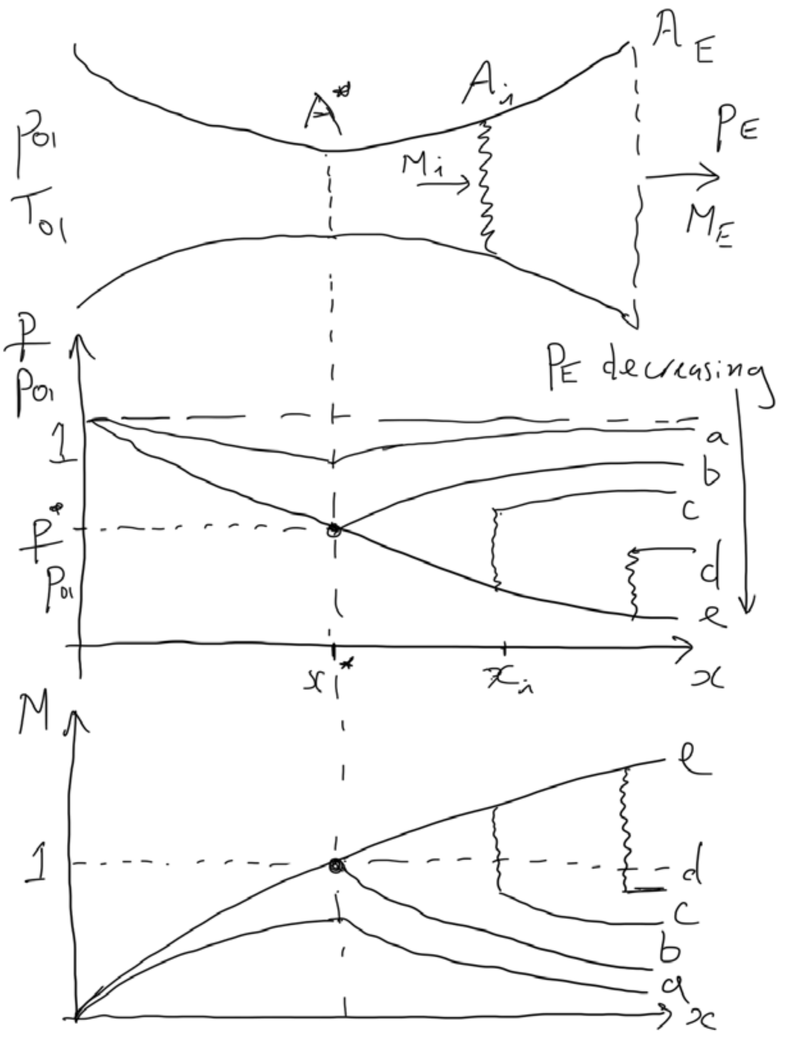
Q1

a)



$p_E = p_a$ – unchoked, $M < 1$ throughout

$p_E \leq p_b$ – choked, $\dot{m} = \dot{m}_{max}$

$p_E = p_c$ – shock in divergence

$p_E = p_d$ – shock at exit

$p_E = p_e$ – supersonic throughout (perfectly expanded)

[30%]

b) i) $\dfrac{\dot{m}\sqrt{c_p T_{01}}}{A^* p_{01}} = 1.281$ – choked.

$$\therefore \dot{m} = 1.281 \times \frac{0.025 \times 200 \times 10^3}{\sqrt{1005 \cdot 283}}$$

$$= \underline{12.0 \text{ kg/s}}$$

$$\frac{\dot{m}\sqrt{c_pT_{01}}}{A_E\,p_E} = \frac{12\times\sqrt{1005\times 283}}{0.04\times 135\times 10^3}$$

$$= 1.1851$$

Tables $\Rightarrow$ $\underline{M_E = 0.5218}$

[20%]

ii) $p_{0E} = p_E \times \left(\frac{p_{0E}}{p_E}\right)$

$$= 135\times 1.2039$$

$$= 162.5\,\text{kPa}$$

$p_{0E} < p_{01}$ $\therefore$ There must be a shock in divergence (case c)

[10%]

iii) $\frac{p_{0E}}{p_{01}} = 0.8126$

$\Rightarrow$ Shock upstream Mach number

$$\underline{M_i = 1.80}$$

$$\frac{\dot{m}\sqrt{c_pT_{01}}}{A_i\,p_{01}} = \frac{\dot{m}\sqrt{c_pT_{01}}}{A^*\,p_{01}} \times \frac{A^*}{A_i}$$

