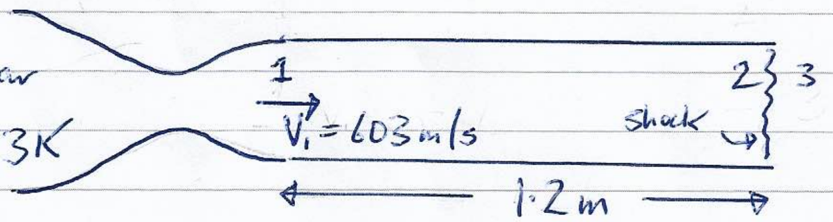
a) $p_0 = 5$ bar

$T_0 = 583K$



(i) $T_1 = T_0 - \dfrac{V_1^2}{2c_p} = 402.1$ K (adiabatic)

$a_1 = \sqrt{\gamma R T_1} = 401.95$ m/s

NOT T_0! $\Rightarrow M_1 = \dfrac{V_1}{a_1} = \underline{\underline{1.500}}$

Alternatively from tables, using $V_1/\sqrt{c_p T_{01}} = 0.7878$

(ii) $T_2 = T_1 + 69K = 471.1K$

Adiabatic $\Rightarrow T_{02} = 583$ K

$T_2/T_{02} = 0.80806$

$M_2 = 1.090$ (Tables, or direct calc.)

Distance to choking at 1 : $\left(\dfrac{4c_f L_{max}}{D}\right)_1 = 0.1361$

— " — 2 : $\left(\dfrac{4c_f L_{max}}{D}\right)_2 = 0.0082$

and $(L_{max})_1 - (L_{max})_2 = 1.2$ m , $D = 0.08$m

$\Longrightarrow c_f = \underline{\underline{2.13(17) \times 10^{-3}}}$

(iii) $\dot{m}$, T_0, A all constant

$\dfrac{\dot{m}\sqrt{c_p T_0}}{A p_{01}} = 1.0891$ $\dfrac{\dot{m}\sqrt{c_p T_0}}{A p_2} = 2.6842 \Rightarrow p_2 = 2.0287$ bar

$p_{01} = p_0$ (nozzle friction loss)

a) (iii) cont'd ...

Shock table : $p_3/p_2 = 1.2195 \Rightarrow p_3 = \underline{\underline{2.47 \text{ bar}}}$

N.B. p_3 same as ambient pressure since flow subsonic.

b) (i) To match higher pressure at exit...

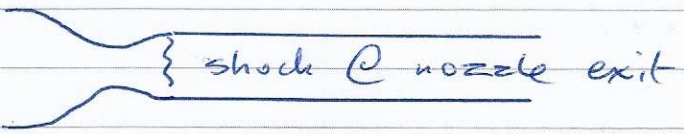
either → – must reduce friction force $\propto \frac{1}{2}\rho V^2$ ①

OK → – must increase pressure rise across shock ②

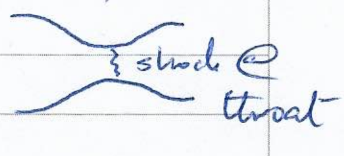
⇒ shock moves upstream – to give more subsonic, low $\frac{1}{2}\rho V^2$, flow ①

– to act on flow with higher Mach no. ②

until we reach



Further rises in back pressure send shock into nozzle, as per standard picture, until ...



after which the nozzle unchokes, and mass flow starts to drop.

(ii) 1% reduction $\Rightarrow \dfrac{\dot{m}\sqrt{c_p T_{01}}}{A p_{01}} = 0.99 \times 1.0891 = 1.0782$

So now $M_1 = 0.600$ (tables); $[4c_f L_{max}/D]_1 = 0.4908$

! Note subsonic sol'n ↑

Use known c_f, D, ΔL_{max} for $[4c_f L_{max}/D]_2 = 0.3629$

Tables : $M_2 = 1.637 \quad \dfrac{\dot{m}\sqrt{c_p T_0}}{A p_2} = 1.4659 \Rightarrow p_2 = 0.7355\, p_{01} = \underline{\underline{3.68 \text{ bar}}}$

a) (i) Differentiate gas law: $dp = \rho R dT + RT d\rho$ (A)

Continuity: $\rho V = \text{const} \Rightarrow \frac{d\rho}{\rho} + \frac{dV}{V} = 0$

Subs. for $d\rho$ in (A): $dp = \rho R dT - \rho RT \frac{dV}{V}$

and $RT = \frac{a^2}{\gamma}$: $\underline{\underline{dp = \rho\left[R dT - \frac{a^2}{\gamma}\frac{dV}{V}\right]}}$

(ii) Frictionless duct $\Rightarrow p + \rho V^2 = \text{const}$

N.B. NOT isentropic. Why?

Differentiate, noting that $\rho V = \text{const}$:

$$dp + \rho V dV = 0$$

Eliminate dp: $\left[\frac{a^2}{\gamma}\frac{1}{V} - V\right] dV = R dT$

At max T, $dT = 0 \Rightarrow V^2 = \frac{a^2}{\gamma}$, $\underline{\underline{M = \frac{1}{\sqrt{\gamma}}}}$

b) (i) Stagnation density $\rho_0 = \frac{p_0}{RT_0} = 2.5489 \text{ kg/m}^3$

$\rho_0 = \rho\left[1 + \frac{\gamma - 1}{2}M^2\right]^{\frac{1}{\gamma-1}} \Rightarrow \underline{\underline{M = 0.400}}$ (or from tabulated ρ/ρ_0)

Tables: $\frac{V}{\sqrt{c_p T_0}} = 0.2490 \Rightarrow \underline{\underline{V = 130.5 \text{ m/s}}}$

b)(ii) Use $F\ (=(p+\rho V^2)A) = \text{const}$

N.B. NOT p_0 = const. Why?

