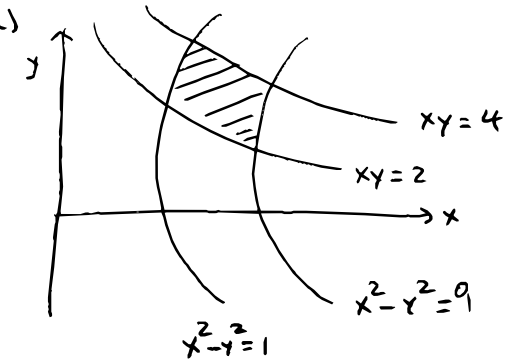
Q1 2P7 2024

(a)



(b) Use the transformation $u = x^2 - y^2$, $v = 2xy$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2)$$

$$\Rightarrow \frac{1}{J} = \frac{1}{4(x^2+y^2)}$$

It is convenient to use $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$

$$\Rightarrow \frac{1}{J} = \frac{1}{4\sqrt{u^2+v^2}}$$

$$\therefore \iint (x^2+y^2)\,dx\,dy = \frac{1}{4}\int_1^9\int_4^8 \sqrt{u^2+v^2}\,\frac{du\,dv}{\sqrt{u^2+v^2}} = [9-1][8-4]\frac{1}{4}$$

$$= 8$$

(c) $\oint \underline{F}\cdot d\underline{l} = \iint \nabla\times\underline{F}$

So to find the line integral we first find $\nabla\times\underline{F}$ and then integrate it over the region R.

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = 3x^2 + 3y^2 = 3(x^2+y^2)$$

Our integral becomes $\iint 3\cdot(x^2+y^2)\,dx\,dy$

$= 3 \times$ (integral of part (b))

$= 24$.

Q2

(a) Show that $\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$

If $\underline{A} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$

$\underline{B} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$

$\underline{C} = c_1 \underline{i} + c_2 \underline{j} + c_3 \underline{k}$

$$\underline{A} \times (\underline{B} \times \underline{C}) = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \times \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \times \left[(b_2 c_3 - b_3 c_2)\underline{i} + (b_3 c_1 - b_1 c_3)\underline{j} + (b_1 c_2 - b_2 c_1)\underline{k} \right]$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \end{vmatrix}$$

$$= (a_2 b_1 c_2 - a_2 b_2 c_1 - a_3 b_3 c_1 + a_3 b_1 c_3)\underline{i}$$
$$+ (a_3 b_2 c_3 - a_3 b_3 c_2 - a_1 b_1 c_2 + a_1 b_2 c_1)\underline{j}$$
$$+ (a_1 b_3 c_1 - a_1 b_1 c_3 - a_2 b_2 c_3 + a_2 b_3 c_2)\underline{k}$$

Collect terms to build $\underline{B}$ & $\underline{C}$:

$$= \left[b_1 (a_1 c_1 + a_2 c_2 + a_3 c_3) - c_1 (a_1 b_1 + a_2 b_2 + a_3 b_3) \right] \underline{i}$$
$$+ \left[b_2 (a_1 c_1 + a_2 c_2 + a_3 c_3) - c_2 (a_1 b_1 + a_2 b_2 + a_3 b_3) \right] \underline{j}$$
$$+ \left[b_3 (a_1 c_1 + a_2 c_2 + a_3 c_3) - c_3 (a_1 b_1 + a_2 b_2 + a_3 b_3) \right] \underline{k}$$

$$= \underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B}) \qquad \text{QED}$$

(b) Expand each triple product with the result of part (a):

$$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$$
$$\underline{B} \times (\underline{C} \times \underline{A}) = (\underline{B} \cdot \underline{A})\underline{C} - (\underline{B} \cdot \underline{C})\underline{A}$$
$$\underline{C} \times (\underline{A} \times \underline{B}) = (\underline{C} \cdot \underline{B})\underline{A} - (\underline{C} \cdot \underline{A})\underline{B}$$

Summing the r.h.s. $= 0$. QED

Q2 (c)

$$\underline{F} = (2xy - z^3)\,\underline{i} + x^2\,\underline{j} - (3xz^2+1)\,\underline{k}$$