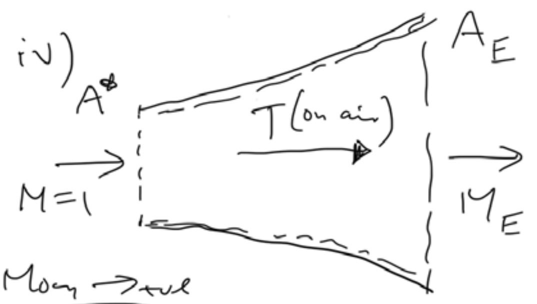
$\therefore$ $A_i = 1.281\times A^* \Big/ \left(\frac{\dot{m}\sqrt{c_pT_{01}}}{A_i\,p_{01}}\right)$

For $M_i = 1.80$, $\frac{\dot{m}\sqrt{c_pT_{01}}}{A_i p_{01}} = 0.8902$

$\Rightarrow$ $A_i = \frac{1.281}{0.8902}\times 0.025 = \underline{0.036\,m^2}$

[20%]

iv)



Mom → +ve

$$T + p^* A^* - p_E A_E = \rho_E A_E V_E^2 - \rho^* A^* V^{*2}$$

$$T = F_E - F^*$$

$$\therefore\ T = \left(\frac{F_E}{\dot m \sqrt{c_p T_0}} - \frac{F^*}{\dot m \sqrt{c_p T_0}} \right) \dot m \sqrt{c_p T_0}$$

$$= (1.1645 - 0.9897) \times 12 \times \sqrt{1005 \times 283}$$

(TABLES)

$$= \underline{1.119\ \text{kN}}$$

Force on nozzle is $\underline{1.12\ \text{kN}}$

+ve ←

[20%]

