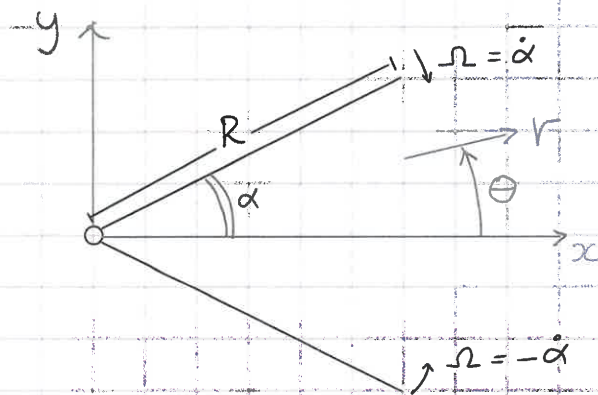


1



3A1 2025 solution.

The behaviour in the bottom half is symmetric to that in the top half. Therefore consider only the top half and infer the bottom half via symmetry.

many students wrongly stated $u_r = 0$

most students wrote this

a) On the top plate, u_r can take any value and $u_\theta = \Omega r$.

$$\Rightarrow u_\theta = r \dot{\alpha} \text{ at } \theta = \alpha \quad (\text{top plate})$$

$$u_\theta = -r \dot{\alpha} \text{ at } \theta = -\alpha \quad (\text{bottom plate})$$

b) $\psi = Axy = Ar \cos \theta r \sin \theta = Ar^2 \cos \theta \sin \theta$

optional, but nearer

From datacard, $u_\theta = -\frac{\partial \psi}{\partial r} = 2Ar \cos \theta \sin \theta = Ar \sin(2\theta)$

At $\theta = \alpha$, $u_\theta = Ar \sin 2\alpha = r \Omega$

$$\Rightarrow A = \frac{\Omega}{\sin 2\alpha}$$

most students derived this well

Whatever the value of the closing speed of the claws, Ω , the flow becomes very fast as $\alpha \rightarrow 0$.

few students commented on this.

c) From Bernoulli, $\frac{p}{\rho} + \frac{1}{2} \underline{u} \cdot \underline{u} + \frac{\partial \phi}{\partial t} = \text{const.}$

where $u_x = Ax$; $u_y = -Ay \Rightarrow \underline{u} \cdot \underline{u} = A^2(x^2 + y^2) = A^2 r^2 = \frac{\Omega^2 r^2}{\sin^2 2\alpha}$

and $\phi = A(x^2 - y^2) = Ar^2(\cos^2 \theta - \sin^2 \theta) = Ar^2 \cos 2\theta$

so $\frac{\partial \phi}{\partial t} = \frac{dA}{dt} r^2 \cos 2\theta$ where $\frac{dA}{dt} = \Omega \frac{d}{dt} (\sin 2\alpha)^{-1} = -\frac{2\Omega \cos 2\alpha}{(\sin 2\alpha)^2} \frac{d\alpha}{dt}$

$$\Rightarrow \frac{\partial \phi}{\partial t} = -\frac{2\Omega r^2}{\sin^2 2\alpha} \cos 2\alpha \cos 2\theta \frac{d\alpha}{dt}$$

Consider the top plate, where $\theta = \alpha$ and $\frac{d\alpha}{dt}$ is negative with size Ω .

$$\Rightarrow \frac{\partial \phi}{\partial t} = \frac{2\Omega^2 r^2}{\sin^2 2\alpha} \cos^2 2\alpha$$

check: ϕ increases with time as the claws snap shut and the speed of the flow increases.

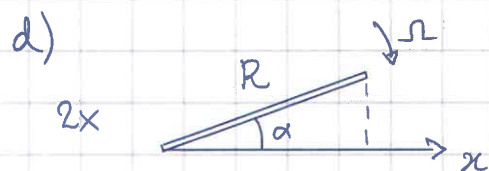
By unsteady Bernoulli, $\frac{p}{\rho} = \text{const.} - \frac{1}{2} \underline{u} \cdot \underline{u} - \frac{\partial \phi}{\partial t}$

$$= \text{const.} - \frac{1}{2} \frac{\Omega^2 r^2}{\sin^2 2\alpha} - \frac{2\Omega^2 r^2}{\sin^2 2\alpha} \cos^2 2\alpha$$

$$= \text{const.} - \frac{\Omega^2 r^2}{\sin^2 2\alpha} \left(\frac{1}{2} + \cos^2 2\alpha \right)$$

Pressure contours are circles centred on the origin and the pressure decreases radially along the plate. The pressure gradient is favourable and therefore boundary layers will not separate. This implies that the inviscid model is good between the claws. Viscous effects will become significant as $\alpha \rightarrow 0$, preventing infinite velocities as $\alpha \rightarrow 0$.

Only 3 students included the unsteady term and nobody managed to calculate its contribution to p/ρ . Full marks were given for a correct $\frac{1}{2} \underline{u} \cdot \underline{u}$ term. Few students commented correctly on the fact that the pressure gradient is favourable so the model is good.



This part was answered well by a few students but badly by the rest.

Consider the momentum flowrate across the dashed line at $x = R \cos \alpha$

$u_x = Ax$, independent of y , where $A = \frac{\Omega}{\sin 2\alpha}$

area = $R \sin \alpha$ per unit depth into page

$\Rightarrow$ momentum flowrate = $\rho u_x \text{ area} = \rho R \sin \alpha \left(\frac{\Omega}{\sin 2\alpha} R \cos \alpha \right)^2$

n.b. $\frac{\cos \alpha}{\sin 2\alpha} = \frac{\cos \alpha}{2 \cos \alpha \sin \alpha} = \frac{1}{2 \sin \alpha}$

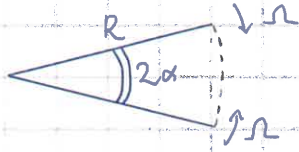
$\Rightarrow$ momentum flowrate = $\frac{\rho R^3 \Omega^2}{4 \sin \alpha}$ on each $\frac{1}{2}$ -plane

$\Rightarrow$ total momentum flowrate = $\frac{\rho R^3 \Omega^2}{2 \sin \alpha}$ per unit depth into page.

This is an effective way to create a fast jet, particularly at the end of the stroke.

Check with alternative for small angles:

The volume contained within the jaws, per unit depth into the page, is:



$$\text{volume} = \pi R^2 \frac{2\alpha}{2\pi} = R^2 \alpha$$

$$\Rightarrow \text{volumetric flowrate across dashed line} = \frac{d}{dt} R^2 \alpha = R^2 \dot{\alpha} = R^2 \Omega$$

When α is small, the dashed line is at $x = R \cos \alpha \approx R$

$$\text{At } x = R, u_x = A x = A R = \frac{\Omega R}{\sin 2\alpha} \approx \frac{\Omega R}{2\alpha}$$

$$\Rightarrow \text{momentum flowrate across dashed line} = \rho R^2 \Omega \frac{\Omega R}{2\alpha} = \frac{\rho R^3 \Omega^2}{2\alpha}$$

M. Tunipar 2025

2.



