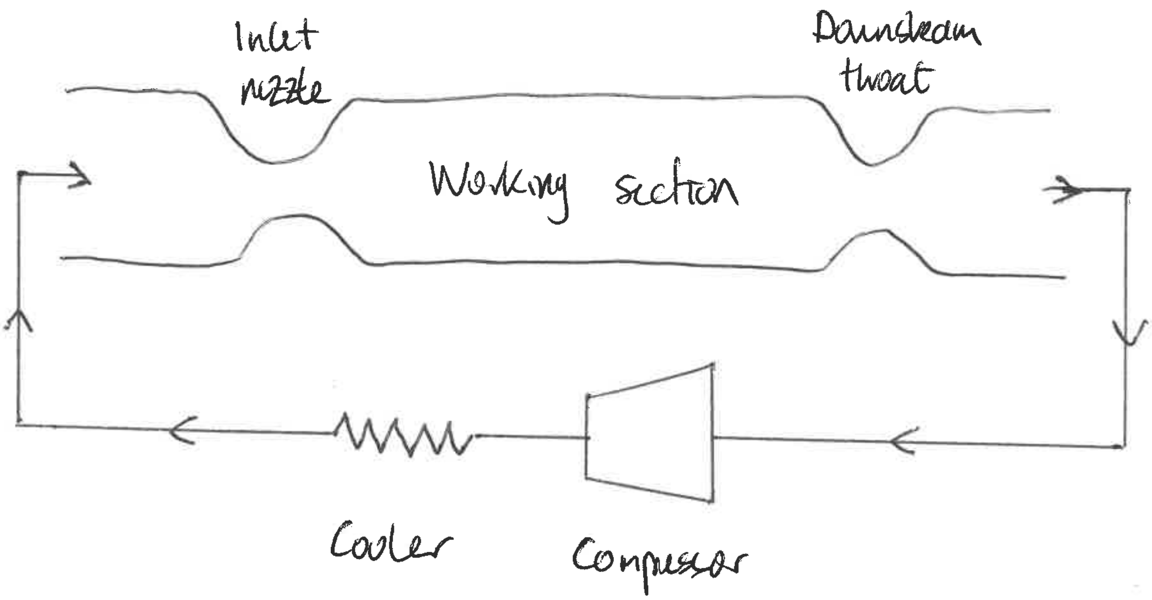
3A3 2025 Crib

1\. a

(i)



Subsonic flow is accelerated through the inlet nozzle to supersonic in the working section. The supersonic flow must be decelerated through the downstream throat back to subsonic, and this will involve a shock wave. The compressor restores the stagnation pressure back to the initial value, while the cooler restores the stagnation temperature. The downstream throat must not be choked.

ii) Losses due to friction/boundary layers and to the shock in/downstream of the downstream throat.

b) i) Working section $M = 1.8$. Tables, $\gamma = 1.4$,

$$\frac{\dot{m}\sqrt{c_p T_0}}{A P_0} = 0.8902$$

Inlet nozzle is choked $$\frac{\dot{m}\sqrt{c_p T_0}}{A_I^* P_0} = 1.281$$

Inlet nozzle throat area $$A_I^* = \frac{0.8902}{1.281} \times 0.3^2$$

$$= \underline{0.06254\,m^2}$$

ii) Downstream throat not choked:

$$\frac{\dot{m}\sqrt{c_p T_0}}{A_D^* P_{0S}} \leq 1.281$$ with shock in the working section

tables at $M = 1.8$: $$\frac{P_{0S}}{P_0} = 0.8127$$

Ratio $$\frac{A_D^*}{A_I^*} = \frac{P_0}{P_{0S}} \quad \therefore A_D^* = \frac{0.06254}{0.8127}$$

$$= \underline{0.07696\,m^2}$$

c) The tunnel is started by increasing the pressure difference until the inlet nozzle is choked and there is a shock at the entrance to the working section. The pressure difference must be increased further to bring the shock to the downstream end of the working section and into the downstream throat.

For continuous operation the shock must sit just downstream of the downstream throat.

During the starting procedure there is a shock in the working section at $M = 1.8$ (worst case).

The compressor must restore the stagnation pressure loss:

Isentropic compressor (and elsewhere except for the shock):

$$\frac{T_2}{T_1} = \left(\frac{P_0}{P_{0S}}\right)^{\frac{\gamma-1}{\gamma}} \quad \therefore \; T_2 = 293\left(\frac{100}{81.27}\right)^{\frac{1.4-1}{1.4}}$$

$$= 310.89\,K$$

Power is $\dot{m} C_p (T_2 - T_1)$

Working section $M = 1.8$ $\quad \dfrac{\dot{m}\sqrt{C_p T_0}}{A\,P_0} = 0.8902$

$$\dot{m} = \frac{0.8902 \times 0.3^2 \times 100 \times 10^3}{\sqrt{1005 \times 293}} = 14.76\,kg/s$$

$\therefore$ Power is $14.76 \times 1005 \times (310.89 - 293) = \underline{265.4\,kW}$

d) With shock in the downstream throat have

$$\frac{\dot{m}\sqrt{C_p T_0}}{A_D^* P_0} = 1.281\,\frac{A_I^*}{A_D^*} = 1.281 \times 0.8127 = 1.041$$