Q2

a) $$\delta h_0 = h\left(1-M^2\right)\frac{\delta V}{V}$$

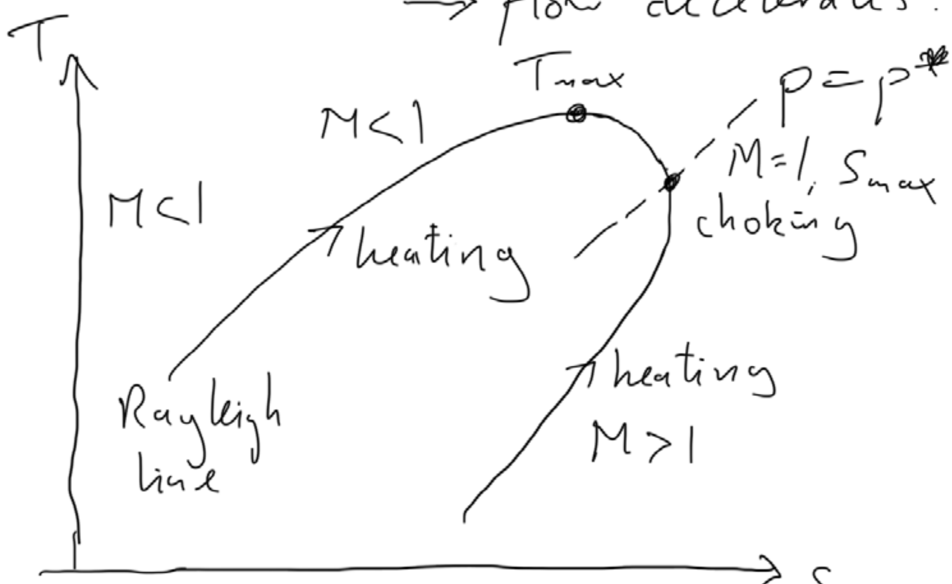
For a subsonic flow, $(1-M^2) > 0$

Heating $\Rightarrow \delta h_0 > 0 \therefore \delta V > 0$ flow accelerates

For a supersonic flow $(1-M^2) < 0$

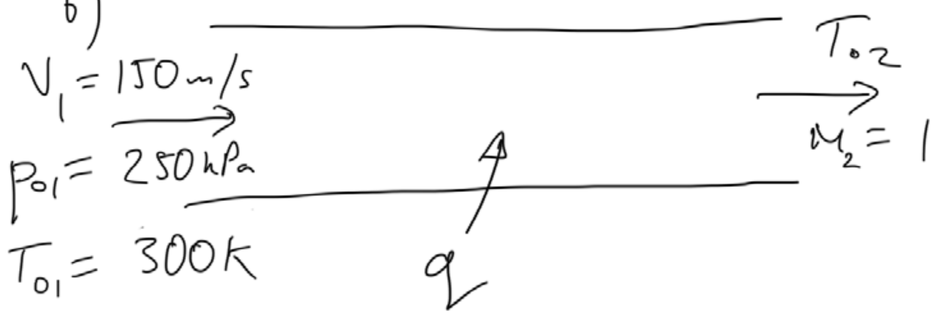
With heating, $\delta h_0 > 0 \therefore \delta V < 0$

$\rightarrow$ flow decelerates.



[30%]

b)



(i) $$\frac{V_1}{\sqrt{c_p T_{01}}} = \frac{150}{\sqrt{1005 \times 300}} = 0.2732$$

$$\therefore \underline{M_1 = 0.44}$$

[10%]

(ii) $q = c_p (T_{02} - T_{01})$

$$\frac{F_1}{\dot{m}\sqrt{c_p T_{01}}} = \frac{F_2}{\dot{m}\sqrt{c_p T_{02}}} \times \sqrt{\frac{T_{02}}{T_{01}}}$$

$$\therefore \frac{T_{02}}{T_{01}} = \left(\frac{F_1 / \dot{m}\sqrt{c_p T_{01}}}{F_2 / \dot{m}\sqrt{c_p T_{02}}} \right)^2$$

$$= \left(\frac{1.2804}{0.9897} \right)^2 = 1.674$$

$$q = c_p T_{01} \left(\frac{T_{02}}{T_{01}} - 1 \right) = 1005 \times 300 \times 0.674$$

$$= \underline{203 \text{ kJ/kg}}$$

[20%]

(iii) $$\frac{F_1}{A p_{01}} = \frac{F_1}{\dot{m}\sqrt{c_p T_{01}}} \times \frac{\dot{m}\sqrt{c_p T_{01}}}{A p_{01}}$$

$$= 1.2804 \times 0.8691$$

$$= 1.1128$$

$$\frac{F_2}{A p_{02}} = \frac{F_2}{\dot{m}\sqrt{c_p T_{02}}} \times \frac{\dot{m}\sqrt{c_p T_{02}}}{A p_{02}}$$

$$= 0.9897 \times 1.281$$

$$= 1.2678$$

$$p_{02} = p_{01} \times \frac{(F_1 / A p_{01})}{(F_2 / A p_{02})} = 250 \times \frac{1.1128}{1.2678}$$

$$= 219.4 \text{ kPa}$$

$$V_2 = M_2 \sqrt{\gamma R T_2}$$

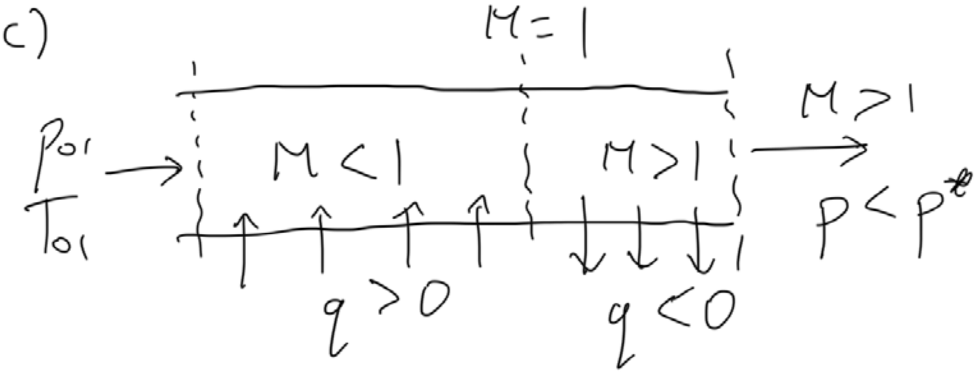
$$T_2 = \frac{T_{02}}{\left(1 + \frac{\gamma - 1}{2} M_2^2\right)} = \frac{1.674 \times 300}{1 + 0.2.1}$$