	Inlet	Choked
M	0.4	1
$F/\dot{m}\sqrt{c_pT_0}$	1.3608	0.9897
	$T_0 = T_{0i}$ = 273.4K	$T_0 = T_{0c}$

$\Rightarrow \dfrac{T_{0c}}{T_{0i}} = 1.8905$

$T_{0c} = 516.9\text{K}$

Also need $\dot{m} = \rho A V = 1.537\ \text{kg/s}$ (from (i))

$\dot{Q} = \dot{m} c_p \Delta T_0 = \underline{\underline{376\ \text{kW}}}$ +ve, hence heat <u>added</u> to flow.

(iii) At $M = \gamma^{-1/2} = 0.845$, $F/\dot{m}\sqrt{c_p T_{0m}} = 1$ & T_0 @ max T

Following same route as (ii): $T_{0m} = 506.3\text{K}$

$\dot{Q} = \dot{m} c_p (T_{0m} - T_{0c}) = \underline{\underline{-16\ \text{kW}}}$

−ve, hence heat <u>removed</u> from flow.

(iv) Cooling can alternatively take the flow away from the choked condition to supersonic:

$F/\dot{m}\sqrt{c_pT_0} = 1$ also @ $M = 1.196$

with $T_0 = T_{0m}$ again
and $\dot{m} = 1.537\ \text{kg/s}$ still

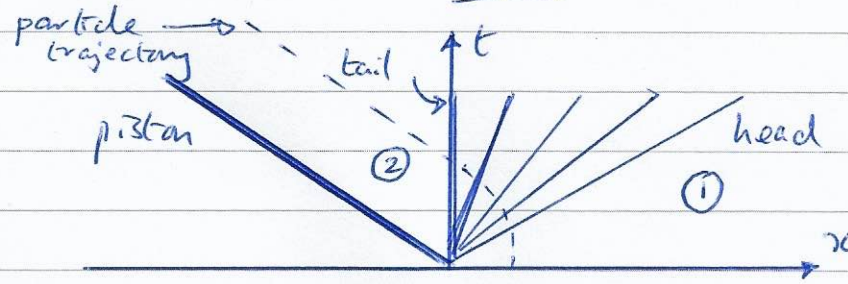
N.B. Two more quantities needed to define gas state.

a) (i) Right-running waves, ~~speed~~ velocity(!) in Laboratory frame is $V+a$, with: V flow velocity
a local sound speed

Head moves into stationary fluid with $V=0$
$a=\sqrt{\gamma RT} = 347.2\,m/s$

So head travels @ $\underline{347.2\,m/s}$ to right

(ii)



$V_1 = 0$
$a_1 = 347.2\,m/s$

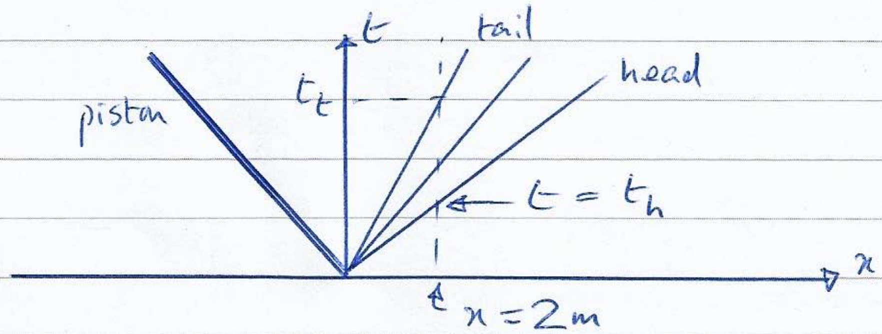
(iii) Across r-r waves: $V_2 - \frac{2}{\gamma-1}a_2 = V_1 - \frac{2}{\gamma-1}a_1$

Tail speed: $V_2 + a_2 = 0$

Solve to obtain $a_2 = 289.3\,m/s$, $V_2 = -289.3\,m/s$

Piston speed is same as flow speed in 2 : $\underline{289\,m/s}$

b) (i)



First variation detected by mike at $t=t_h = 2/347.2 = 5.76\,ms$
Duration of 30ms $\Rightarrow t_t = 35.76\,ms$

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Question number 3 Sheet number 2

b)(i) cont'd

Hence tail speed $= 210.03576 = 55.9\ m/s$

Across r-r waves $V_2 - \frac{2}{\gamma-1} a_2 = V_1 - \frac{2}{\gamma-1} a_1 = -\frac{2}{\gamma-1} a_1$