$\Rightarrow$ Tables give $M = 1.575$

with $\dfrac{P_{0S}}{P_0} = 0.904$ have $\dfrac{T_2}{T_1} = \left(\dfrac{1}{0.904}\right)^{\frac{\gamma-1}{\gamma}} \Rightarrow T_2 = 301.6\,K$

Power is $14.76 \times 1005 \times (301.6 - 293) = \underline{127.6\,kW}$

2 a) Continuity $\rho V = \text{const}$; energy $h + \frac{1}{2}V^2 = \text{const}$

Ideal gas $P = \rho R T$, $h = C_p T$, $a^2 = \gamma P/\rho$

$\Rightarrow P = \frac{\gamma-1}{\gamma}\rho h$, $\rho V^2 = \gamma P M^2$

and $d(\rho V) = 0$, $dh + V dV = 0$

Impulse function $\frac{F}{A} = P + \rho V^2$ $\therefore d\left(\frac{F}{A}\right) = dP + \rho V dV + V \cancel{d(\rho V)}^{\,0}$

$$d\left(\frac{F}{A}\right) = \frac{\gamma-1}{\gamma}(\rho dh + h d\rho) + \rho V dV$$

$$= -\frac{\gamma-1}{\gamma}\rho V dV + \frac{P}{\rho}d\rho + \rho V dV$$

$$= \frac{1}{\gamma}\rho V dV - P\frac{dV}{V} = PM^2\frac{dV}{V} - P\frac{dV}{V}$$

$$\therefore d\left(\frac{F}{A}\right) = P(M^2-1)\frac{dV}{V} \quad \text{as required}$$

b) $D = 0.5\text{m}$, $C_f = 0.0025$, $P_0 = 1.5\text{bar}$, $T_0 = 293\text{K}$

$M = 1.8$, Tables $\frac{\dot{m}\sqrt{C_p T_0}}{A P_0} = 1.2014$, $\frac{4 C_f L_{max}}{D} = 0.0648$

i) $\frac{4C_f}{D} = 0.02$ $\therefore L_{max} = 3.24\text{m}$

ii) $A = \frac{\pi D^2}{4} = 0.1964\text{m}^2$

$$\dot{m} = \frac{1.2014 \times 0.1964 \times 10^5 \times 1.5}{\sqrt{1005 \times 293}} = 65.2\text{kg/s}$$

c) Pipe length increased to 4m : shock in pipe at $M = 1.26$

tables : $\frac{4 C_f L_2}{D} = 0.0517$, $M_s = 0.8071$

ahead of shock

Behind shock $\frac{4 C_f L_2'}{D} = 0.066$ at $M_s = 0.8071$

$\therefore L_2' = 3.3\,m$

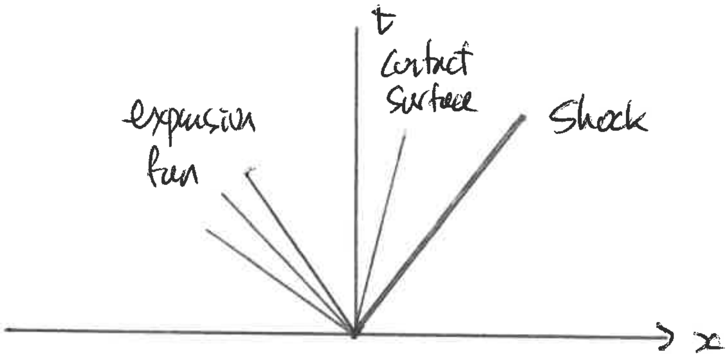
$\therefore$ shock is 0.7 m from the pipe entrance.

Check : $L_2 = \frac{0.0517}{0.02} = 2.585\,m$

$L_{max} - L_2 = 3.24 - 2.585 = 0.655\,m$ close!

3 a)

i)



ii) $\frac{P_s}{P} = 1.8$: Tables have $M = 1.3$

iii) Tables at $M = 1.3$ have $\frac{T_s}{T} = 1.1909$

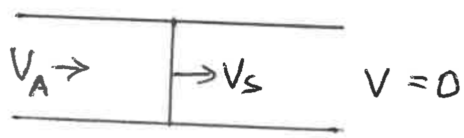
$\Rightarrow T_s = 1.1909 \times 293$

$= 348.9\,K$

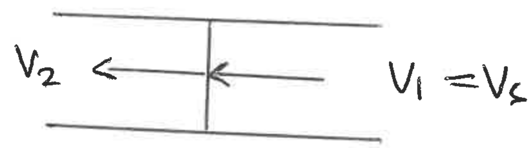
corresponding speed of sound is $\sqrt{\gamma R T_s} = 1.4 \times 285 \times 348.9$

$= 373.1\ m/s$

iv) lab frame



Shock frame



$V_2 = M_2 a_2 = 0.786 \times 373.1 = 293.2\ m/s$
(tables)