$$= 418.5 \text{ K}$$

$$V_2 = \sqrt{1.4 \times 287 \times 418.5} = 410 \text{ m/s}$$

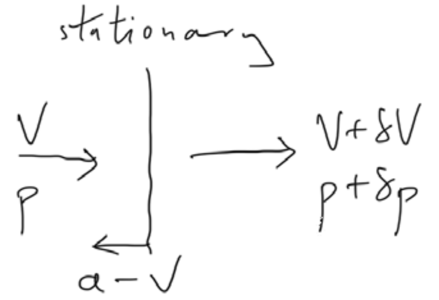
[30%]

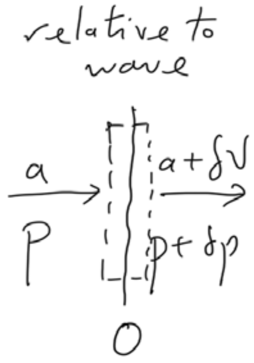
c)



To accelerate to a supersonic Mach number (M>1) heating can be applied until choking (part (b)) then the flow needs to be cooled so that $\delta V > 0$ with $M > 1$ (part (a)). In addition, exit static pressure must be low enough to support supersonic exit flow. [10%]

Q3 a)





Momentum →+ve $\quad -\delta p = \rho a \,\delta V$

Isentropic: $p \propto T^{\frac{\gamma}{\gamma-1}} \propto a^{\frac{2\gamma}{\gamma-1}} \Rightarrow \frac{\delta p}{p} = \frac{2\gamma}{\gamma-1}\frac{\delta a}{a}$

Speed of sound: $a^2 = \frac{\gamma p}{\rho}$

$\therefore \delta p = \frac{2\gamma}{\gamma-1} \cdot \frac{a^2 \rho}{\gamma} \cdot \frac{\delta a}{a} = \frac{2}{\gamma-1} \rho a \,\delta a$

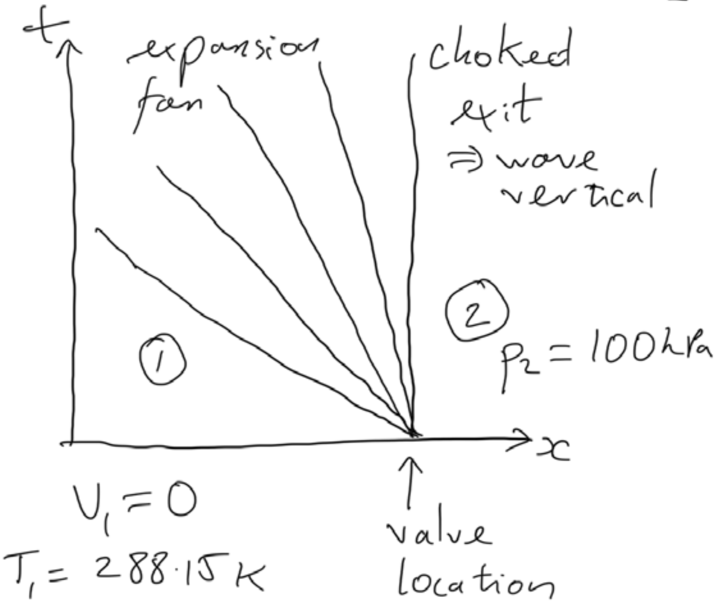
$= -\rho a \,\delta V$

$\therefore \delta V + \frac{2}{\gamma-1}\delta a = 0$

$\delta\left(V + \frac{2a}{\gamma-1}\right) = 0 \rightarrow \underline{V + \frac{2a}{\gamma-1} = \text{const.}}$

[30%]

b) (i)