Tail velocity $V_2 + a_2 = 55.9$

Solve to obtain $a_2 = 298.65\ m/s$, $V_2 = -242.75\ m/s$

Piston speed $\underline{\underline{243\ m/s}}$

Expansion wave system is isentropic,

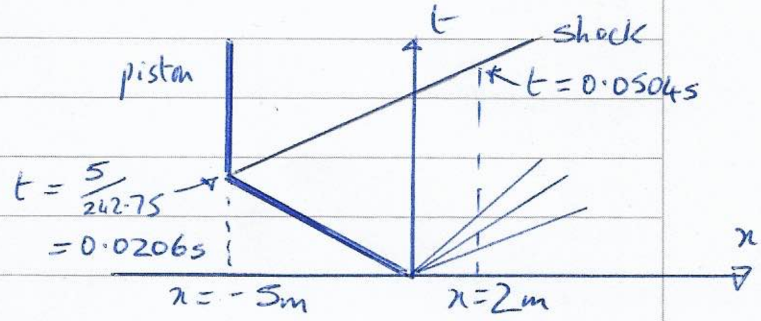
so... $\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} = \left(\frac{a_2}{a_1}\right)^{2\gamma/\gamma-1}$

BUT can't say p_0 = const! Why?

giving $p_2 = \underline{\underline{0.348}}\,(40)$ bar

(ii) When the piston stops, a shock-wave is generated. The change in pressure detected by the mike marks the shock arrival.

New wave diagram:



Shock travels 7m in 0.0298 s $\Rightarrow$ speed is 234.90 m/s

Lab frame: $V = 0$ (3) | $\rightarrow 234.9\ m/s$ | $\leftarrow 242.75\ m/s$ (2) 0.3484 bar

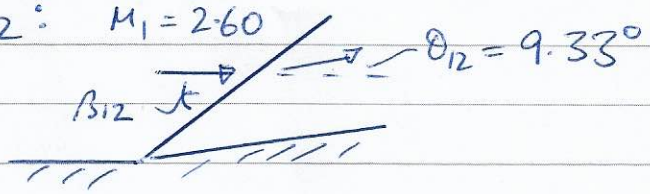
In shock frame, $M = 1.60$

Shock frame: $\xleftarrow{234.9\ m/s}$ | $\xleftarrow{477.65\ m/s}$ $a_2 = 298.65\ m/s$

Tables: $p_3/p_2 = 2.820$

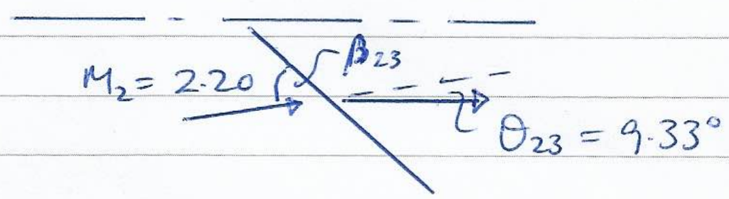
$p_3 = \underline{\underline{0.982\ \text{bar}}}$

a) (i) Across S_{12}:



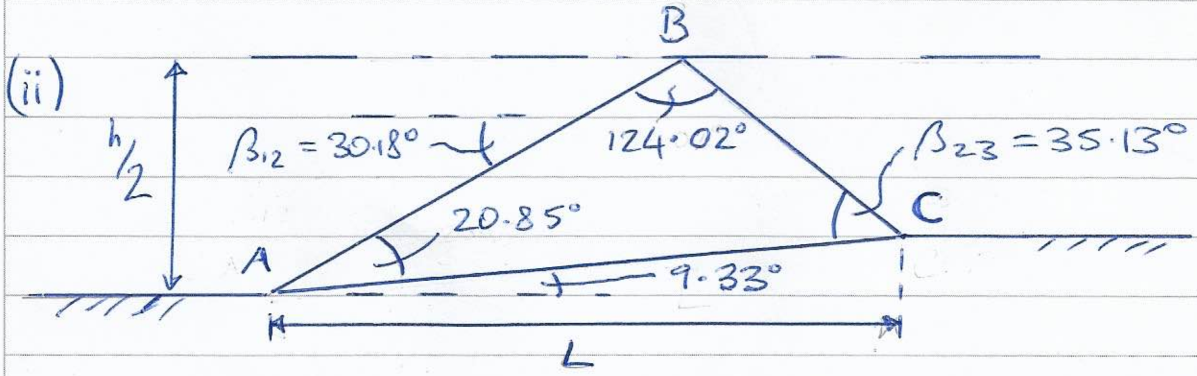
Tables: $\underline{\underline{M_2 = 2.20}}$ $(\beta_{12} = 30.18°)$

Across S_{23}:



Tables: $\underline{\underline{M_3 = 1.85}}$ $(\beta_{23} = 35.13°)$

(ii)



