{\partial \phi} = 0 \tag{46}$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_\phi) - 0 = 0 \tag{47}$$

Which means that the vertical component of the angular momentum is conserved.

b(iii) Starting from two different locations with different values of ϕ, the two motions will be identical but with an offset in ϕ.