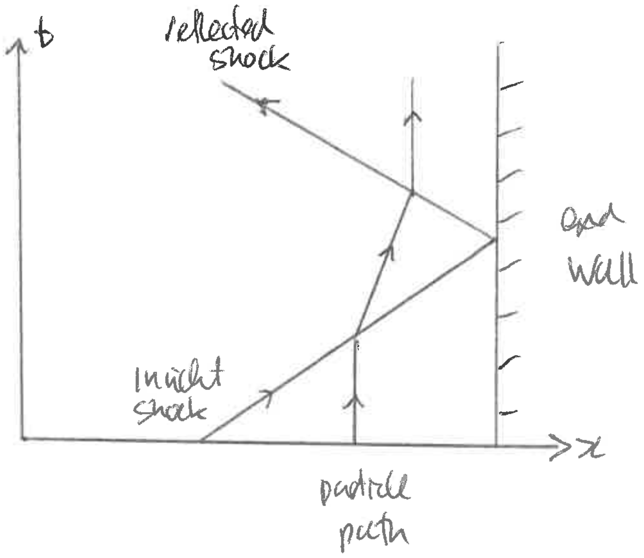
shock speed $V_s = M a_1 = M\sqrt{\gamma R T_1} = 1.3 \times \sqrt{1.4 \times 285 \times 293}$

$= 444.5\ m/s$

air speed $V_A = V_s - V_2 = 151.3\ m/s$

b)

i)



ii)

lab frame: $V_A \rightarrow$ $\quad V_R \leftarrow$ | $V = 0$

shock frame: $V_1 = V_A + V_R$ | $V_2 = V_R$

Tables $\quad \dfrac{\rho_2}{\rho_1} = \dfrac{(\gamma+1)M_1^2}{2\left(1+\frac{\gamma-1}{2}M_1^2\right)} = \dfrac{V_1}{V_2} = \dfrac{V_A+V_R}{V_R}$

$$\frac{(V_A+V_R)-V_A}{V_A+V_R} = 1 - \frac{V_A}{V_A+V_R} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2}$$

Have $\quad \dfrac{V_A}{V_A+V_R} = \dfrac{M_A}{M_1}$, hence $\quad M_1^2 - \frac{1}{2}(\gamma+1)M_A M_1 - 1 = 0$

Also have $\quad M_A = \dfrac{V_A}{a_s} = \dfrac{151.3}{373.1} = 0.405$

$$\therefore M_1 = \frac{1}{2}\left[(1.2\times0.405) \pm \sqrt{(1.2\times0.405)^2+4}\right] = \underline{1.272}$$

iii) Tables for $M_1 = 1.272 \Rightarrow T_2/T_1 = 1.173$ for $T_1 = 348.9K$

$\therefore T_2 = 409.3K$ and $a_2 = \sqrt{\gamma R T_2}$

ie $a_2 = \sqrt{1.4\times285\times409.3} = \underline{404.1 \text{ m/s}}$

4.

a) Conservation: Mass: $\nabla\cdot(\rho V)=0 \Rightarrow \nabla\cdot V+\frac{1}{\rho}V\cdot\nabla\rho=0$

Momentum: $-\frac{1}{\rho}\nabla p = V\cdot\nabla V$ Isentropic: $p=k\rho^{\gamma} \Rightarrow \nabla p = a^2\nabla\rho$

Energy: $h_0 = c_p T+\frac{1}{2}(u^2+v^2)=\frac{a^2}{\gamma-1}+\frac{1}{2}(u^2+v^2)$

Eliminate ∇p & $\nabla\rho \Rightarrow$ $\nabla\cdot V = \frac{-1}{\rho}V\cdot\nabla\rho = \frac{-1}{\rho a^2}V\cdot\nabla p = \frac{1}{a^2}V\cdot(V\cdot\nabla V) \Rightarrow a^2\nabla\cdot V - V\cdot(V\cdot\nabla V)=0$ ①

b) Flow is irrotational: $\nabla\times V=0 \Rightarrow V=\nabla\Phi=\nabla(U_\infty x+\phi)$ and $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$

Consider equation ① in terms of components $V=(u,v)$

$$a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left[u e_x+v e_y\right]\cdot\left[\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)e_x+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)e_y\right]=0$$

$$a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-u\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-v\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=0$$

$$(a^2-u^2)\frac{\partial u}{\partial x}-uv\frac{\partial u}{\partial y}-vu\frac{\partial v}{\partial x}+(a^2-v^2)\frac{\partial v}{\partial y}=0$$

and consider small perturbations: $V=\left(U_\infty+\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)$

$$\Rightarrow \left[a^2-\left(U_\infty+\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}-2\left[U_\infty+\frac{\partial\phi}{\partial x}\right]\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}+\left[a^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}=0$$

which for $\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\ll U_\infty$ becomes $\beta^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ for $\beta=\sqrt{1-M_\infty^2}$

c) $a=\sqrt{\gamma RT}$ $\gamma=1.4$ $R=287$ For $T=+15°C=288K$ $a=340$ $\Rightarrow M=0.5$, $V=170$ m/s