- a) The speed induced by one vortex on the other is $\Gamma/4\pi a$. They rotate about their mid-point, at radial distance a . Therefore the angular velocity is:

$$\Omega = \frac{\Gamma}{4\pi a^2} \quad (\text{anticlockwise})$$

Most students completed this successfully

b) $\psi = \frac{\Omega r^2}{2} = \frac{\Omega(x^2 + y^2)}{2}$; $u_\theta = -\frac{\partial \psi}{\partial r} = -\Omega r$; $u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = 0$

This is solid body rotation clockwise

$$\omega = \nabla \times \underline{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -2\Omega \neq 0$$

most students completed this

Streamfunctions are defined such that they automatically satisfy continuity but do not place any constraints on rotationality. } few students wrote this

- c) The x axis connects the vortices and rotates with the vortices. The streamfunction in this frame is the sum of the two vortices at $\pm a$ and a clockwise solid body rotation:

$$\psi = \frac{\Gamma}{4\pi a^2} \frac{r^2}{2} - \frac{\Gamma}{2\pi} (\ln r_1 + \ln r_2) = \frac{\Gamma}{2\pi} \left\{ \frac{r^2}{4a^2} - \ln r_1 - \ln r_2 \right\}$$

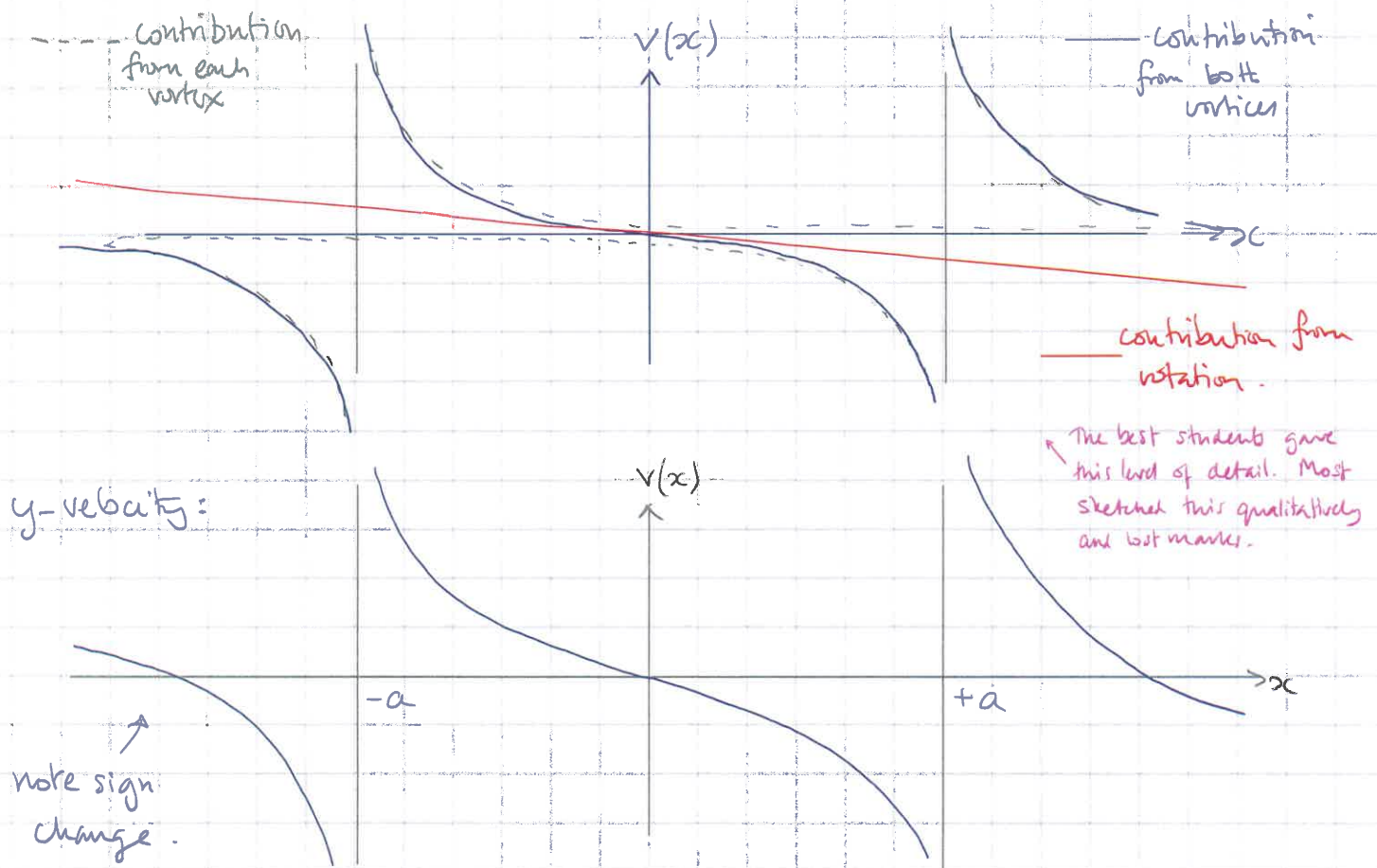
many students missed out the solid body rotation.

where r is the distance from the origin and r_1 and r_2 are the distances from the two vortices.

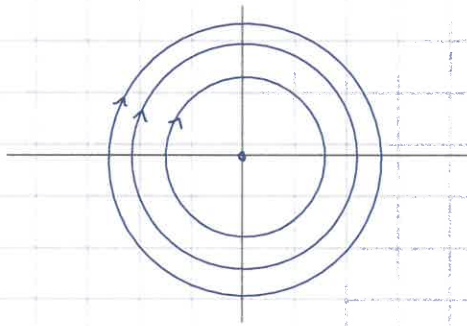
- (d) (i) Each vortex contributes $\frac{\Gamma}{2\pi r}$ and the solid body contributes $-\frac{\Gamma r}{4\pi a^2}$

$$\Rightarrow v(x) = \frac{\Gamma}{2\pi(x+a)} + \frac{\Gamma}{2\pi(x-a)} - \frac{\Gamma x}{4\pi a^2}$$

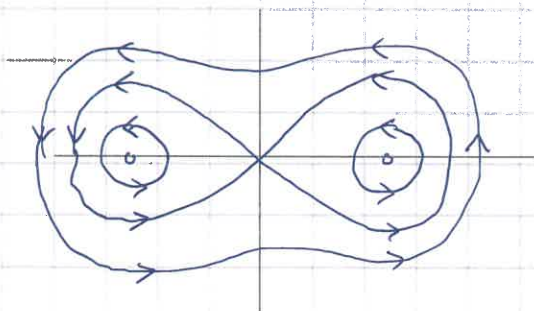
few students did this carefully.



