Version MPJ/4

EGT2
ENGINEERING TRIPOS PART IIA

Monday 27 April 2026 9.30 to 12.40

Module 3A1

FLUID MECHANICS I

Answer not more than ***five*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Attachments:

- Incompressible Flow Data Card (2 pages);
- Boundary Layer Theory Data Card (1 page);
- 3A1 Data Sheet for Applications to External Flows (2 pages).

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The far-field flow beneath a stationary helicopter hovering far above the ground can be crudely modelled in two dimensions by a vertically-aligned doublet of strength $2\pi\mu$ centred on the hub of the helicopter rotor blades.

(a) The helicopter is hovering distance d above the ground. Show that the complex potential for this flow is

$$F(z) = -\frac{2d\mu}{z^2 + d^2}$$

[20%]

(b) The helicopter now starts to move horizontally at speed U. Sketch the streamlines

(i) when $U = 0$,

(ii) when $U \ll \mu/d^2$,

(iii) when $U \gg \mu/d^2$.

[25%]

(c) A non-dimensional speed can be defined as $V \equiv Ud^2/\mu$. What is the minimum value of V at which the helicopter will avoid sucking dust from the ground into its rotor blades? [45%]

(d) Discuss the defects of this model and suggest how it could be improved. [10%]

2 Consider a two-dimensional, incompressible, inviscid, and irrotational flow over a Joukowski airfoil. A uniform flow of speed U approaches the airfoil at an angle of attack α. A circulation Γ develops around the airfoil such that the Kutta condition becomes satisfied at the trailing edge.

The flow around the airfoil is obtained by a conformal mapping from the flow around a cylinder of radius $a(1+c)$, centred at $z=-ac$. This conformal mapping is

$$\zeta = z + \frac{a^2}{z},$$

where $z = x + \mathrm{i}y$ is a point in the circle plane and $\zeta = \xi + \mathrm{i}\eta$ is a point in the airfoil plane.

(a) Without calculations, sketch the flow in the airfoil plane. [10%]

(b) Show that the complex potential for a uniform flow at angle of attack α around a circular cylinder centred on the origin with radius R and circulation Γ is [20%]

$$F(z) = Uz\mathrm{e}^{-\mathrm{i}\alpha} + \frac{UR^2}{z}\mathrm{e}^{\mathrm{i}\alpha} - \frac{\mathrm{i}\Gamma}{2\pi}\ln z + \text{constant}$$

(c) State the Kutta condition for the Joukowski airfoil and describe the physical phenomenon that it models. [10%]

(d) Derive an expression for the circulation, Γ, around the Joukowski airfoil as a function of U, a, c, and α. [40%]

(e) Calculate the chord length, L, which is defined as the distance from the leading edge to the trailing edge. Then, for a thin Joukowski airfoil at small angle of attack, α, show that the lift coefficient is approximately $C_L = 2\pi\alpha$. [20%]

3 The Burgers vortex is an exact steady solution of the incompressible Navier–Stokes equations in cylindrical coordinates (r, θ, z), with velocity field

$$u_r = -\alpha r, \qquad u_\theta = v_\theta(r), \qquad u_z = 2\alpha z,$$

where the flow is axisymmetric, $\alpha > 0$ is a constant strain rate, and $v_\theta(r)$ is a function that will be calculated later. You may use the following identities in polar coordinates:

$$\begin{aligned}
\nabla &\equiv \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{f} &\equiv \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z} \\
\nabla^2 &\equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

(a) Sketch streamlines of the flow in the $(r, z)-$plane and show that the velocity field satisfies the incompressibility condition. [20%]

(b) The steady vorticity transport equation for an incompressible Newtonian fluid is

$$(\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}$$

where ν is the kinematic viscosity. Identify the physical meaning of each term. Explaining your reasoning, show that the axial vorticity $\omega_z(r)$ for the Burgers vortex satisfies [30%]

$$-\alpha r \frac{\mathrm{d}\omega_z}{\mathrm{d}r} = 2\alpha \omega_z + \frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}\omega_z}{\mathrm{d}r}\right)$$

(c) Show that the vorticity distribution

$$\omega_z(r) = \omega_0 \exp\left(-\frac{\alpha r^2}{2\nu}\right)$$

satisfies the second equation in part (b) and determine a characteristic radius of the vortex core. [10%]

(d) The azimuthal velocity is related to the vorticity through

$$v_\theta(r) = \frac{1}{r}\int_0^r \omega_z(\tilde{r})\, \tilde{r}\, d\tilde{r}, \qquad \text{where } \tilde{r} \text{ is a dummy variable.}$$

Sketch $v_\theta(r)$, indicating its behaviour as $r \to 0$ and $r \to \infty$. [30%]

(e) Explain why this flow is steady despite the fact that energy is being lost continuously through viscous dissipation. [10%]

4 Consider the development of a steady laminar high Reynolds flow along a flat plate aligned with the x-axis, where L is the length of its development and U is the free-stream velocity. The Reynolds number $\mathrm{Re} = UL/\nu$, where ν is the kinematic viscosity of the fluid.