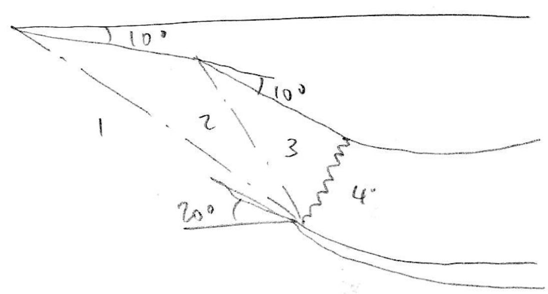
For $T=-40°C=233K$ $a=306$ $\Rightarrow V=170$, $M=0.556$

$C_L=0.2$

$C_L=\frac{C_{L0}}{\sqrt{1-M_\infty^2}}$ $\therefore C_{L0}=C_L\sqrt{1-M_\infty^2}=0.2\times\sqrt{1-0.556^2}=0.166$

5.

a)



$M_1 = 2.05$

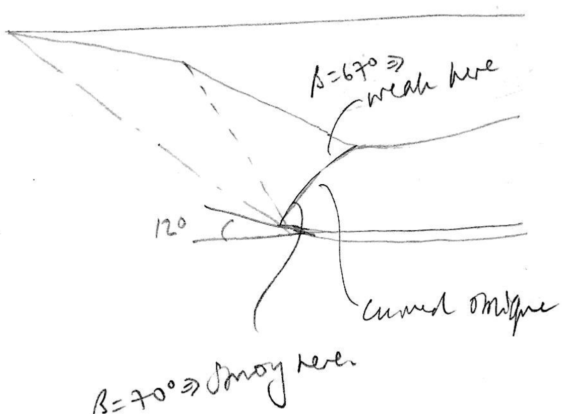
$M_2 = 1.69$ $\quad \frac{p_{02}}{p_{01}} = 0.98$ $\quad \frac{p_2}{p_1} = 1.72$

$M_3 = 1.35$ $\quad \frac{p_{03}}{p_{02}} = 0.99$ $\quad \frac{p_3}{p_2} = 1.65$

$M_4 = 0.76$ $\quad \frac{p_{04}}{p_{03}} = 0.97$ $\quad \frac{p_4}{p_3} = 1.96$

94% $\quad$ 5.56

b)



weak:

$M_4 \sim 0.95$ $\quad \frac{p_{04}}{p_{03}} = 0.99$ $\quad \frac{p_4}{p_3} = 1.63$

strong

$M_4 \sim 0.91$ $\quad \frac{p_{04}}{p_{03}} = 0.98$ $\quad \frac{p_4}{p_3} = 1.71$

Avg: $\quad$ 0.93 $\quad$ 0.985 $\quad$ 1.67

95.5% $\quad$ 4.65

ie. 16% reduction

N.B. 8° turning of curved shock is close to θ_{max} and flow is subsonic behind both strong and weak sections of shock.

c) Advantages:

- Extra 1.5 p.p. pressure recovery, so more efficient shock system
- All other things being equal, lower nacelle frontal area ⇒ reduced drag.

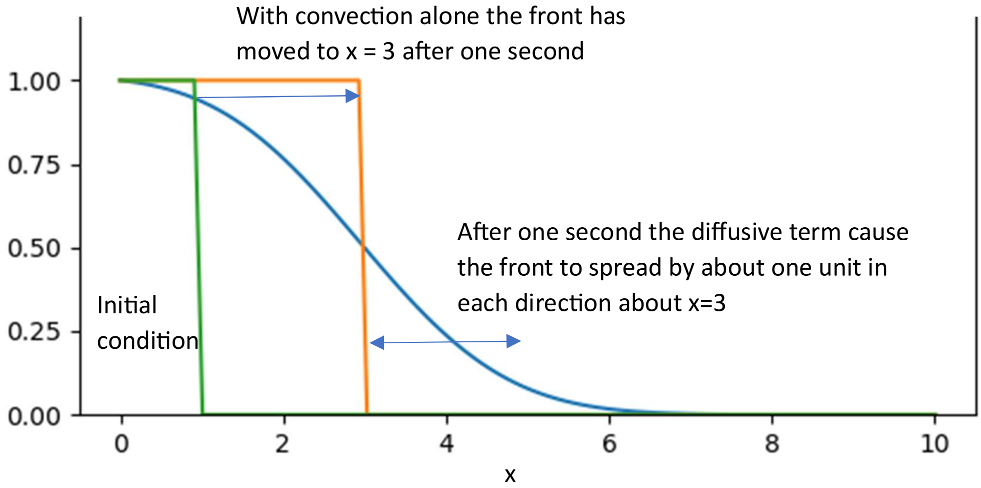
d) Reduced static pressure rise: (needs to be replaced by):

- Increased static diffusion between shock system & engine face
  ↳ may mean longer diffuser ⇒ adds weight / cost
- Increased pressure rise in engine ⇒ extra compressor stage?
  ⇒ adds weight / cost / complexity.