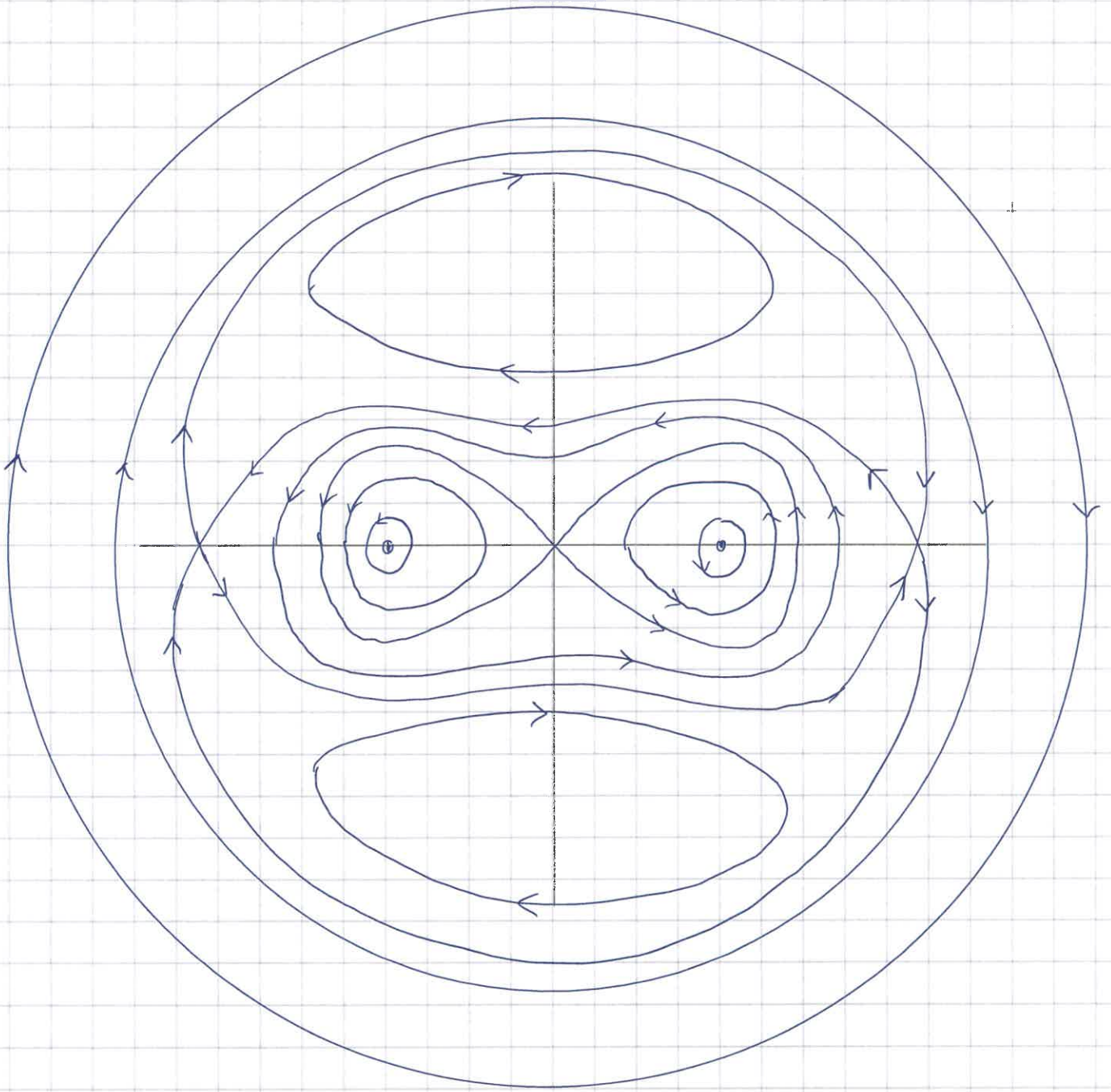
- ii) For $|r| \gg a$ the flow is dominated by solid body rotation $\propto r^{\frac{3}{2}}$. These are circles centred on the origin that become closer together as r increases.



- iii) For $0 < |r| < 2a$, the flow is dominated by the vortices:



iv)

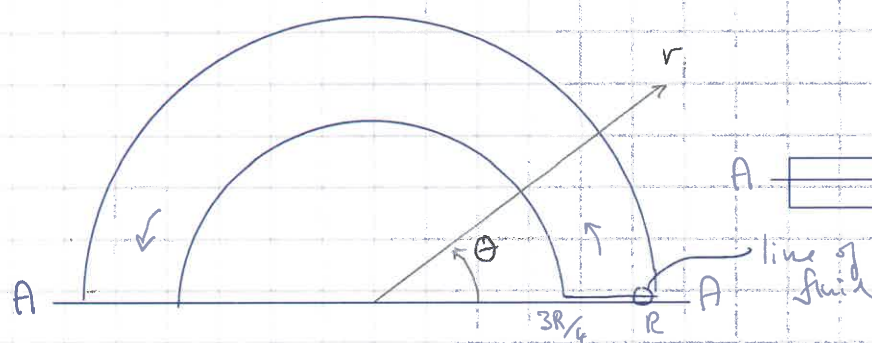


Many students drew the streamlines around the vortices but only one correctly deduced that there are 3 stagnation points (as in (d)(i)) and a recirculating loop above and below the vortices.

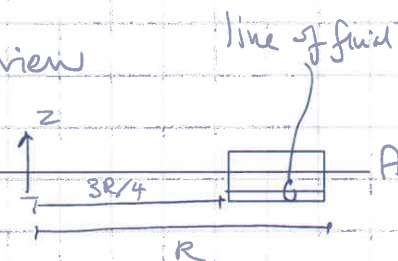
M. Turner 2025

3.

Top view



Side view



- (a) The flow is inviscid and irrotational, and traces out a circular arc. A line vortex centred on the origin is appropriate:

$$u_r = 0, \quad u_\theta = \frac{k}{r} \quad \text{where } k \text{ is a constant.}$$

Although this is straight-forward, many students tried to derive this from first principles. Curious.

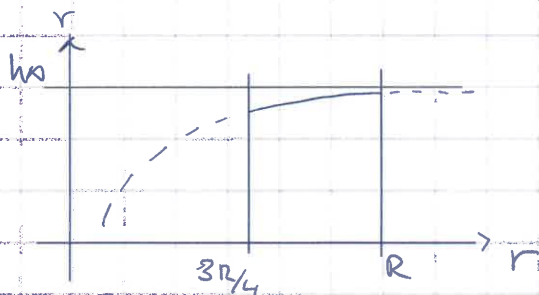
The pressure on the bottom of the channel is given by steady Bernoulli:

$$p(r; z=0) = p_0 - \frac{1}{2} \mathbf{u} \cdot \mathbf{u} = p_0 - \frac{k^2}{r^2}$$

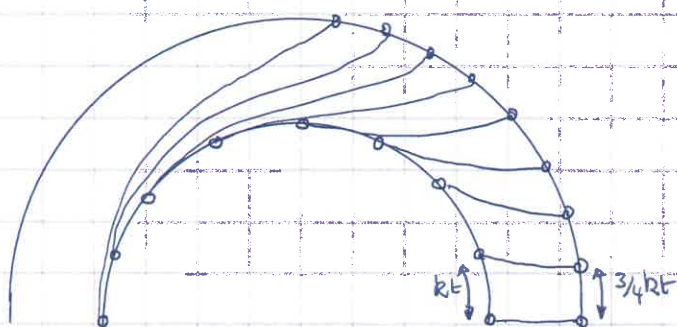
This was straight-forward but many students tried to derive this from streamline curvature.