$V_1 = 0$

$T_1 = 288.15\,\text{K}$

$p_1 = 450\,\text{kPa}$

[20%]

ii)
$$\frac{p_1}{p_2} = 4.5, \quad \frac{T_1}{T_2} = (4.5)^{\frac{\gamma-1}{\gamma}} = 1.537$$

$$\therefore T_2 = \frac{288.15}{1.537} = 187.5\,K$$

$$a_1 = \sqrt{1.4 \times 287 \times 288.15} = 340.3\,m/s$$

$$a_2 = \sqrt{1.4 \times 287 \times 187.5} = 274.5\,m/s$$

Riemann invariant:

$$\cancel{V_1}^{0} + \frac{2a_1}{\gamma-1} = V_2 + \frac{2a_2}{\gamma-1}$$

$$\therefore V_2 = \frac{2}{\gamma-1}(a_1 - a_2)$$

$$= \frac{2}{0.4}(340.3 - 274.5)$$

$$= 329\,m/s$$

$V_2 > a_2 \;\therefore$ Exit is choked [20%]

iii)
With choked exit, $V_2 = a^*$

$$\frac{2a_1}{\gamma-1} = a^* + \frac{2a^*}{\gamma-1} = a^* \frac{(\gamma+1)}{(\gamma-1)}$$

$$a^* = \frac{2a_1}{\gamma+1} = \frac{2}{2.4}.340.3$$

$$V^* = 283.6\,m/s$$

$$\dot{m} = \rho^* A V^*$$

$$\frac{\rho^*}{\rho_1} = \left(\frac{T^*}{T_1}\right)^{\frac{1}{\gamma-1}} = \left(\frac{a^*}{a_1}\right)^{\frac{2}{\gamma-1}}$$

$$= \left(\frac{283.6}{340.3}\right)^{\frac{2}{\gamma-1}} = 0.402$$

$$\therefore \rho^* = 0.402 \times \rho_1$$

$$= 0.402 \times \frac{450 \times 10^3}{287 \times 288.15} = 2.187\,kg/m^3$$

$$A = \frac{\pi d^2}{4} = \frac{\dot{m}}{\rho^* V^*}$$

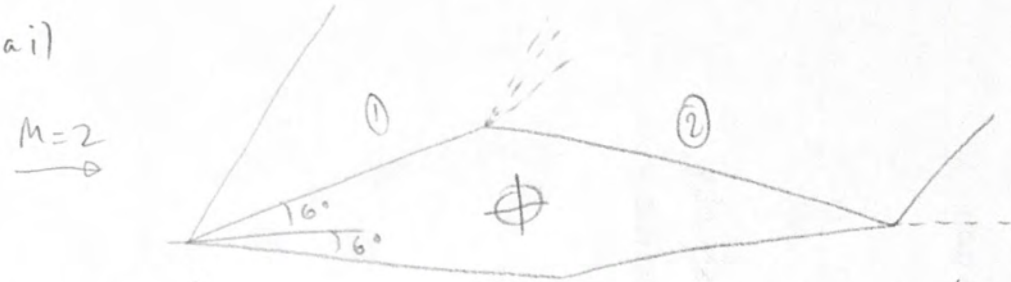
$$\therefore d = \sqrt{\frac{4\dot{m}}{\pi \rho^* V^*}} = \sqrt{\frac{4 \times 100}{\pi \times 2.187 \times 283.6}}$$

$$= \underline{0.453\ m}$$

[30%]

Q4

a i)



Shock/expansion system
symmetrical about centreline.

$M_1 = 1.786$ $\quad \frac{p_1}{p_\infty} = 1.387$ $\quad \frac{p_{01}}{p_{0\infty}} = 0.996$ $\quad \nu_1 = 20.32 \Rightarrow \nu_2 = 32.32 \; (\Rightarrow M_2 = 2.223)$ $\quad \frac{p_\infty}{p_{0\infty}} = 0.128$

$\frac{p_2}{p_{02}} = 0.0888$ $\quad p_{02} = p_{01}$

max $\nu = 490$.

$M = 2 \Rightarrow \nu = 26.38$.

ii)

	p	α	sin α	R (→)
①	4120	6°	0.105	64.9
②	2050	6°	0.105	32.3

$D = 97.2\,N/m$

$\Rightarrow C_{D_p} = \frac{D}{\frac{1}{2}\rho V^2 A} = \frac{97.2}{8270 \times 0.298} = 0.0394.$