6a)

$$\left.\frac{\partial u}{\partial t}\right|_{x,t} + 2\left.\frac{\partial u}{\partial x}\right|_{x,t} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x,t}$$

For this equation the wave is convected with a speed of 2 units per second. The diffusion coefficient is 1, so after time t the rough distance things have diffused is $t \sim \frac{l^2}{1}$, i.e. after one second, the sharp front will have diffused approximately $l \sim 1\ unit$



(note that this shows an actual solution; however, a hand-drawn and very rough sketch with these features would be acceptable)

b) Using standard finite difference approximation

$$u_i^{N+1} = u_i^N - 2\left(u_{i+1}^N - u_{i-1}^N\right)\frac{\Delta t}{2\Delta x} + \left(u_{i+1}^N - 2u_i^N + u_{i-1}^N\right)\frac{\Delta t}{\Delta x^2}$$

The solution is second order accurate in space, owing to the central difference used for the first derivative. Students would be expected to remember these. However, it will be also be apparent when deriving the equivalent PDE below, or if students had derived the central difference estimates of the spatial derivatives.

c) Starting with the taylor series,

$$u_i^{N+1} = u_i^N + \left.\frac{\partial u}{\partial t}\right|_{x,t}\Delta t + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t}\Delta t^2 + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_{x,t}\Delta t^3 \ldots$$

$$u_{i-1}^N = u_i^N - \left.\frac{\partial u}{\partial x}\right|_{x,t}\Delta x + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,t}\Delta x^2 - \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t}\Delta x^3 \ldots$$

$$u_{i+1}^N = u_i^N + \left.\frac{\partial u}{\partial x}\right|_{x,t} \Delta x + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x^2 + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^3 \ldots$$

The update formula can be rearranged to

$$\frac{u_i^{N+1} - u_i^N}{\Delta t} = -2\frac{\left(u_{i+1}^N - u_{i-1}^N\right)}{2\Delta x} + \frac{\left(u_{i+1}^N - 2u_i^N + u_{i-1}^N\right)}{\Delta x^2}$$

From the taylor series in time

$$\frac{u_i^{N+1} - u_i^N}{\Delta t} = \left.\frac{\partial u}{\partial t}\right|_{x,t} + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,\,t} \Delta t + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_{x,t} \Delta t^2$$

From taylor series in x

$$u_{i+1}^N - u_{i-1}^N = 2\left.\frac{\partial u}{\partial x}\right|_{x,t} \Delta x + \frac{2}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^3 \ldots$$

$$u_{i+1}^N + u_{i-1}^N = 2u_i^N + \left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x^2 + \frac{2}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x,t} \Delta x^4 \ldots$$

$$u_{i+1}^N - 2u_i^N + u_{i-1}^N = \left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x^2 + \frac{2}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x,t} \Delta x^4 \ldots$$

Putting back into update formula

$$\left.\frac{\partial u}{\partial t}\right|_{x,t} + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,\,t} \Delta t + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_{x,t} \Delta t^2$$
$$= -2\left(\left.\frac{\partial u}{\partial x}\right|_{x,t} + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^2 \ldots\right) + \left(\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} + \frac{2}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x,t} \Delta x^2 \ldots\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{x,t} + 2\left.\frac{\partial u}{\partial x}\right|_{x,t} + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,\,t} \Delta t + O(\Delta t^2)$$
$$= \left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} - 2\frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^2 + \frac{2}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x,t} \Delta x^2 + O(\Delta x^4)$$

$$\left.\frac{\partial u}{\partial t}\right|_{x,t} + a\left.\frac{\partial u}{\partial x}\right|_{x,t} + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,\,t} \Delta t + O(\Delta t^2)$$
$$= \left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} - 2\frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^2 + \frac{2}{4!}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x,t} \Delta x^2 + O(\Delta x^4)$$

From the original equation (acceptable as higher order terms would just end up being neglected)

$$\left.\frac{\partial u}{\partial t}\right|_{x,t} = -2\left.\frac{\partial u}{\partial x}\right|_{x,t} + \left.\frac{\partial^2 u}{\partial x^2}\right|_{x,t}$$

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t} = \left[-2\left.\frac{\partial}{\partial x}\right|_{x,t} + \left.\frac{\partial^2}{\partial x^2}\right|_{x,t}\right]\left(-2\left.\frac{\partial u}{\partial x}\right|_{x,t} + \left.\frac{\partial^2 u}{\partial x^2}\right|_{x,t}\right)$$

$$= 4\frac{\partial^2 u}{\partial x^2}\bigg|_{x,t} - 4\frac{\partial^3 u}{\partial x^3}\bigg|_{x,t} + \frac{\partial^4 u}{\partial x^4}\bigg|_{x,t}$$

The second term is a third derivative which doesn't contribute to dissipation, whilst we are told we can ignore the fourth derivatives contribution to dissipation.