There are no shear forces so the pressure at the bottom must balance the weight of water above it

$$\Rightarrow h(r) = h_0 - \frac{\text{const}}{r^2}$$



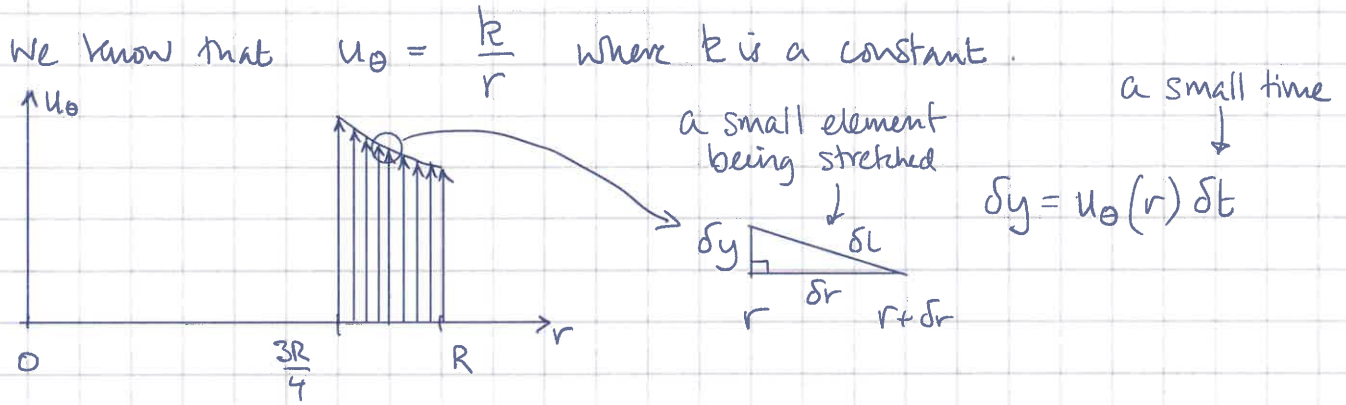
- (b) $u_\theta = \frac{k}{r}$ so after time, t , the fluid has moved arclength $tu_\theta = \frac{k t}{r}$



Most students answered this well

(c) $\omega(z) = \frac{\partial u}{\partial z}$ and is non-zero. It is initially uniform along the line.

As the line convects, it is stretched. The local vorticity will increase such that $\frac{\omega(r)}{\delta r}$ is constant locally, where δr is the local length of the line:



$$\delta y = k \delta t \left(\frac{1}{r} - \frac{1}{r + \delta r} \right) \approx k \delta t \frac{\delta r}{r^2} \quad k \delta t$$

$$\delta L = (\delta y^2 + \delta r^2)^{1/2} = \delta r \left(\left(\frac{\delta y}{\delta r} \right)^2 + 1 \right)^{1/2} = \delta r \left(1 + \left(\frac{k \delta t}{r^2} \right)^2 \right)^{1/2}$$

Taylor expansion: $(1 + x^2)^{1/2} \approx 1 + x^2 + \dots$

$$\Rightarrow \text{for small } \delta t, \quad \delta L = \delta r \left(1 + \frac{k^2 \delta t^2}{r^2} \right) \Rightarrow \frac{\delta L}{\delta r} = 1 + \frac{k^2 \delta t^2}{r^2}$$

but we know that $\frac{\omega}{\delta r} = \frac{(\omega + \delta \omega)}{\delta L}$

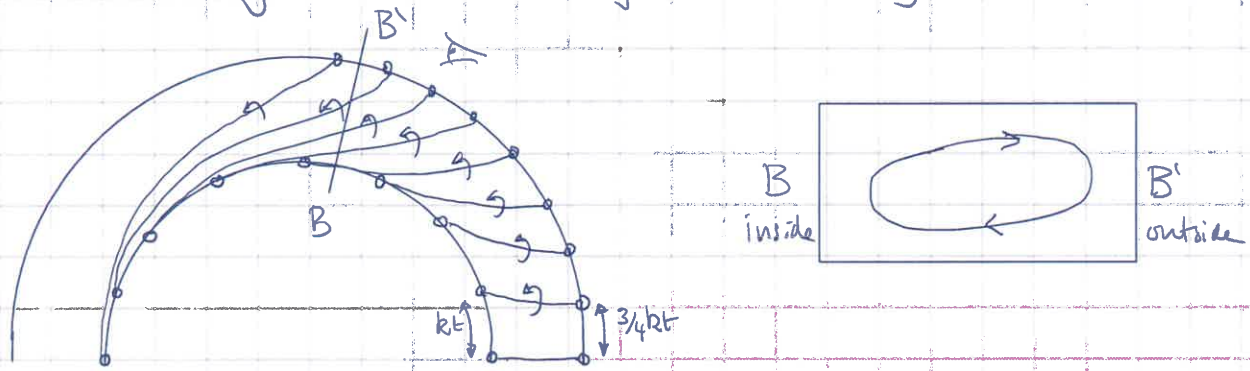
$$\Rightarrow (\omega + \delta \omega) = \frac{\delta L}{\delta r} \omega \Rightarrow \omega + \delta \omega = \omega \left(1 + \frac{k^2 \delta t^2}{r^2} \right)$$

$$\Rightarrow \delta \omega = \frac{k^2 \delta t^2}{r^2} \omega$$

$$\Rightarrow \frac{d\omega}{dt} = \frac{k^2}{r^2} \omega \quad \text{where } \omega \text{ is the vorticity of the vortex line.}$$

This was a difficult question and, although many students showed that they understood the principles, nobody made a decent attempt at solving the problem.

(d) The vorticity increases as the flow moves along the channel



The vortex lines pass through the cross-section (e.g. BB above) causing a secondary flow from the outside to the inside. *This was well answered*

(e) The pressure gradient in the boundary layer matches that in the free stream. The pressure is high on the outside and low on the inside. In the boundary layer, where the flow is slow and does not need as much centripetal force to retain circular motion, the flow will be accelerated towards the centre. This is consistent with (d).

Only a few students answered this well.

M. Tunip 2025

4. (a) At the wall, u and v are zero so $\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2}$

Now, $\frac{u}{u_0} = f(\eta) = \sin\left(\frac{\pi\eta}{2}\right)$ where $\eta = y/\delta$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = u_0 f''(0) \frac{\partial^2 \eta}{\partial y^2} = \frac{u_0 f''(0)}{\delta^2} \quad \text{but } f''(0) = -\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi\eta}{2}\right) = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \frac{dp}{dx} = 0$$

Most students answered this well but marks were lost for not showing reasoning

$$(b) \text{ i) } \delta^* = \delta \int_0^1 (1-f) d\eta = \delta \int_0^1 \left(1 - \sin\frac{\pi\eta}{2}\right) d\eta = \delta \left[\eta - \frac{2\cos\frac{\pi\eta}{2}}{\pi}\right]_0^1 = \delta \left(1 - \frac{2}{\pi}\right)$$

$$\text{ii) } \theta = \delta \int_0^1 f(1-f) d\eta = \delta \int_0^1 \left(\sin\frac{\pi\eta}{2} - \sin^2\frac{\pi\eta}{2}\right) d\eta = \delta \left[-\frac{2\cos\frac{\pi\eta}{2}}{\pi} - \frac{1}{2}\right]_0^1 = \delta \left(\frac{2}{\pi} - \frac{1}{2}\right)$$

$$\text{iii) } H = \delta^*/\theta = \frac{\frac{\pi-2}{\pi}}{\frac{4-\pi}{2\pi}} = \frac{2(\pi-2)}{4-\pi}$$

$$\text{iv) } C_f' = \frac{\tau_w}{\frac{1}{2}\rho u_1^2} = \frac{\mu \frac{\partial u}{\partial y}|_0}{\frac{1}{2}\rho u_1^2} = \frac{2\mu}{u_1} \frac{\partial f}{\partial \eta}|_0 = \frac{2\mu}{u_1} \frac{\partial \eta}{\partial y} f'(0) = \frac{2\mu}{u_1 \delta} f'(0) = \frac{2\mu}{u_1 \delta} \frac{\pi}{2}$$