$h = 24\,km$ $\quad \frac{p_\infty}{p_{SL}} = 0.0293$

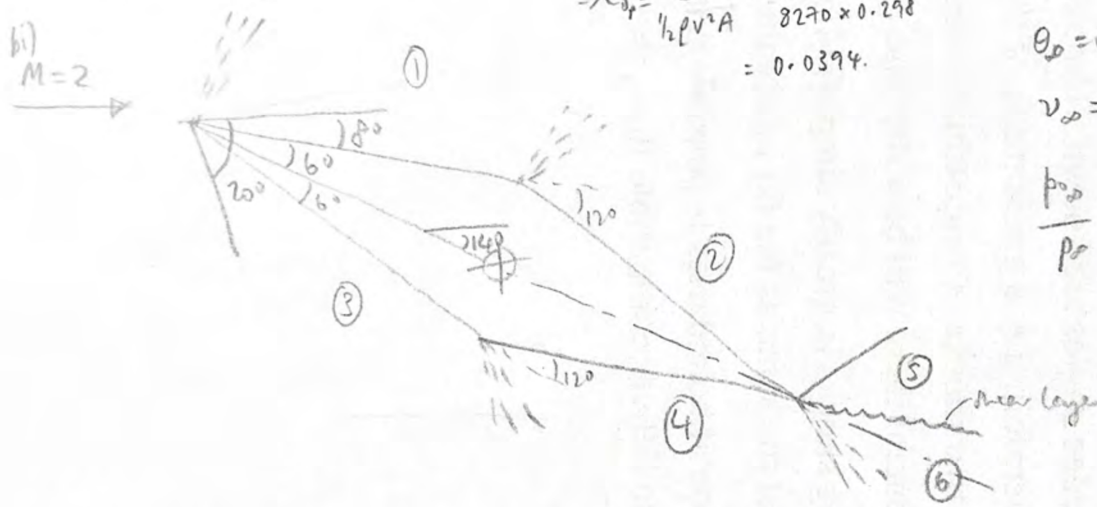
$\theta_\infty = 0°$ $\quad p_{SL} = 1.01325 \times 10^5\,Pa$

$\nu_\infty = 26.38$

$p_\infty = 2970\,Pa$

$\frac{p_{0\infty}}{p_\infty} =$ $\quad p_{0\infty} = 23.2 \times 10^3\,Pa$

b i)



$\theta_1 = -8°$

$\theta_2 = -20°$

$\theta_3 = -20°$

$\theta_4 = -8°$

ii)

$\nu_\infty + \theta_\infty = \nu_1 + \theta_1 \Rightarrow \nu_1 = 34.38° \Rightarrow M_1 = 2.303$

$\frac{T_\infty}{T_{SL}} = 0.765 \therefore T_\infty = 220\,K$

$p_\infty = 2970\,Pa$ $\quad \frac{p_1}{p_{01}} = 0.0797$ $\quad p_{01} = p_{0\infty} \Rightarrow p_1 = 1850\,Pa$

$\frac{\rho_\infty}{\rho_{SL}} = 0.0383 \therefore \rho_\infty = 0.0469\,kg/m^3$

$\nu_1 + \theta_1 = \nu_2 + \theta_2 \Rightarrow \nu_2 = 46.38 \Rightarrow M_2 = 2.830$

$\frac{p_2}{p_{02}} = 0.0352 \Rightarrow p_2 = 817\,Pa$

$a_\infty = \sqrt{\gamma R T_\infty} = 297\,m/s$

$V = 594\,m/s$

$M_\infty = 2$ $\quad \theta = 20°$ $\quad \frac{p_3}{p_\infty} = 2.843$ $\quad M_3 = 1.210$ $\quad \frac{p_{03}}{p_{0\infty}} = 0.893 \Rightarrow p_3 = 8440\,Pa$

$\nu_3 = 3.81$ $\quad p_{03} = 20.7 \times 10^3\,Pa$

$\nu_3 - \theta_3 = \nu_4 - \theta_4 \Rightarrow \nu_4 = 15.81 \Rightarrow M_4 = 1.632$ $\quad \frac{p_4}{p_{04}} = 0.224 \Rightarrow p_4 = 4640\,Pa$

	p	α	cos α	R ↑
①	1850	8	0.990	41.2
②	817	20	0.940	17.3
③	8440	20	0.940	178.5
④	4640	8	0.990	103.4

$\Rightarrow L = 340\,N$

iii) $p_5 = p_6$ for wake. Try flow returns to free stream direction:
so flow must leave trailing edge above free stream direction. Try +1°:
∴ flow leaves ~ 0.5° ABOVE free stream direction.