(a) How does the magnitude of the boundary layer thickness, δ, scale with the magnitude of L? [10%]

(b) Within the boundary layer, what are the orders of magnitude of the x-velocity component, u, and its derivatives, $\partial u/\partial x$, and $\partial u/\partial y$? Explain your reasoning. [10%]

(c) By considering the continuity equation for the boundary layer, deduce the order of magnitude of the y-velocity component, v. [20%]

(d) Apply Prandtl's order of magnitude argument to the x-direction momentum equation to deduce an approximate equation for the boundary layer. [30%]

(e) Figure 1 shows an inviscid flow impinging symmetrically on a 240° corner. Find an expression for the freestream velocity, U, along the x-axis and deduce an expression for the growth in the boundary layer thickness, δ, with streamwise distance, x. [30%]

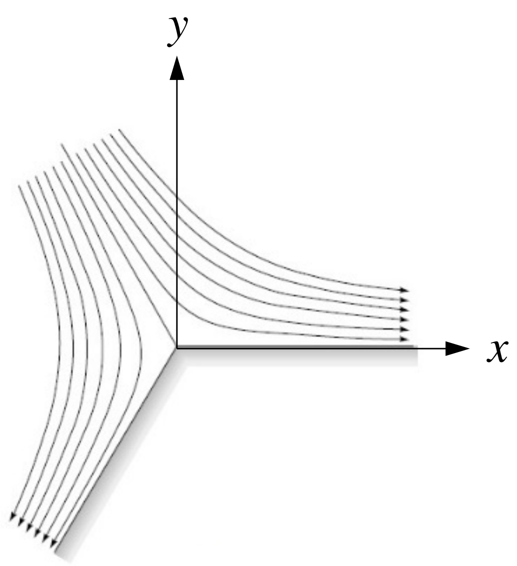


Fig. 1

5 A boundary layer developing in a wind tunnel is found to have a self-similar velocity profile given by

$$\frac{u}{U} = f(\eta),$$

where $f = 1 - e^{-\eta}$ and $\eta = y/\delta^*$. The free-stream velocity is U and the displacement thickness is δ^*.

(a) Find the momentum thickness, θ, in terms of δ^*. Also find the shape factor, H, and the local skin-friction coefficient, C'_f. [30%]

(b) Using the results from part (a), write down the momentum integral equation for this flow. [20%]

(c) Find an expression for the growth in the displacement thickness, δ^*, with streamwise distance, x, for the case of zero pressure gradient. [20%]

(d) Find the variation of U with x that will lead to a constant δ^*. Describe a realistic external flow that would give this variation in U. [30%]

6 A symmetric airfoil of chord 1 m is operating at an incidence of 3° in still air with a density of 1.2 kg m^{-3}. The flight speed is 30 ms^{-1}.

(a) What is the lift per unit span? [5%]

(b) There is a sudden gust generating a downdraft measuring 2 ms^{-1}. What is the lift per unit span when the airfoil is inside the gust? What is the change in lift coefficient? [5%]

(c) The airfoil is equipped with a 30% trailing-edge flap. By what angle does this need to be deployed in order to maintain the lift of part (a) while inside the gust? [60%]

(d) The vehicle reduces its flight speed to 10 ms^{-1} in order to land. How do you assess the ability of the flap to compensate for a similar downward gust? You do not need to perform any calculations. [30%]

7 An aircraft is equipped with a rectangular wing which has semi-span s and aspect ratio π.

(a) The spanwise circulation distribution can be expressed by a Fourier series with the coefficients G_1 and G_3, where $G_3 = G_1/8$. All other coefficients are negligible. Express the lift coefficient in terms of G_1. [10%]

(b) The wing is twisted to make the lift distribution elliptical. By how much (in percent) does this reduce the induced drag coefficient? What is the difference in geometric angle between the wing root and wing tip as a function of G_1? [50%]

(c) How might the stall behaviour of the wing be affected by this wing twist? [20%]

(d) As an alternative, the wing is redesigned to be untwisted and tapered towards the tip, with a taper ratio of 40% . How will this affect the wing efficiency and stall behaviour? [20%]