(c) If $\frac{dp}{dx} = 0$ then, by Bernoulli in the free stream, $\frac{du_1}{dx} = 0$ so the

momentum integral equation becomes $\frac{d\theta}{dx} = \frac{C_f'}{2}$

Substitute in for θ and C_f' : $\left(\frac{2}{\pi} - \frac{1}{2}\right) \frac{d\delta}{dx} = \frac{1}{2} \frac{2\mu}{u_1 \delta}$ and solve for δ :

$$\Rightarrow \left(\frac{4-\pi}{2\pi}\right) \frac{1}{2} \frac{d\delta^2}{dx} = \frac{1}{2} \frac{2\mu}{u_1} \Rightarrow \frac{d\delta^2}{dx} = \frac{\mu}{u_1} \left(\frac{2\pi^2}{4-\pi}\right)$$

Integrate this in x from the leading edge, where $\delta^2 = 0$

$$\Rightarrow \delta^2 = \frac{\mu}{u_1} \left(\frac{2\pi^2}{4-\pi}\right) x \quad \Rightarrow \delta = \left(\frac{\mu x}{u_1}\right)^{1/2} \left(\frac{2\pi^2}{4-\pi}\right)^{1/2}$$

$$(d) \quad C_f' = \frac{2\mu}{u_1 \delta} = \frac{2\mu}{u_1} \left(\frac{u_1}{\mu x}\right)^{1/2} \left(\frac{4-\pi}{2\pi^2}\right)^{1/2} = \left(\frac{\mu}{u_1 x}\right)^{1/2} \left(\frac{4-\pi}{2}\right)^{1/2} = 0.655 \sqrt{\frac{1}{Re_x}}$$

This differs from the exact value of 0.664 by only 3%

This was well answered by around half the students.

J. Li 2025

5. (a)(i) At the wall, $\nu \frac{\partial^2 u}{\partial y^2} \Big|_w = \frac{1}{\rho} \frac{dp}{dx}$, but $\frac{1}{\rho} \frac{dp}{dx} = -U \frac{dU}{dx}$ in free stream

$$\Rightarrow m = \frac{\theta^2}{U} \frac{\partial^2 u}{\partial y^2} \Big|_w = - \frac{\theta^2}{\nu} \frac{dU}{dx}$$

This was well answered but full reasoning was required for full marks.

(ii) Momentum integral equation: $\frac{d\theta}{dx} + (2+H) \frac{\theta}{U} \frac{dU}{dx} = \frac{\tau_w}{\rho U^2}$

$$\tau_w = \mu \frac{\partial u}{\partial y} \Big|_w$$

Background $\left\{ \begin{aligned} \times \frac{U\theta}{\nu} &\Rightarrow \frac{U\theta}{\nu} \frac{d\theta}{dx} + (2+H) \frac{\theta^2}{\nu} \frac{dU}{dx} = \frac{\theta}{\nu} \frac{\tau_w}{\rho U} = \frac{\theta}{\nu U} \tau_w \\ &\quad \underbrace{\frac{1}{2} \frac{d\theta^2}{dx}}_{-m \text{ from (a)}} \end{aligned} \right.$

$$\Rightarrow U \frac{d\theta^2}{dx} = 2\nu [(2+H)m + L]$$

where $L \equiv \frac{\theta}{U} \frac{\tau_w}{\rho U} \Big|_w$

$$\Rightarrow U \frac{d\theta^2}{dx} = \nu [2L + 2(2+H)m] = \nu (0.45 + 6m) \text{ from experiments}$$

Now we need to integrate this:

$$U \frac{d\theta^2}{dx} = 0.45\nu - 6\theta^2 \frac{dU}{dx} \quad \text{from (a)}$$

$$\Rightarrow 6\theta^2 \frac{dU}{dx} + U \frac{d\theta^2}{dx} = 0.45\nu$$

$$\Rightarrow \frac{d(U^6 \theta^2)}{dx} = 0.45\nu U^5$$

Integrate this from location $x=a$:

$$\int_{x=a}^x \frac{d}{dx} (U^6 \theta^2) dx = \int_{x=a}^x 0.45\nu U^5 dx$$

$$\Rightarrow \theta^2(x) = \frac{0.45\nu}{U^6} \int_a^x U^5 dx$$

Around 1/2 the students answered this with no problem. Those who did not know how to apply an integration factor (U^6) struggled.

$$(b) (i) U(x) = U_0 \frac{a}{x} \Rightarrow \frac{dU}{dx} = -\frac{U_0 a}{x^2}$$

$$\begin{aligned} \theta^2(x) &= \frac{0.45\nu}{U^6} \int_a^x U^5 dx = \frac{0.45\nu}{U_0^6} \frac{x^6}{a^6} U_0^5 a^5 \int_a^x x^{-5} dx \\ &= \frac{0.45\nu}{U_0^6} \frac{x^6}{a^6} U_0^5 a^5 \left[-\frac{1}{4} x^{-4} \right]_a^x \\ &\quad \frac{1}{4} \left(\frac{1}{a^4} - \frac{1}{x^4} \right) \end{aligned}$$

$$\Rightarrow \theta^2 = \frac{0.1125\nu}{U_0 a} x^2 \left(\frac{x^4}{a^4} - 1 \right)$$

This was well answered by those who had answered (a)(ii)

$$\Rightarrow m = -\frac{\theta^2}{\nu} \frac{dU}{dx} = \frac{\theta^2 U_0 a}{\nu x^2} = 0.1125 \left(\frac{x^4}{a^4} - 1 \right)$$

$$(ii) m = 0.09 \text{ at separation} \Rightarrow \frac{x^4}{a^4} = 1 + \frac{0.09}{0.1125} = 1.8$$

$$\Rightarrow x = 1.158a$$

$$\text{pressure coefficient} = \frac{p - p_0}{\frac{1}{2} \rho U_0^2} = 1 - \left(\frac{U}{U_0} \right)^2 = 1 - \left(\frac{a}{x} \right)^2 = 0.255$$

This shows that a laminar boundary layer can only withstand a very small pressure increase before separating.

← few students wrote a satisfactory comment about their answer.