$$AB = h/2 \left(\sin 30.18°\right)^{-1} = 0.995h$$

Sine rule: $$AC = AB \frac{\sin 124.02}{\sin 35.13} = 1.433h$$

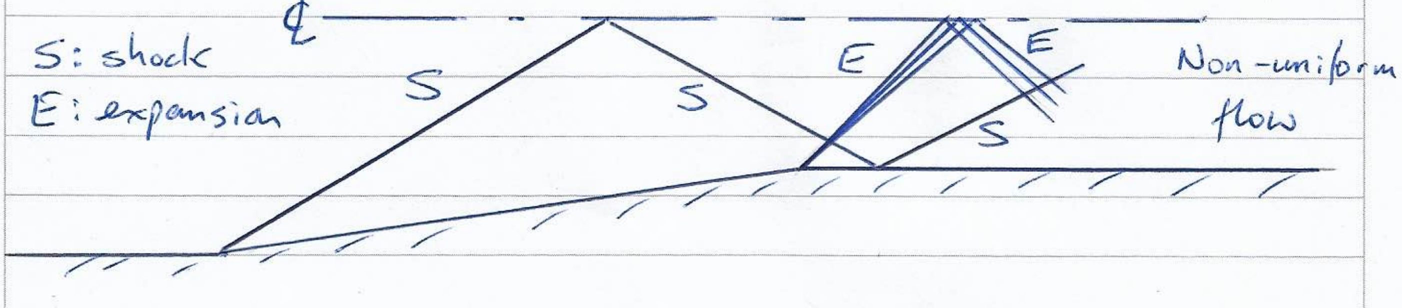
$$L = AC \cos 9.33 = \underline{\underline{1.41\,h}}$$

Measurement from a reasonably careful scale drawing also acceptable.

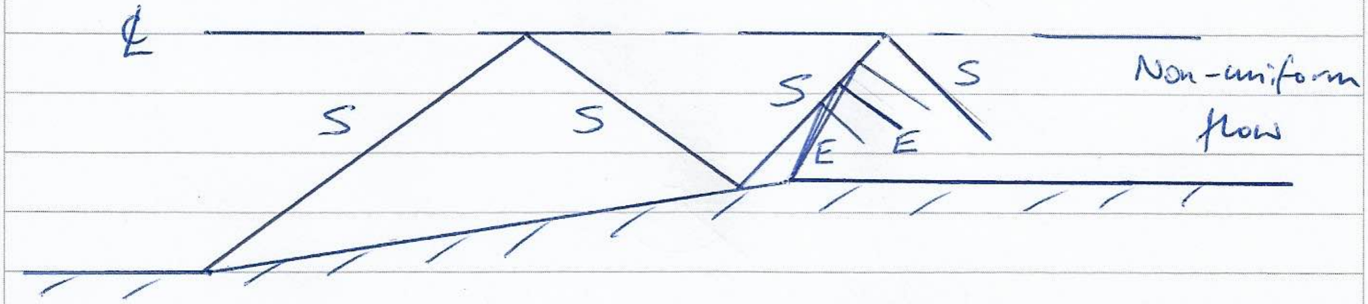
(iii) Tables: $p_2/p_1 = 1.8272$

b) (i) Increased $M \rightarrow$ decreased β_{12}

S: shock
E: expansion

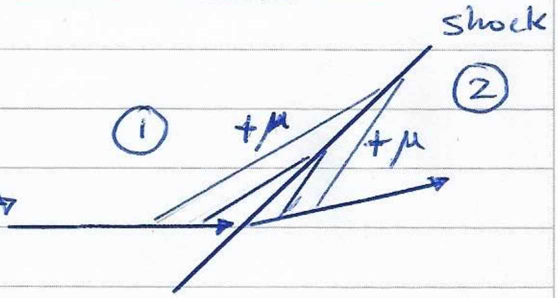


(ii) Decreased $M \rightarrow$ increased β_{12}



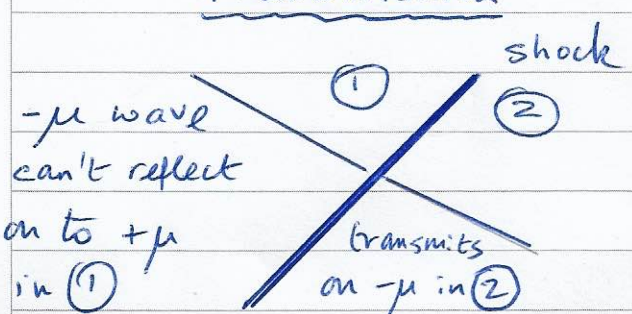
A NOTE ON INTERACTIONS BETWEEN SHOCKS AND EXPANSIONS

Oblique shock angles are such that similarly inclined characteristics intersect the shock on both sides →

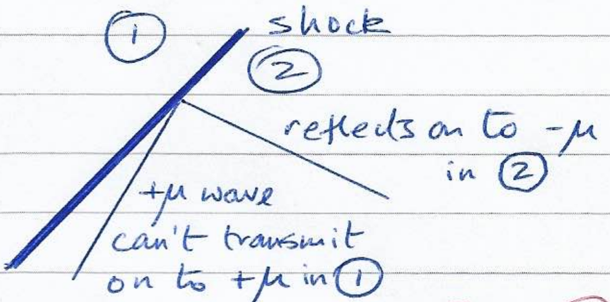


This observation determines whether a wave is transmitted or reflected:

Transmission



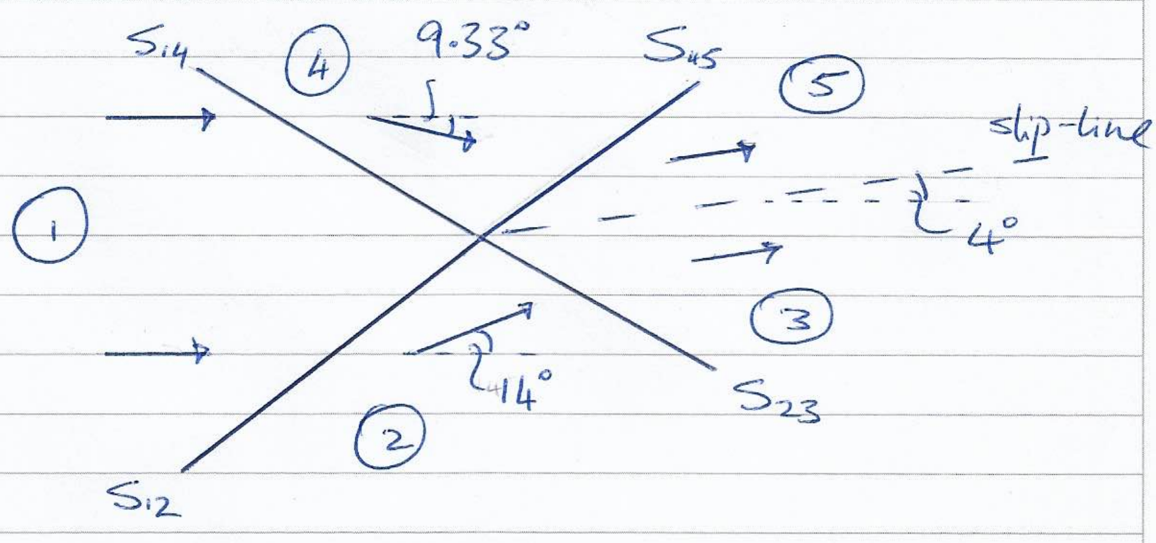
Reflection



c) The flow is now asymmetric, with a slip-line emanating from the shock intersection.

Across a slip-line ... velocities parallel, speeds differ
pressures <u>equal</u>

Assume the researcher is correct:



S_{12}: $M_2 = 2.00$ $p_2/p_1 = 2.3955$

S_{23}: $\theta_{23} = 10°$ $p_3/p_2 = 1.7066$ $\boxed{p_3 = 4.088\, p_1}$

S_{14} unchanged: $M_4 = 2.20$ $p_4/p_1 = 1.8272$

S_{45}: $\theta_{45} = 13.33°$ $p_5/p_4 = 2.1037$ $\boxed{p_5 = 3.844\, p_1}$

p_3 is too high and p_5 too low, so in reality S_{23} is weaker than assumed, and S_{45} stronger

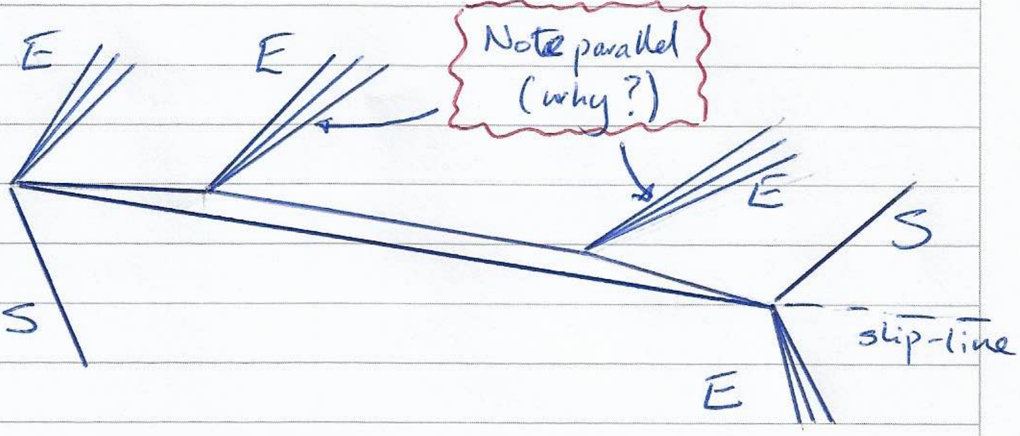
$\Rightarrow$ less flow turning through S_{23} and more through S_{45}

$\Rightarrow$ researcher's estimate is too <u>low</u>, angle $> 4°$.

Question number 5 Sheet number 1

a)

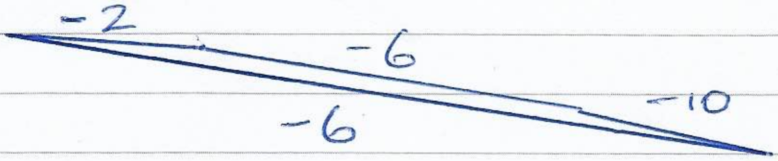
S: shock
E: expansion



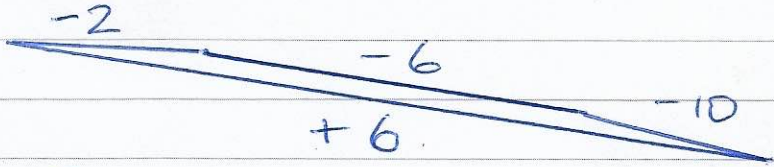
b) Linearised method of ch'ics gives $\dfrac{p-p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \pm \dfrac{2\theta}{\sqrt{M_\infty^2-1}}$

Notes: + across $+\mu$ (upper surface)
− across $-\mu$ (lower surface)
θ in radians