8 (a) Consider a simplified vehicle with the cross-section shown in Fig. 2. Assuming the flow along the symmetry plane to be two-dimensional and inviscid, sketch where you expect stagnation points to occur. Plot a graph showing the approximate pressure coefficient distribution along the upper and lower surfaces as a function of x. [20%]

(b) Using further sketches, describe how the flow field and pressure coefficient distribution in part (a) will change when viscous flow is considered. [20%]

(c) An old vehicle has the front shape shown in Fig. 3(a). To improve the aerodynamic performance, the nose is altered by the addition of the geometry shown in Fig. 3(b). The drag coefficient is found to reduce by 2%. Explain how this reduction is achieved. [20%]

(d) As an alternative to the nose modification, the basic shape of Fig. 3(a) is altered with the addition of a front spoiler, as shown in Fig. 3(c). The drag coefficient is found to reduce by 2% relative to the baseline value. Explain how this reduction is achieved. [20%]

(e) In a final test, the modified nose and the front spoiler are both added, as shown in Fig. 3(d). The drag coefficient is found to reduce by 8% relative to the baseline value. Explain why the drag reduction of the combined modifications is greater than the sum of the drag reductions of the individual modifications. [20%]

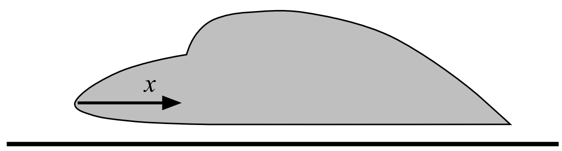
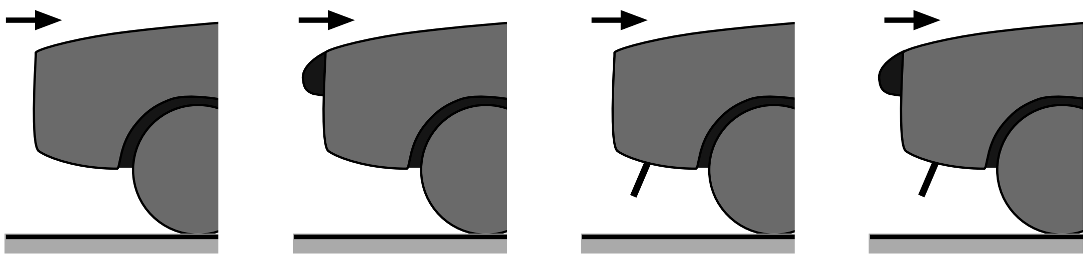


Fig. 2



(a) (b) (c) (d)

Fig. 3

END OF PAPER

THIS PAGE IS BLANK

6. (a) 178 N/m
6. (b) −48 N/m, 0.417
6. (c) 5.7°
7. (a) $(\pi^2/4)G_1$
7. (b) 4.7%, $G_1/2$

Module 3A1: Fluid Mechanics I

INCOMPRESSIBLE FLOW DATA CARD

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl } \boldsymbol{u}$

Vorticity equation (inviscid) $\dfrac{D\omega}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem (inviscid) $\dfrac{D\Gamma}{Dt} = 0, \;\; \Gamma = \oint \boldsymbol{u} \cdot \boldsymbol{dl} = \int \boldsymbol{\omega} \cdot \boldsymbol{dS}$

For an irrotational flow

velocity potential ϕ $\boldsymbol{u} = \nabla \phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow: $\dfrac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} = \text{constant}$ throughout flow field

TWO-DIMENSIONAL FLOW

Streamfunction ψ

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

Lift force Lift / unit length $= \rho U(-\Gamma)$

For an irrotational flow

complex potential *F(z)* $F(z) = \phi + i\psi$ is a function of $z = x + iy$

$$\frac{dF}{dz} = u - iv$$

TWO-DIMENSIONAL FLOW (continued)

Summary of simple 2 - D flow fields

	ϕ	ψ	$F(z)$	$\boldsymbol{u}$
Uniform flow (x - wise)	Ux	Uy	Uz	$u = U,\ v = 0$
Source at origin	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	$\frac{m}{2\pi}\ln z$	$u_r = \frac{m}{2\pi r},\ u_\theta = 0$
Doublet (x - wise) at origin	$-\frac{\mu\cos\theta}{2\pi r}$	$\frac{\mu\sin\theta}{2\pi r}$	$-\frac{\mu}{2\pi z}$	$u_r = \frac{\mu\cos\theta}{2\pi r^2},\ u_\theta = \frac{\mu\sin\theta}{2\pi r^2}$
Vortex at origin	$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$	$-\frac{i\Gamma}{2\pi}\ln z$	$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3 - D flow fields