J. Li 2025

a) At wing tips, local lift $\ell = 0$, hence $\alpha_{\text{eff}} = 0$

Therefore: $\alpha_{\text{geo}}(y=s) = \alpha_d$

From data sheet (elliptic lift distr.):

$$\alpha_d = \frac{\Gamma_0}{4Us}$$

$$L = \frac{\pi}{2} \rho U \Gamma_0 s$$

$$A_R = \frac{4s^2}{S} \quad (\text{here } = 2\pi)$$

Steady flight: $L = W = W_L \cdot S$ (W_L : wing loading)

$$\therefore \Gamma_0 = \frac{2L}{\rho \pi U s} = \frac{2W_L S}{\pi \rho U s} = \frac{8W_L s^2}{\pi \rho U s A_R} = \frac{8W_L s}{\pi \rho U A_R}$$

$$\text{and } \alpha_d = \frac{2W_L}{\pi \rho U^2 A_R} = \dots = 0.152$$

$$\alpha_d = 0.87^\circ = \alpha_{\text{geo}}(y=s)$$

b) At root, local lift $\ell = \rho U \Gamma_0 = \frac{1}{2} \rho U^2 c_\ell(y=0) c$

$$\therefore c_\ell(y=0) = \frac{2\Gamma_0}{Uc} \quad (\text{rect. wing, } c \text{ is uniform})$$

For symmetric airfoil: $c_\ell = 2\pi \alpha_{\text{eff}}$

$$\therefore \alpha_{\text{eff}}(y=0) = \frac{\Gamma_0}{\pi U c} = \frac{8W_L s}{\rho \pi^2 U^2 c A_R}$$

note: rect. wing, so $A_R = \frac{2s}{c}$

$$\alpha_{\text{eff}}(y=0) = \frac{4W_L}{\rho \pi^2 U^2} = \dots = 0.061$$

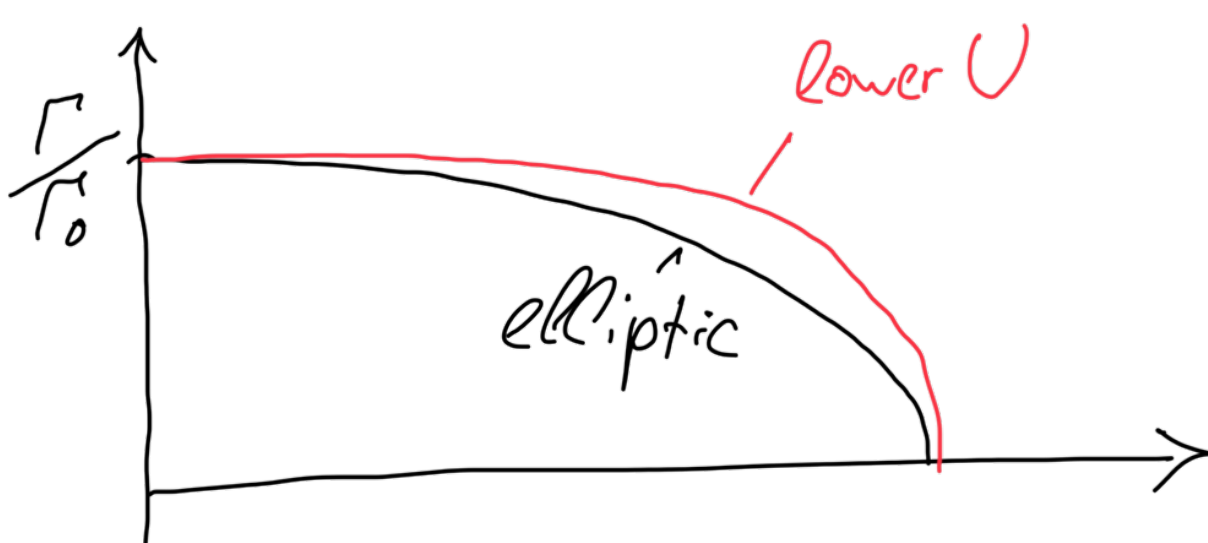
$$\alpha_{\text{eff}}(y=0) = 3.48^\circ \quad \text{so } \alpha_{\text{geo}}(y=0) = 4.35^\circ$$

c) recall from above:

$$\text{wing twist: } \alpha_{\text{geo, root}} - \alpha_{\text{geo, tip}} = \left(\frac{4W_L}{\rho \pi^2} \right)^{\text{const.}} \cdot \frac{1}{U^2} \sim \frac{1}{U^2}$$

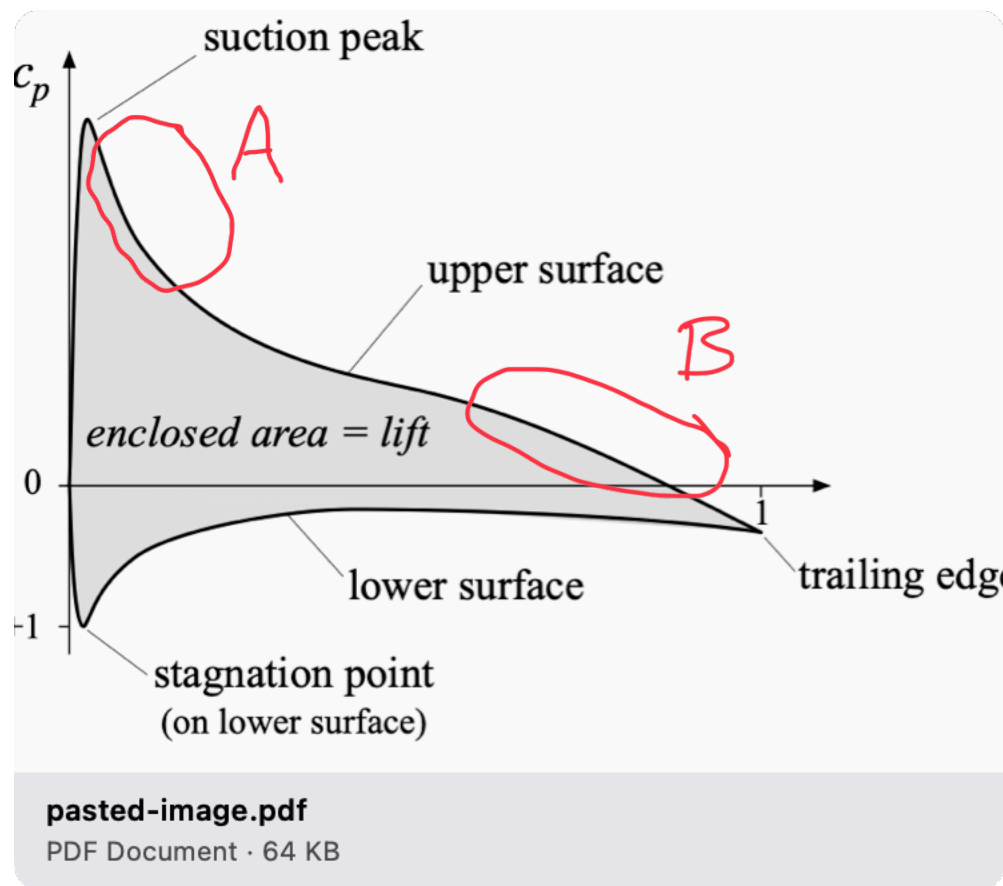
Thus, for reduced U wing twist would have to be increased to maintain elliptic lift distribution.

This means that at lower U , the wing is insufficiently twisted, giving too low α_{eff} at root and too high α_{eff} at tip.