$\underline{F}$ is conservative if $\nabla \times \underline{F} = 0$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy - z^3 & x^2 & -3xz^2-1 \end{vmatrix}$$

$$= (0-0)\,\underline{i} - (-3z^2+3z^2)\,\underline{j} + (2x-2x)\,\underline{k}$$

$$= 0$$

We seek a scalar potential Φ such that $\underline{F} = -\nabla\Phi$. So

$$\frac{\partial \Phi}{\partial x} = -2xy + z^3 \Rightarrow \Phi = -x^2y + z^3x + F_1(y,z)$$

$$\frac{\partial \Phi}{\partial y} = -x^2 \Rightarrow \Phi = -x^2y + F_2(x,z)$$

$$\frac{\partial \Phi}{\partial z} = 3xz^2+1 \Rightarrow \Phi = xz^3 + z + F_3(x,y)$$

$$\Rightarrow \Phi = -x^2y + xz^3 + z + C$$

(d) From divergence theorem,

$$\iint \underline{n}\cdot\nabla\underline{F}\,dA = \iiint \nabla\cdot\underline{F}\,d\tau$$

$d\tau$ = volume element

dA = area element

$$\nabla\cdot\underline{F} = 2y + 0 - 6xz$$

$$\Rightarrow \iiint \nabla\cdot\underline{F}\,d\tau = \int_0^1\int_0^1\int_0^1 (2y - 6xz)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (2y - 3z)\,dy\,dz = \int_0^1 (1-3z)\,dz = 1 - \frac{3}{2} = -\frac{1}{2}$$

Q3

(a) Integrating the p.d.e. over x, y:

$$\iint \frac{\partial T}{\partial t}\, dx\, dy = \frac{\partial T}{\partial t} \cdot 1$$

$$\iint \nabla^2 T\, dx\, dy = \oint \underline{n} \cdot \nabla T\, dA$$ i.e. the integral of the flux over the boundary

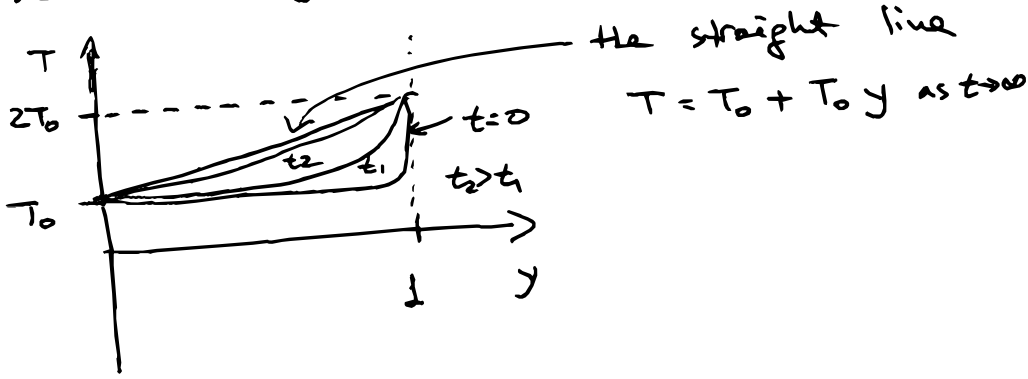
$$= 0$$ due to the zero grad b.c.'s

$$\iint f(x,y,t)\, dx\, dy = T_0 e^{-t}$$

$$\therefore \frac{dT}{dt} = T_0 e^{-t} \Rightarrow T = T_0 \text{ at } t \to \infty$$

Physically, we can understand this as the area-averaged temperature. Due to diffusion, at long times the temperature will be homogeneous inside the domain.

(b) We ignore x-dependence since $T = T_0$ for all x and because the b.c.'s at $x=0$ & $x=1$ do not change. So, in the y-direction:



(c) θ is the normalized temperature of part (b): $\theta = \dfrac{T(y) - T_0}{2T_0 - T_0}$. So at long times, we will reach the steady state solution $\theta = y$. The pde for this θ is the same as for θ – (a straight line in y), so it is easier to seek the general solution of a new variable that has b.c.'s $\theta = 0$ at both $y=0$ & $y=1$.

