

EGT1
ENGINEERING TRIPOS PART IB

Tuesday 11 June 2024 2 to 4.10

Paper 7

MATHEMATICAL METHODS

Teaching office note - This paper originally appeared with an incorrect rubric - this was corrected in the exam.

The correct rubric is given at <https://teaching.eng.cam.ac.uk/node/6144>

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

Answer not more than **two** questions from this section.

1 Consider the region $\mathcal{R}$ bounded by the curves $xy = 2$, $xy = 4$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 9$.

(a) Sketch $\mathcal{R}$. [5]

(b) Evaluate the double integral $\iint (x^2 + y^2) dx dy$ over the region $\mathcal{R}$. [10]

(c) If $d\mathbf{l}$ is the infinitesimal line element along the boundary of $\mathcal{R}$, find $\oint \mathbf{F} \cdot d\mathbf{l}$ when $\mathbf{F} = -y^3 \mathbf{i} + x^3 \mathbf{j}$. [10]

2 (a) Show that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C}$ for any vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. [5]

(b) Using the result of part (a) or otherwise, show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$$

[5]

(c) Show that the vector field $\mathbf{F} = (2xy - z^3)\mathbf{i} + x^2\mathbf{j} - (3xz^2 + 1)\mathbf{k}$ is conservative and find a scalar potential ϕ such that $\mathbf{F} = -\nabla\phi$. [10]

(d) Evaluate the flux of $\mathbf{F}$ out of a cubic volume defined by $0 < x < 1$, $0 < y < 1$, and $0 < z < 1$. [5]

3 Consider an unsteady two-dimensional heat transfer problem described by the following partial differential equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + f(x, y, t)$$

where T is the temperature, t is time and $f(x, y, t)$ is a localised source of heat. The domain is defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$. At time $t = 0$, the temperature is $T = 0$.

(a) The boundary conditions are zero flux at all boundaries of the domain. If $f(x, y, t) = T_0 e^{-t}$ for all (x, y) , show that as $t \rightarrow \infty$, then $T(x, y) \rightarrow T_0$. [5]

(b) At a sufficiently long time after T has reached the steady state of part (a), suddenly certain boundary conditions change while $f(x, y, t)$ remains 0. At $x = 0$ and $x = 1$ the boundary condition remains $\partial T / \partial x = 0$, and at $y = 0$, $T = T_0$. However, at $y = 1$, the temperature is suddenly raised to $T = 2T_0$. Sketch $T(x, y)$ at different times. [5]

(c) The variable θ obeys the partial differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2}$$

The boundary conditions are that $\theta = 0$ at $y = 0$ and $\theta = 1$ at $y = 1$. The initial condition is $\theta = 0$ for $y < 1$ and $\theta = 1$ at $y = 1$. Using separation of variables and not otherwise, show that the general solution of the above p.d.e. is

$$\theta = y + \sum_{n=1}^{\infty} C_n e^{-(n\pi)^2 t} \sin(n\pi y)$$

[15]

SECTION B

Answer not more than **two** question from this section.

4 Consider the matrix

$$A = \begin{bmatrix} 2 & t & 2e^s \\ 1 & 1 & -1 \\ 5 & 5 & 1 \end{bmatrix},$$

which is constructed with $a = [2 \ t \ 2e^s]$, $b = [1 \ 1 \ -1]$ and $c = [5 \ 5 \ 1]$ as rows.

- (a) Determine the condition on real numbers t and s , such that matrix A has a full rank. [5]
- (b) Let $t = 2$ and $s = 0$ from now onwards. Are the vectors a , b and c linearly independent? Ensure that your response is consistent with your answer to part (a). [5]
- (c) Determine whether matrix A^T has a nontrivial null space, and if so then find it. [5]
- (d) Find an orthonormal basis, $u_1, \dots, u_k$, for the space spanned by vectors a , b and c . Here k is the dimension of the space spanned by a , b and c . [5]
- (e) Find an orthonormal basis, $n_1, \dots, n_m$, for the space orthogonal to the span of a , b and c . Here m is the dimension of this space. [5]

- 5 Consider a 3×3 matrix A given by

$$A = \begin{bmatrix} -7/25 & 0 & 24/25 \\ 0 & 2 & 0 \\ 24/25 & 0 & 7/25 \end{bmatrix}$$

- (a) Find the eigenvalues and normalized eigenvectors of A . [7]

- (b) The matrix A can be written as

$$A = \mu_1 v_1 v_1^T + \mu_2 v_2 v_2^T + \mu_3 v_3 v_3^T,$$

where v_1, v_2 and v_3 are the normalized eigenvectors that you found in part (a). Determine the real numbers μ_1, μ_2 and μ_3 in the above expression. [5]

- (c) The matrix A^2 may also be written as

$$A^2 = \beta_1 v_1 v_1^T + \beta_2 v_2 v_2^T + \beta_3 v_3 v_3^T,$$

for some real numbers β_1, β_2 and β_3 . Determine β_1, β_2 and β_3 . [4]

- (d) Determine A^{2n} for any positive integer n . [9]

6 In table tennis, the first player to reach 11 points wins the game, except that you have to win by a margin of at least 2; therefore if a score of 10:10 is reached, then the first player to get ahead by 2 points wins. Assume player A wins a point with probability $p = 0.55$ (and loses with probability $q = 1 - p$), independently for all points in a game.

(a) Calculate the probability for player A that a score of 10:10 is reached. [4]

(b) Calculate the conditional probability that A wins, given that a score of 10:10 was reached. [7]

(c) Instead of using the scoring rules from table tennis, in this question the players simply play 20 points. Approximate the discrete distribution of player A winning k out of the 20 points with a Gaussian distribution with the same mean and variance as those of that discrete distribution. Use this Gaussian to approximate the probability of player A winning at least 11 out of the 20 points. [8]

(d) Player C and D play 5 games, C wins them all. At the $\alpha = 5\%$ significance level, would you reject the null hypothesis $\mathcal{H}_0$ that they're equally good? [6]

END OF PAPER

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