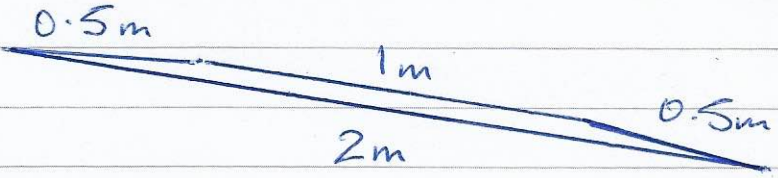
$\theta = \dfrac{\pi}{180} \times$:



$\dfrac{p-p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \dfrac{\pi}{180} \times \dfrac{2}{\sqrt{M_\infty^2-1}} \times$:



Surface lengths :
($\cos 4^\circ \simeq 1$)



(i) Add resolved force contributions for drag:

$$D = \underbrace{\frac{1}{2}\rho_\infty U_\infty^2 . \frac{\pi}{180} \frac{2}{\sqrt{M_\infty^2-1}}}_{0.01781} \underbrace{\Big[0.5.(-2)(-\sin 2^\circ) + 1(-6)(-\sin 6^\circ) + 0.5(-10)(-\sin 10^\circ) + 2(+6)\sin 6^\circ \Big]}_{2.785}$$

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 . 2} = \underline{\underline{0.0248}}$$

2 chord

N.B. $\sin\theta \simeq \theta$ also OK

b)(ii) Work with forces $\perp$ chord, again with $\cos 4^\circ \simeq 1$
(though here any error cancels with same approx for lengths!)

$$\uparrow F = \frac{1}{2}\rho_\infty U_\infty^2 \times \frac{\pi}{180} \times \frac{2}{\sqrt{M_\infty^2 - 1}} \times :$$

$$M = \frac{1}{2}\rho_\infty U_\infty^2 \times \frac{\pi}{180} \times \frac{2}{\sqrt{M_\infty^2-1}} \times \left[0.25.1 + 1.6 + \underset{\text{moment arm}}{1.75}.\underset{\text{force}}{5} + 1.12 \right]$$

$$C_M = \frac{M}{\frac{1}{2}\rho_\infty U_\infty^2 c^2} = \underline{\underline{0.120}}$$

c) Linearised theory admits superposition,
i.e. we can calculate ΔC_M from change in geometry:

$$\Delta\theta = \frac{\pi}{180} \times :$$

$$\Delta\left(\frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}\right) = \frac{\pi}{180} \times \frac{2}{\sqrt{M_\infty^2-1}} \times :$$

$$\Delta M = \frac{1}{2}\rho_\infty U_\infty^2 \times \frac{\pi}{180} \times \frac{2}{\sqrt{M_\infty^2-1}} \times 1.75\,(8+8).0.5$$

$$\Rightarrow \Delta C_M = \underline{\underline{0.062}}$$

d) Implications:
- significant change in pitching moment affects balance, and hence tail design;
- significant increase in drag expected from additional pressure forces, hence need more thrust to avoid performance degradation.

a) Check initial condition: $u(x,0) = \cos[k(x-0)] = \cos kx$ ✓

Check equation: $\frac{\partial u}{\partial t} = ka \sin[k(x-a)]$

$$\frac{\partial u}{\partial x} = -k \sin[k(x-a)]$$

So $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, and the equation is satisfied.

b)(i) Substitute the claimed solution into truncated equivalent PDE:

$$\underbrace{\frac{dA}{dt}\cos[k(x-at)] + kaA\sin[k(x-at)]}_{\partial u/\partial t} + a\underbrace{\left\{-kA\sin[k(x-at)]\right\}}_{\partial u/\partial x} = \underbrace{-\nu_{eff}k^2A\cos[\ldots]}_{-\partial^2 u/\partial x^2}$$

sin[ ] terms cancel, and remaining equation is required to be true for all x,t...

$$\frac{dA}{dt} = -\nu_{eff}k^2 A \Rightarrow A = A_0 e^{-\nu_{eff}k^2 t}$$

Initial condition $A(0) = 1 \Rightarrow \underline{\underline{A = e^{-\nu_{eff}k^2 t}}}$

(ii) For the scheme to be stable, ν_{eff} must be +ve (otherwise we see exponential growth). Hence we require $\alpha \leq 1$ or, equivalently, $a\Delta t \leq \Delta x$.

When stable with $\nu_{eff} > 0$, A decays from its initial value of 1, when it should stay constant. This inaccuracy is mitigated by keeping α near 1 (if possible) and/or minimising Δx.

c) (i) First derive the new equivalent PDE:

$$u_i^{n+1} - u_i^{n-1} = -\frac{a\Delta t}{\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) \Rightarrow \frac{u_i^{n+1}-u_i^{n-1}}{\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{\Delta x}$$

Now $u_i^{n+1} = u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^3}{6} + \frac{\partial^4 u}{\partial t^4}\frac{\Delta t^4}{24} + O(\Delta t^5)$

and $u_i^{n-1} = u_i^n - \cdots + \cdots - \cdots + \cdots + O(\Delta t^5)$

So $\frac{u_i^{n+1}-u_i^{n-1}}{\Delta t} = 2\left(\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^2}{6} + O(\Delta t^4)\right)$

Applying the same approach to the term $(u_{i+1}^n - u_{i-1}^n)/\Delta x$:

$$2\left(\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^2}{6} + O(\Delta t^4)\right) = -2a\left(\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^2}{6} + O(\Delta x^4)\right)$$

leading to the following truncated equivalent PDE:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} - \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}$$

For the claimed solution....

$$\frac{\partial u}{\partial t} = kc\sin\left[k(x-ct)\right] \qquad \frac{\partial^3 u}{\partial t^3} = -(kc)^3\sin\left[k(x-ct)\right]$$

$$\frac{\partial u}{\partial x} = -k\sin\left[k(x-ct)\right] \qquad \frac{\partial^3 u}{\partial x^3} = k^3\sin\left[k(x-ct)\right]$$

Substitute into equivalent PDE ...