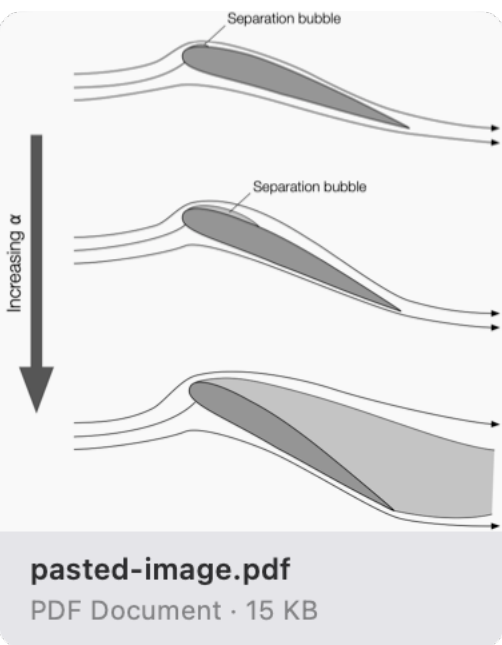
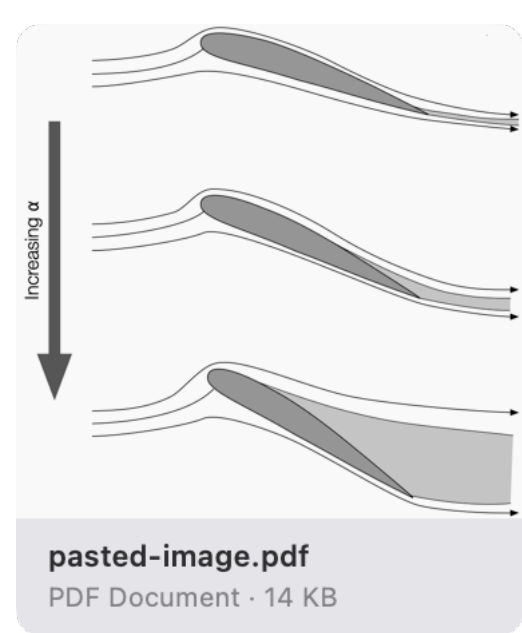
The wing is more tip-loaded

a) From lecture notes:

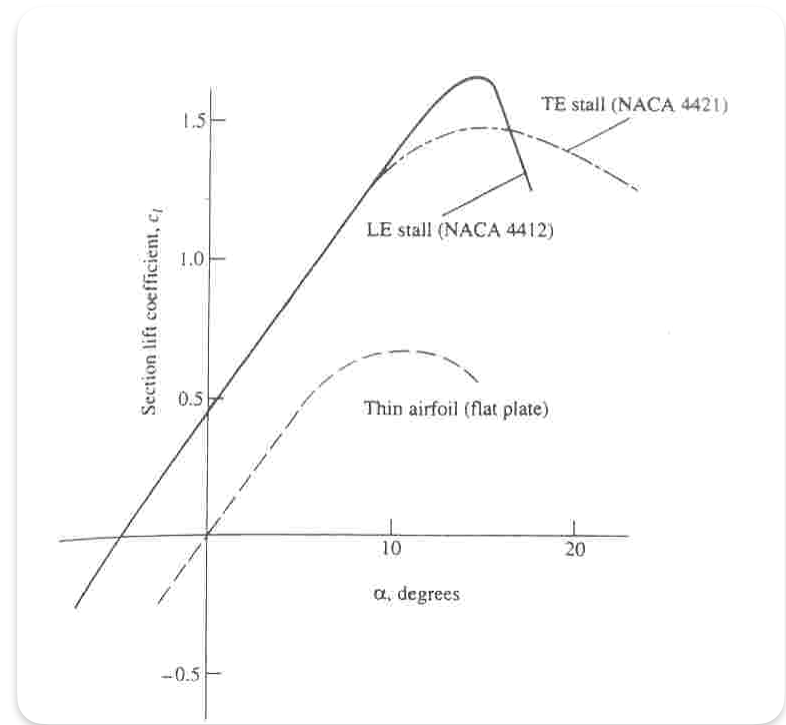


Potential separation regions:
A: Behind suction peak (upper surface) and,
B: towards trailing edge (upper surface)
see lecture notes for further details

b) From lecture notes:



c) From lecture notes:



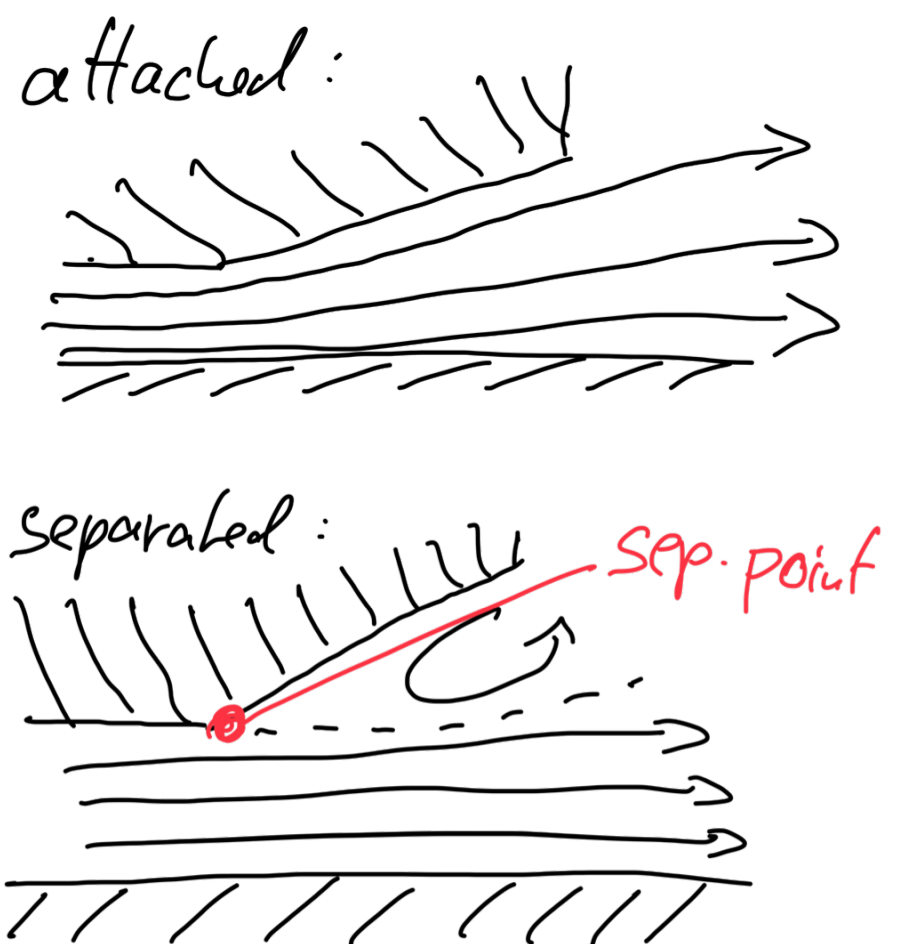
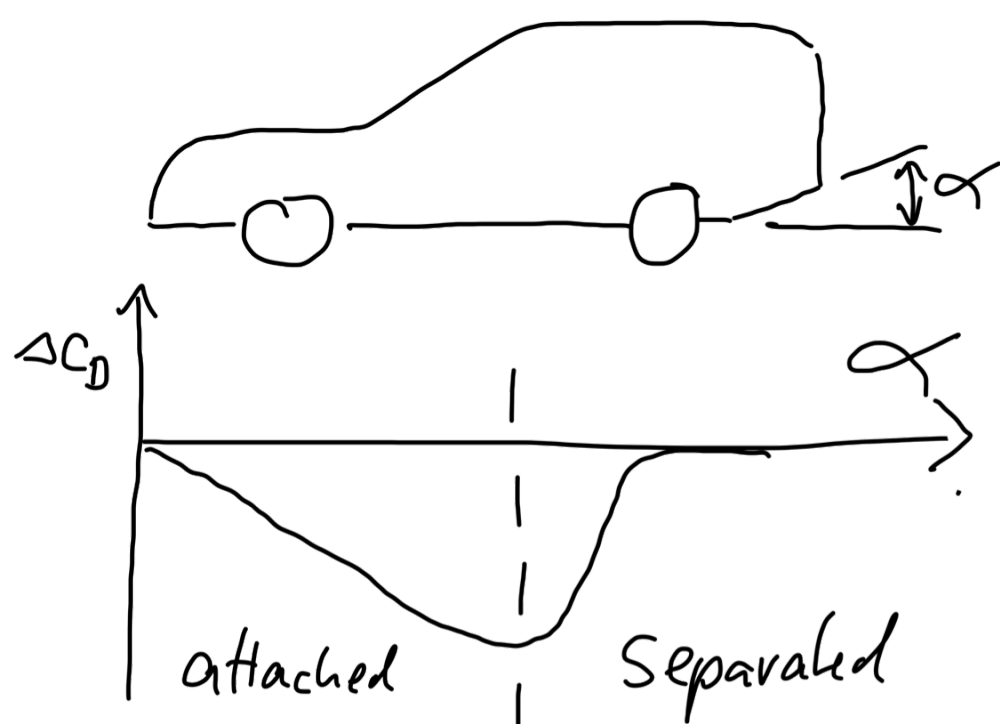
NACA 4412 is typical for a thin aerofoil (12% thick) with a sharp stall (as often caused by 'bubble burst') while NACA 4421 is 21% thick and features a more gentle, 'trailing edge' type stall. Ignore the 'flat plate' curve).
For a trainer aircraft one would prefer a gentle stall - and therefore TE stall behaviour