$$\frac{\partial u}{\partial t}\bigg|_{x,t} + 2\frac{\partial u}{\partial x}\bigg|_{x,t} + O(\Delta t^2) = \frac{\partial^2 u}{\partial x^2}\bigg|_{x,\,t} - \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}\bigg|_{x,\,t}\Delta t + O(\Delta x^2)$$

$$\frac{\partial u}{\partial t}\bigg|_{x,t} + 2\frac{\partial u}{\partial x}\bigg|_{x,t} + O(\Delta t^2) = \frac{\partial^2 u}{\partial x^2}\bigg|_{x,\,t} - \frac{1}{2!}4\frac{\partial^2 u}{\partial x^2}\bigg|_{x,t}\Delta t + O(\Delta x^2)$$

$$\frac{\partial u}{\partial t}\bigg|_{x,t} + 2\frac{\partial u}{\partial x}\bigg|_{x,t} + O(\Delta t^2) = \frac{\partial^2 u}{\partial x^2}\bigg|_{x,\,t}(1 - 2\Delta t) + O(\Delta x^2)$$

The magnitude of the nuermical dissipation coefficient is $2\Delta t$. Therefore, the magnitude of the ratio of numerical to physical dissipation is

$$\frac{1}{2\Delta t}$$

d) For cfl stability need $\frac{\Delta t}{\Delta x} \leq 1$, so if running at half the minimum time step, then the numerical dissipation coefficient is $2\Delta t = \frac{2\Delta x}{2} = \Delta x$.

The % reduction in the dissipation caused by negative numerical dissipation is then

$$1 - \frac{(1 - \Delta x)}{1} = \Delta x$$

So for a grid space of $\Delta x = 0.01\ m$ the dissipation is reduced by 1%

7

(a) The following numerical scheme is used to produce a numerical solution,

$$u_i^{N+1} = u_i^{N-1} + \left(u_{i+1}^N + u_i^N - 2u_{i-1}^N \right) \frac{\Delta t}{\Delta x}$$

$$\frac{u_i^{N+1} - u_i^{N-1}}{\Delta t} = \left(u_{i+1}^N + u_i^N - 2u_{i-1}^N \right) \frac{1}{\Delta x}$$

A short cut is to notice that there appears to be a central and a backward difference superimposed.

$$\left(u_{i+1}^N + u_i^N - 2u_{i-1}^N \right) = (u_{i+1}^N - u_{i-1}^N) + (u_i^N - u_{i-1}^N)$$

$$u_{i-1}^n = u_i^n - \left.\frac{\partial u}{\partial x}\right|_{x,t} \Delta x + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x^2 - \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^3 + \cdots$$

$$u_{i+1}^n = u_i^n + \left.\frac{\partial u}{\partial x}\right|_{x,t} \Delta x + \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x^2 + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^3 + \cdots$$

$$\left(u_{i+1}^N - u_{i-1}^N \right) = 2\left.\frac{\partial u}{\partial x}\right|_{x,t} \Delta x + \frac{2}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^3$$

$$\left(u_i^N - u_{i-1}^N \right) = \left.\frac{\partial u}{\partial x}\right|_{x,t} \Delta x - \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x^2 + \frac{1}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^3$$

$$\left(u_{i+1}^N - u_{i-1}^N + u_i^N - u_{i-1}^N \right) \frac{\Delta t}{\Delta x} = 3\left.\frac{\partial u}{\partial x}\right| \Delta t_{x,t} - \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta t \Delta x + \frac{3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta t \Delta x^2$$

Then

$$u_i^{N+1} = u_i^{N-1} + \left(u_{i+1}^N + u_i^N - 2u_{i-1}^N \right) \frac{\Delta t}{\Delta x}$$

Becomes

$$\frac{u_i^{N+1} - u_i^{N-1}}{\Delta t} = 3\left.\frac{\partial u}{\partial x}\right|_{x,t} - \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x + \frac{3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^2$$

$$\frac{u_i^{N+1} - u_i^{N-1}}{\Delta t} = 3\left.\frac{\partial u}{\partial x}\right|_{x,t} - \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x + \frac{3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^2$$

Noting that the time derivative approximation can be formed using a central difference in time,

$$\frac{u_i^{N+1} - u_i^{N-1}}{\Delta t} = 2\left.\frac{\partial u}{\partial t}\right|_{x,t} + \frac{2}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_{x,t} \Delta t^2 \ldots$$

So that

$$2\left.\frac{\partial u}{\partial t}\right|_{x,t} + \frac{2}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_{x,t} \Delta t^2 \ldots = 3\left.\frac{\partial u}{\partial x}\right|_{x,t} - \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x \quad + \frac{3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta x^2$$

As dt and dx -> 0 this becomes

$$2\left.\frac{\partial u}{\partial t}\right|_{x,t} + \frac{2}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_{x,t} \Delta \boldsymbol{t}^{\mathbf{2}} \ldots = 3\left.\frac{\partial u}{\partial x}\right|_{x,t} - \frac{1}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,\,t} \Delta x \quad + \frac{3}{3!}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x,t} \Delta \boldsymbol{x}^{\mathbf{2}}$$

$$2\left.\frac{\partial u}{\partial t}\right|_{x,t} - 3\left.\frac{\partial u}{\partial x}\right|_{x,t} = 0$$

Where the leading order error terms are highlighted. The scheme is unusually second order in time and only first order in space.