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Question number 6 Sheet number 3

$$k(c-a) = (kc)^3 \frac{\Delta t^2}{6} - a k^3 \frac{\Delta x^2}{6}$$

This confirms $c \simeq a$, so $c^3 = a^3 +$ higher order

and $k(c-a) = \frac{k^3}{6}\left[a^3 \Delta t^2 - a \Delta x^2\right]$ to leading order

$$c - a = \frac{k^2 a (\alpha^2 - 1)\Delta x^2}{6}$$

Alternatively, note that $\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$ + higher order

so $\frac{\partial^3 u}{\partial t^3} = -a^3 \frac{\partial^3 u}{\partial x^3}$ + higher order,

and substitute for $\partial^3 u / \partial t^3$ before inserting claimed sol'n.

(ii) The solution to the equivalent problem travels at speed c rather than a $\Rightarrow$ false convection.

Furthermore, the factor k^2 in $c-a$ shows that the speed error depends on wavelength $\Rightarrow$ dispersion.

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Question number 7 Sheet number 1

a) (i) $\frac{dp_0}{dr} = 0$, $\frac{dV_x}{dr} = \frac{1}{\sqrt{2}}\frac{dV}{dr}$, $\frac{dV_\theta}{dr} = \frac{1}{\sqrt{2}}\frac{dV}{dr}$

so $$0 = \frac{1}{2} V\frac{dV}{dr} + \frac{V}{2r}\left(V + r\frac{dV}{dr}\right)$$

$$V\frac{dV}{dr} + \frac{V^2}{2r} = 0$$

$$\frac{dV}{V} = -\frac{1}{2}\frac{dr}{r}$$

Integrate $\ln V = \text{const} - \frac{1}{2}\ln r$, or $V = c/r^{1/2}$

$V = V_H$ @ $r = r_H$ $\Rightarrow$ $\underline{\underline{V = V_H\left(\frac{r_H}{r}\right)^{1/2}}}$

(ii) $$\dot{m} = \int_{r=r_H}^{r=r_T} \rho V_x \, dA \quad \text{with } dA = 2\pi r dr$$

N.B Axial component

$$= \frac{2\pi\rho}{\sqrt{2}} V_H r_H^{1/2} \int_{r_H}^{r_T} r^{1/2} dr$$

$$\underline{\underline{= \frac{2\sqrt{2}\pi}{3}\rho V_H r_H^{1/2}\left[r_T^{3/2} - r_H^{3/2}\right]}}$$

b) $$\delta u = \frac{\partial u}{\partial t}\delta t + \frac{\partial u}{\partial x}\delta x$$

Along $x = X(t)$, $\delta x = \dot{X}(t)\,\delta t \Rightarrow \delta u = \underline{\underline{\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\dot{X}\right)\delta t}}$

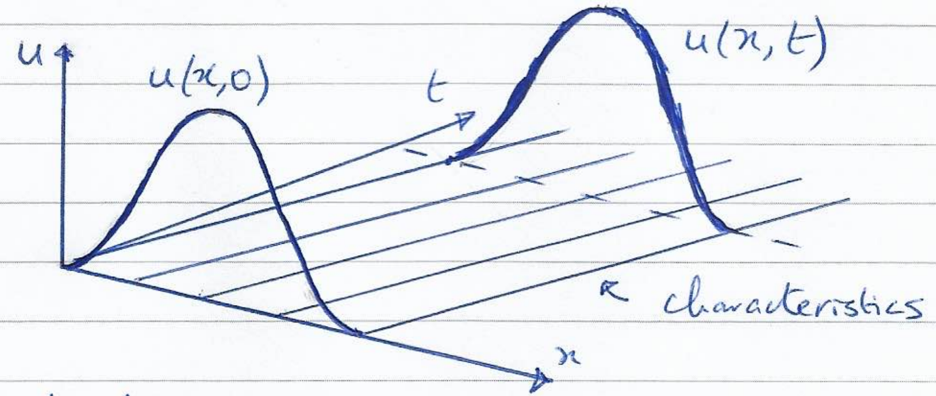
c) Along a characteristic, $\delta u = 0$

i.e., from (b), $\frac{\partial u}{\partial t} + \dot{X}\frac{\partial u}{\partial x} = 0$, where $x = X(t)$ is the equation of the characteristic