d) For bubble burst it may be possible to trip the flow to turbulent (if it is still laminar at separation). Alternatively, one could re-design the nose of the aerofoil to make it less 'sharp'

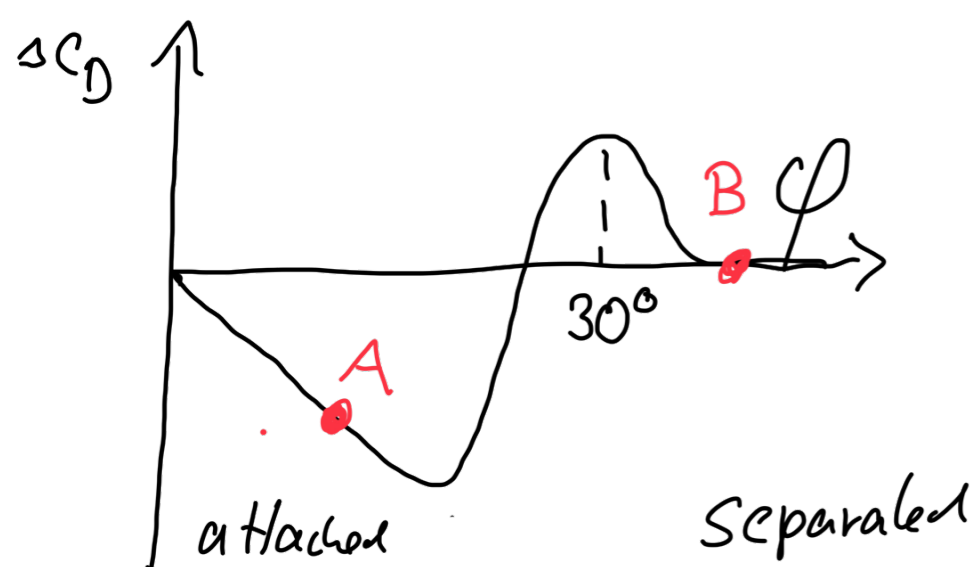
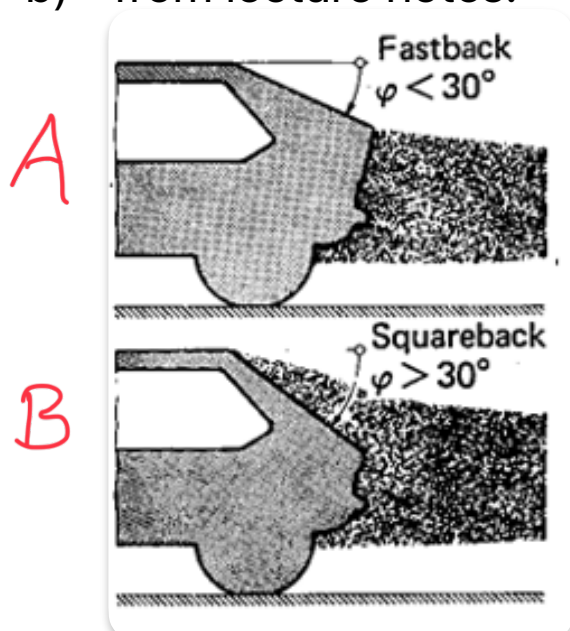
For TE stall, vortex generators could be added to re-energise the boundary layer. If the BL is laminar at separation, turbulence trips could be employed. Reducing the thickness of the airfoil can also help.

e) The model will have a much lower Re and thus the flow is more likely to be laminar (in locations where it would be turbulent on the full scale aircraft). This will increase the danger of 'bubble burst' stall, because the delayed transition will cause bubbles to be greater than at full scale, or generate laminar separation where the full scale flow would remain attached. Therefore, the model should incorporate transition trips and ideally also feature thicker airfoils.

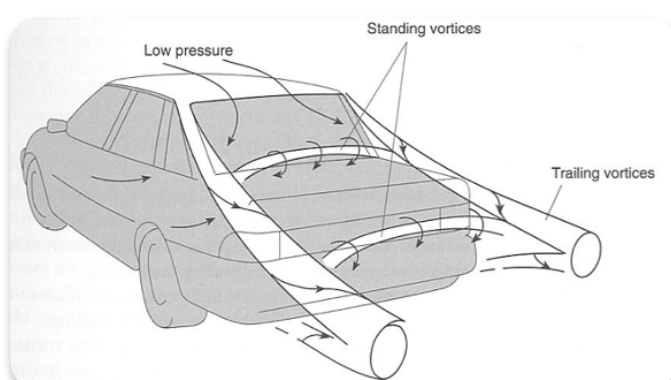
a) See lecture notes.



b) from lecture notes:



c)



At around 30° , there is a maximum in drag. This is caused by the presence of a pair of counter-rotating vortices originating from the edge between the rear and the sides (the "C-pillars"). Although the flow along the centreline is still attached, the vortices induce additional low pressure regions which create a spike in drag.

d) Both modifications will cause a change in lift force.

The Underbody diffuser generally produces a downforce. It increases with diffuser angle until a maximum is reached just before the flow separates. After separation the downforce reduces.

The boat-tail at the top will lead to increased lift. This increases with boat-tail angle, reaching a maximum around 30° (where the vortices are present, further increasing lift). For very steep angles (when the flow is completely separated as shown above for the "squareback") the additional lift reduces again to almost zero.