If $\theta = F(y)\,G(t)$, then $\frac{1}{F}\frac{\partial^2 F}{\partial y^2} = k$

$$\text{and} \quad \frac{1}{G}\frac{\partial G}{\partial t} = k$$

k is the separation constant. It cannot be +ve as the solution would diverge.

So we write $k = -\lambda^2$, $\lambda > 0$.

$\therefore G = e^{-\lambda^2 t}$ and

$F = A \sin \lambda y + B \cos \lambda y$ as the basic solution

We discard the cos term because $\theta = 0$ at $y = 0$.

We also want $\theta = 0$ at $y = 1$ (our modified θ) which happens if $\sin(\lambda) = 0 \iff \lambda = n\pi$

Therefore our general solution is

$$\theta = y + \sum_{n=1}^{\infty} C_n \sin(n\pi y)\, e^{-(n\pi)^2 t}$$

QED

NOTE: C_n are coefficients to be determined by enforcing the b.c.'s.

4. (a) The determinant of A is $2(1+5) - t(1+5) + 2e^s(5-5) = 6(2-t)$. It is zero when $t = 2$. So A is full rank when $t \neq 2$.

   (b) Since a, b and c form the rows of A and A is not full ranked when $t = 2$, the vectors are linearly dependent.

   (c) Since A is not full ranked, nor is A^T. So it must have a non-trivial null space. Since $3a/2 + 2b - c = 0$, the null space is spanned by the vector $[3/2, 2, -1]$.

   (d) Simplest is $u_1 = [1, 1, 0]/\sqrt{2}$ and $u_2 = [0, 0, 1]$. Obtained by Gram-Schmidt orthogonalization, or just by inspection. The general solution is $[c/\sqrt{2}, c/\sqrt{2}, s]$ and $[-s/\sqrt{2}, -s/\sqrt{2}, c]$, where $c^2 + s^2 = 1$.

   (e) $n_1 = [1, -1, 0]/\sqrt{2}$. Also obtained by G-S orthogonalization.

5. (a) For eigenvalues, $\det(A - \lambda I) = 0$, yields, $(\lambda - 2)(\lambda^2 - 1) = 0$. So $\lambda_{3,2,1} = 2, -1, 1$. The corresponding eigenvectors are $v_3 = [0, 1, 0]^T$, $v_2 = [-4/5, 0, 3/5]^T$ and $v_1 = [3/5, 0, 4/5]^T$.

   (b) $\mu_1 = \lambda_1 = 1$, $\mu_2 = \lambda_2 = -1$ and $\mu_3 = \lambda_3 = 2$. They are just the eigenvalues. Can be verified by taking a dot product with the eigenvectors and using orthonormality.

   (c) Taking the dot product again yields, $\beta_k = \lambda_k^2$.

   (d) Noticing that $A^n = \lambda_1^n v_1 v_1^T + \lambda_2^n v_2 v_2^T + \lambda_3^n v_3 v_3^T$,

$$A^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

   because $\lambda_{1,2}^2 = 1$ so $\lambda_{1,2}^n = 1$ for even n.

6. (a) Binomial $B(n{=}20, k{=}10, p{=}0.55) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \simeq 0.1593$.

   (b) The probability of winning w, is the sum of the probability of winning the next two points and w times the probability of winning exactly one of the next two, ie $w = p^2 + 2wpq \Rightarrow w = \frac{p^2}{1-2pq} \simeq 0.5990$. Alternately, via geometric series $w = \sum_{k=0}^{\infty} (2pq)^k p^2 = \frac{p^2}{1-2pq}$, any number of pairs of points going either way $2pq$, and two wins p^2.

   (c) Expectation and variance of $B(n, p)$ is $\mu = np$ and $\sigma^2 = npq$. So, we want $\int_{-\infty}^{11.5} \mathcal{N}(\mu, \sigma^2) dx = \Phi(\frac{11.5-\mu}{\sigma}) \simeq 0.5889$ (table or calculator).

   (d) Under the null hypothesis, the probability of winning a game is $p = 0.5$. Thus, the probability of the observed outcome or something more extreme is $2 \times 0.5^5 = 0.0625$ (two sided), which we would **not reject** at $\alpha = 0.05$.