$\therefore$ By comparison with the general form of the convection eqn,

$$\dot{X} = f(u)$$

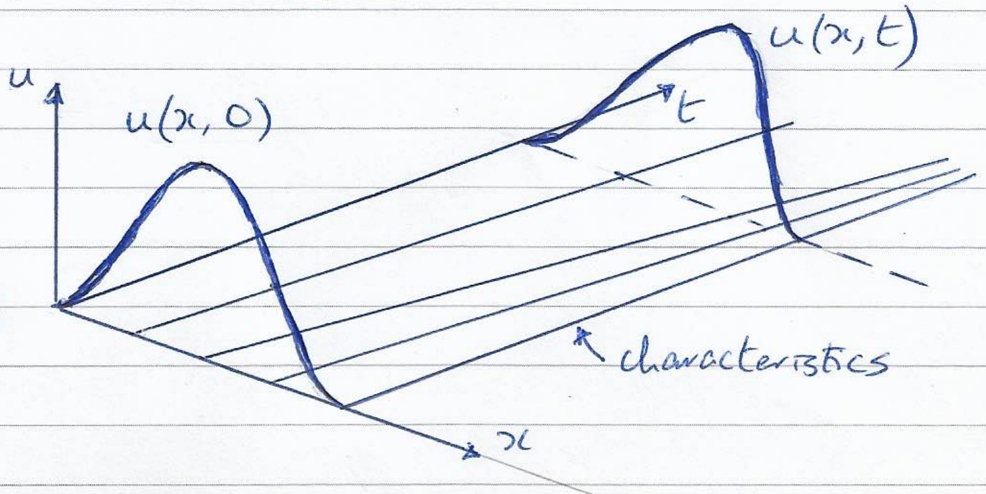
(i) $f(u) = a \Rightarrow X = x_0 + at$



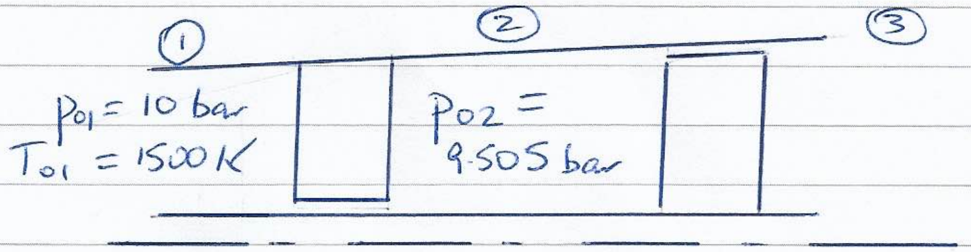
Convects downstream at speed a; no change in shape.

(ii) $f(u) = u$, and by definition $u = \text{const}$ on a ch'ic

$\Rightarrow X = x_0 + u(x_0,0)t$



Distorts, with higher points moving faster.



$\dot{W}_x = 7.24$ MW

a)(i) Conditions at ②: $T_{02} = T_{01}$ (adiabatic, no work)

$$0.1(p_{02} - p_2) = p_{01} - p_{02} \implies p_2 = 4.555$$

$$p_2 / p_{02} = 0.4792 \implies M_2 = 1.1007$$

so $T_2 = T_{02}\left(1 + \frac{\gamma-1}{2} M_2^2\right)^{-1} = 1248.2\,K$; $a_2 = \sqrt{\gamma R T_2} = 691.0$ m/s

$\boxed{\text{Don't forget } \gamma = 1.333 \text{ here!}}$

Thus $V_2 = a_2 M_2 = 760.6$ m/s

$$\rho_2 = p_2 / R T_2 = 1.2715 \text{ kg/m}^3$$

$$\dot{m} = \rho_2 A_2 V_2 \cos\alpha_2 = \underline{\underline{37.4(1)}} \text{ kg/s}$$

$\pi(r_T^2 - r_H^2) = 0.11310$

(ii) Quantities from calc: stagnation-pressure loss, flow angle

p_0 loss: depends on non-ideal effects, so difficult to calculate accurately; $0.1 \to 0.11$ gives $p_2 = 5.005$ bar
$0.1 \to 0.09$ gives $p_2 = 4.005$ bar

α_2: calculation likely less twitchy, but even 1° error changes $\cos\alpha_2$ by 5% (via direct calc)

$\implies$ errors in region of 10% could easily arise from either of the quantities estimated numerically.

b) (i) $\eta_{is} = \frac{\text{actual work}}{\text{ideal work}} = \frac{h_{01} - h_{03}}{h_{01} - h_{03i}}$

$$h_{01} - h_{03} = \dot{W}_x / \dot{m} = 193.5 \times 10^3 \text{ J/kg}$$

$$h_{03i} = c_p T_{03i} = c_p T_{01} \left(\frac{p_{03}}{p_{01}}\right)^{\frac{\gamma-1}{\gamma}} = c_p . 1314.9$$

so $h_{01} - h_{03i} = 212.7 \times 10^3$ J/kg, and $\eta_{is} = \underline{\underline{0.910}}$

(ii) Not possible consider detail control-volume analysis (1st or 2nd law) without additional information. However, we <u>do</u> know that the cooling air mixes with the main flow, and that this will involve irreversible heat and momentum transfer. These phenomena will tend to <u>decrease</u> the efficiency.

c) Rotational speed measurement would yield blade speed, and hence allow us to apply the Euler work eqn:

$$w_x = U \, \Delta V_\theta \qquad \text{over rotor.}$$

In conjunction with continuity ($\dot{m} = \rho_3 A_3 V_{x3}$), this allows conditions at rotor exit to be determined. (In practice, some iteration required.)

Other viable answers, given suitable justification:

flow coefficient; stage-loading coefficient; α_{2r}; blade forces