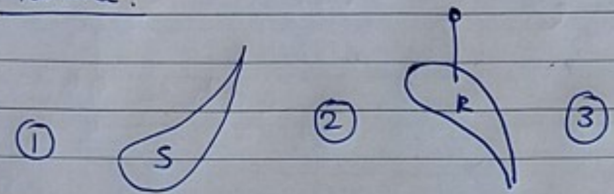
7 b)

i) $\Gamma = \text{Const} \quad \therefore \quad U = \text{Const}$

$V_x = \text{Const}$

$$\therefore V_{\theta,2} = V_x \tan \alpha_2 \quad , \quad V_{\theta,3} = V_x \tan \alpha_{3,rel} + U$$

Turbine:

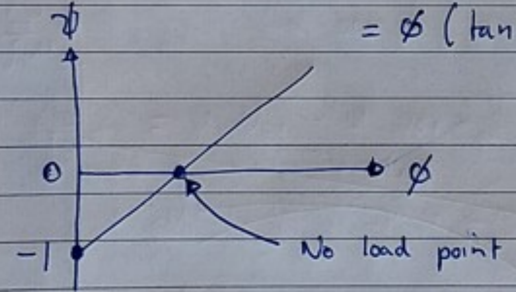


Euler : $\Delta h_0 = U_2 V_{\theta,2} - U_3 V_{\theta 3}$

$$= U(V_x \tan \alpha_2 - V_x \tan \alpha_{3,rel} - U)$$

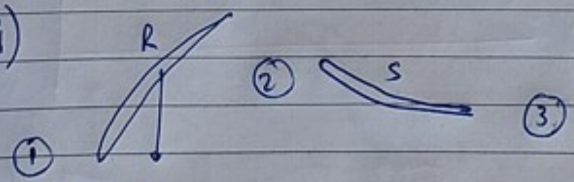
$$\therefore \psi = \frac{\Delta h_0}{U^2} = \frac{V_x}{U}\left(\tan \alpha_2 - \tan \alpha_{3,rel}\right) - 1$$

$$= \phi \left(\tan \alpha_2 - \tan \alpha_{3,rel}\right) - 1$$



Note: +ve and roughly constant in turbine

ii)

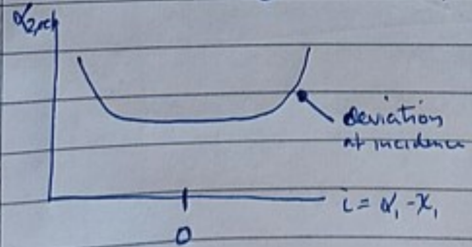


Assume : $\alpha_1 = \text{Const}$

$\alpha_{2,rel} = \text{Const}$

$\alpha_3 = \text{Const}$

Constant angle assumption not as good in compressor



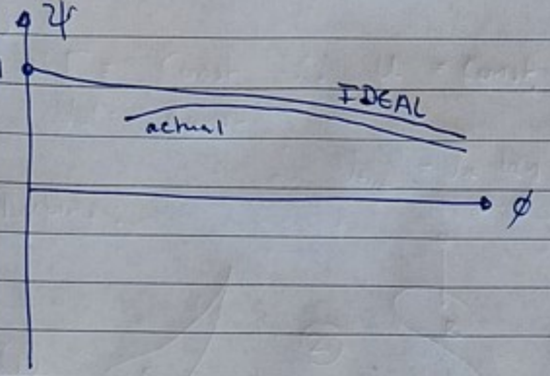
$$\Delta h_0 = U_2 V_{\theta,2} - U_1 V_{\theta 1}$$

$$= U(V_x \tan \alpha_{2,rel} + U - V_x \tan \alpha_1)$$

$$\psi = \frac{\Delta h_0}{U^2} = \phi(\tan \alpha_{2,rel} - \tan \alpha_1) + 1$$

7b) CTD

ii)



Not required but helpful for part iii + iv

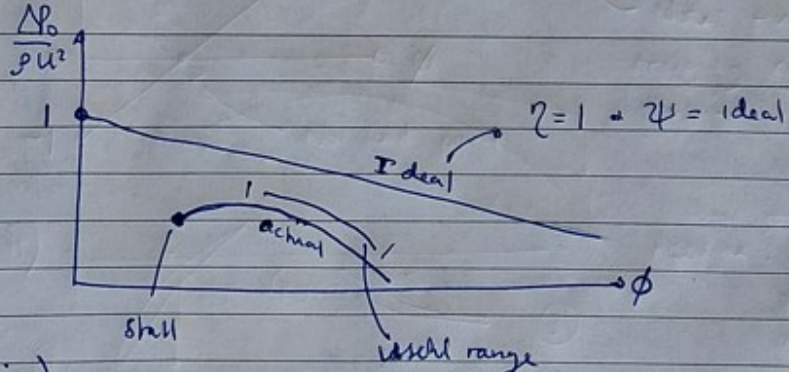
iii) $\frac{\Delta P_0}{\rho U^2}$

Gibbs eqn: $Tds = dh - \frac{dP}{\rho}$

$\therefore \frac{dP_0}{\rho} = dh_{0,s}$ or $\frac{\Delta P_0}{\rho} = \Delta h_{0,s}$