	ϕ	$\boldsymbol{u}$
Source at origin	$-\frac{m}{4\pi r}$	$u_r = \frac{m}{4\pi r^2},\quad u_\theta = 0,\quad u_\psi = 0$
Doublet at origin (with θ the angle from the doublet axis)	$-\frac{\mu\cos\theta}{4\pi r^2}$	$u_r = \frac{\mu\cos\theta}{2\pi r^3},\quad u_\theta = \frac{\mu\sin\theta}{4\pi r^3},\quad u_\psi = 0$

Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U-u)u}{U^2} dy = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U^2-u^2)u}{U^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U}\right)^2\right) \frac{u}{U} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U}\theta\frac{dU}{dx} = \frac{\tau_w}{\rho U^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= -\frac{1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

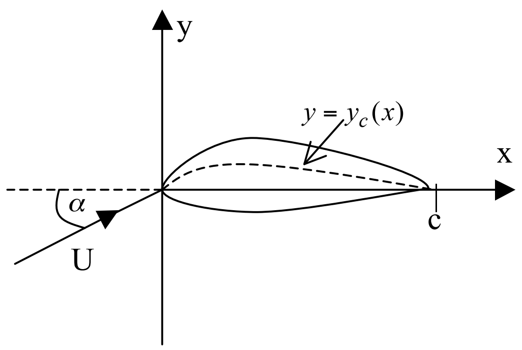
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c}, \quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

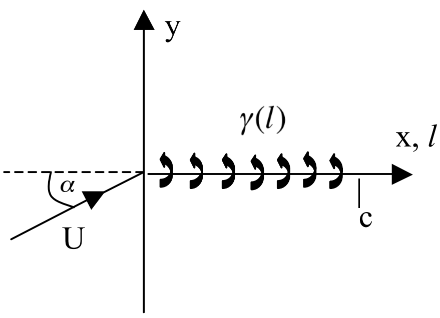
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S}, \quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $c_p = \pm\gamma / U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0) / \frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $l = c(1+\cos\phi)/2, \quad x = c(1+\cos\theta)/2$

Incidence solution: $$\gamma(l) = -2U\alpha\frac{1-\cos\phi}{\sin\phi}, \; c_l = 2\pi\alpha, \; c_m = c_l/4$$

Camber solution: $$\gamma(l) = -U\left[g_0\frac{1-\cos\phi}{\sin\phi} + \sum_{n=1}^{\infty} g_n \sin n\phi\right], \text{ where}$$

$$g_0 = \frac{1}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)d\theta, \; g_n = \frac{2}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)\cos n\theta d\theta\,;$$

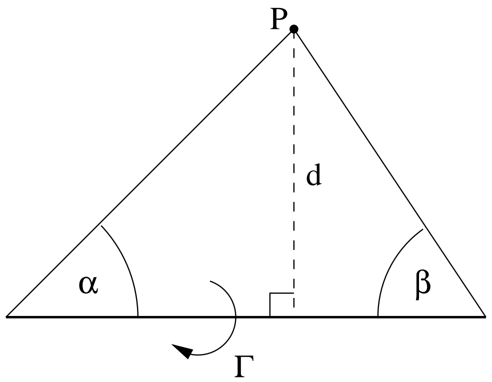
or, equivalently: $$-2\frac{dy_c}{dx} = g_0 + \sum_{n=1}^{\infty} g_n \cos n\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right), \; c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_{o}^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta}\, d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



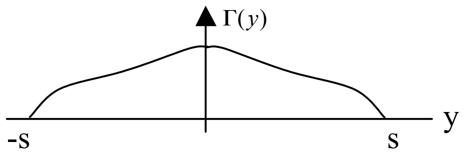
The Biot-Savart integral for a straight element of circulation Γ gives a contribution to the velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y)\, dy$

Downwash angle: $\alpha_d(y) = \dfrac{1}{4\pi U} \displaystyle\int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y-\eta}\, d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y)\alpha_d(y)\, dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{\text{n odd}} G_n \sin n\theta$, with $y = -s\cos\theta$;

equivalently, $G_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} \frac{\Gamma(y)}{Us} \sin n\theta\, d\theta$

Relation between lift and induced drag:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

Elliptic lift distribution: $\Gamma(y) = \Gamma_0\left(1 - \dfrac{y^2}{s^2}\right)^{1/2}$, $L = \dfrac{\pi}{2}\rho U \Gamma_0 s$, $\alpha_d = \dfrac{\Gamma_0}{4Us}$, $\delta = 0$