Efficiency: $\eta = \frac{\Delta h_{0,s}}{\Delta h_0}$ (compressor)

$$\therefore \frac{\Delta P_0}{\rho U^2} = \frac{\Delta h_0}{U^2} \times \frac{\Delta h_{0,s}}{\Delta h_0} = \psi \eta$$



iv) See above

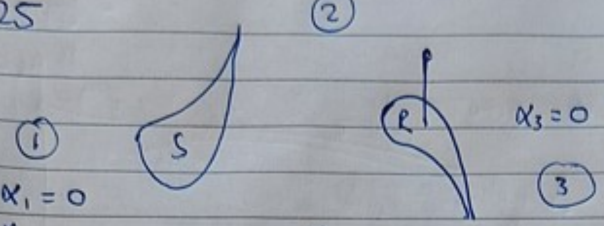
Differences:

- $\eta < 1$ $\therefore$ always below ideal
- at low ϕ loss of turning + increase in loss $\rightarrow$ stall
- at high ϕ increase in loss $\rightarrow$ loss of efficiency

2025

8

a)



$\alpha_1 = 0$

$M_1 = 0.2$ $M_2 = 0.9$ $\alpha_3 = 0$

$T_{0,1} = 1200K$

$P_{0,1} = 10$ bar

$\psi = 1.3$

$r_1 = r_2 = r_3$

$V_{x,1} = V_{x,2} = V_{x,3}$

air : $\gamma = 1.4$, $C_p = 1010$, $R = 287$

a) $\dfrac{V_1}{\sqrt{C_p T_{01}}} = f(M_1) = 0.126 \longrightarrow V_x = 138.7\,m/s$

$= T_{01}$ $\dfrac{V_2}{\sqrt{C_p T_{01}}} = f(M_2) = 0.528 \longrightarrow V_2 = 581.28\,m/s$

$V_2 \cos\alpha_2 = V_x \longrightarrow \underline{\underline{\alpha_2 = 76.2^\circ}}$

b) Euler : $\Delta h_0 = U_2 V_{\theta,2} - U_3 V_{\theta,3}$ Note : $U_2 = U_3 = U$

$\Delta h_0 = U V_{\theta,2}$ $V_{\theta 3} = 0$

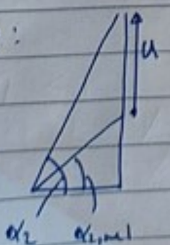
$\psi = \dfrac{\Delta h_0}{U^2} = \dfrac{V_{\theta,2}}{U} = \dfrac{V_x \tan\alpha_2}{U} = \phi \tan\alpha_2$

$\phi = \dfrac{\psi}{\tan\alpha_2} = \underline{0.32}$, $U = \dfrac{V_x}{\phi} = 434.22\,m/s$

$\Delta T_0 = \dfrac{\Delta h_0}{C_p} = \dfrac{\psi U^2}{C_p} = 242.69K$, $T_{0,3} = 957.3^\circ K$

c)

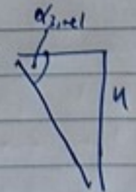
2:



$\tan\alpha_{2,rel} = \tan\alpha_2 - \dfrac{1}{\phi}$

$\alpha_{2,rel} = +43.4^\circ$

3:



$\tan\alpha_{3,rel} = -\dfrac{1}{\phi}$

$\alpha_{3,rel} = -72.3^\circ$

8 CTD

d) $T_3 = T_{0,3} - \frac{V_3^2}{2C_p} = 957.3 - \frac{138.7^2}{2 \times 1010} = 947.8\,°K$

$$V_{3,rel} = \frac{V_x}{\cos\alpha_{3,rel}} = 456.2\ m/s$$

$$M_{3,rel} = \frac{V_{3,rel}}{\sqrt{\gamma R T_3}} = \frac{456.2}{\sqrt{1.4 \times 287 \times 947.8}} = 0.739$$

e) $Y_p = \frac{P_{01} - P_{02}}{P_{02} - P_2} = 0.08$, $\frac{P_2}{P_{02}} = f(M_2) = 0.5913$

$\therefore$

$$Y_p = \frac{P_{01}/P_{02} - 1}{1 - P_2/P_{02}} \longrightarrow \frac{P_{01}}{P_{02}} = 0.08(1 - 0.5913) + 1 = 1.0327$$

$$\underline{P_{02} = 9.683\ bar}$$

f) $P_{0,3} = 4.2\ bar$

$$\eta_{TT,rot} = \frac{T_{02} - T_{03}}{T_{02} - T_{0,3,s}} \ (\text{Turbine}) = \frac{1 - \frac{T_{03}}{T_{02}}}{1 - \frac{T_{03,s}}{T_{02}}}$$

$T_{03}/T_{02} = 0.798$

$\frac{T_{03,s}}{T_{02}} = 0.788 = \left(\frac{P_{0,3}}{P_{0,2}}\right)^{\frac{\gamma-1}{\gamma}}$

$\therefore \eta_{rotor} = 95.3\%$

This is high compared to a stage efficiency as it neglects the losses in the stator.

For example: $\eta_{stage} = \frac{1 - \frac{T_{03}}{T_{01}}}{1 - \left(\frac{P_{03}}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}}} = 92\%$