$\theta_{upper} = 20° \Rightarrow \frac{p_5}{p_2} \sim 3.5 \Rightarrow p_5 \sim 2860\,Pa$

$\theta_{lower} = 8° \Rightarrow \nu_6 = 23.81 \Rightarrow \frac{p_6}{p_{04}} \sim 0.147 \Rightarrow p_6 \sim 3040\,Pa$

$\theta_u = 21° \Rightarrow \frac{p_5}{p_2} \sim 3.76 \Rightarrow p_5 \sim 3070\,Pa$

$\theta_{lower} = 9° \Rightarrow \nu_6 = 24.81 \Rightarrow \frac{p_6}{p_{04}} \sim 0.140 \Rightarrow p_6 \sim 2900\,Pa$

Q5

a) Momentum:
$$\rho V \frac{\partial V}{\partial s} = -\frac{\partial p}{\partial s} = -a^2 \frac{d\rho}{\partial s} \quad \text{(flow isentropic)}$$

$$V\frac{\partial V}{\partial s} = -a^2 \frac{1}{\rho}\frac{\partial \rho}{\partial s} \Rightarrow \text{from continuity: } \frac{V^2}{V}\frac{\partial V}{\partial s} = -a^2 \frac{1}{V}\frac{\partial V}{\partial s} - a^2 \frac{\partial \theta}{\partial n}$$

using geometry from
Prandtl-Meyer datebook.

$$\frac{V^2 - a^2}{V}\frac{\partial V}{\partial s} - \frac{\partial \theta}{\partial n} = 0$$

$$\left[\frac{V^2}{a^2} - 1\right]\frac{1}{V}\frac{\partial V}{\partial s} - \frac{\partial \theta}{\partial n} = 0$$

$$\boxed{\left[M^2 - 1\right]\frac{1}{V}\frac{\partial V}{\partial s} - \frac{\partial \theta}{\partial n} = 0.} \quad (1)$$

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by definition:
$$h_0 = h + \frac{V^2}{2} \Rightarrow dh_0 = dh + VdV$$

$$Tds = dh - \frac{dp}{\rho} \Rightarrow VdV = -\frac{dp}{\rho} \quad \text{for adiabatic, isentropic flow.}$$

$$\rho V^2 \frac{\partial \theta}{\partial s} = -\frac{\partial p}{\partial n} \Rightarrow \boxed{V\frac{\partial \theta}{\partial s} = \frac{\partial V}{\partial n}} \quad (2)$$

---

b) Re-write (1) as $\sqrt{M^2-1}\,\frac{1}{V}\frac{\partial V}{\partial s} - \frac{1}{\sqrt{M^2-1}}\frac{\partial \theta}{\partial n} = 0$

Prandtl-Meyer: $d\nu = \sqrt{M^2-1}\,\frac{dV}{V} \Rightarrow \frac{\partial \nu}{\partial s} - \frac{1}{\sqrt{M^2-1}}\frac{\partial \theta}{\partial n} = 0$

and (2) becomes:
$$\frac{\partial \theta}{\partial s} - \frac{1}{\sqrt{M^2-1}}\frac{\partial \nu}{\partial n} = 0$$

subtracting gives $\frac{\partial}{\partial s}(\nu - \theta) + \frac{1}{\sqrt{M^2-1}}\frac{\partial}{\partial n}(\nu-\theta) = 0$

$$d(\nu-\theta) = \frac{\partial}{\partial s}(\nu-\theta)\,ds + \frac{\partial}{\partial n}(\nu-\theta)\,dn$$
$$= \left[\frac{\partial}{\partial s}(\nu-\theta) + \frac{dn}{ds}\frac{\partial}{\partial n}(\nu-\theta)\right] ds$$

so, for $\frac{dn}{ds} = \frac{1}{\sqrt{M^2-1}}$ term in square brackets is zero, i.e. $d(\nu-\theta) = 0$

Q6

(1) The differential equation

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{F}(u(x,t),x,t) = 0$$

$$u_i^{n+1} = u_i^n - \Delta t\left(\alpha F_i^n + \beta\, \mathrm{F_i}^{\mathrm{N-1}}\right)$$

(a)

$$u_i^{N+1} = u_i^N + \Delta \mathrm{t}\left.\frac{\partial u}{\partial t}\right|_{N,i} + \frac{(\Delta \mathrm{t})^2}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{N,i} + \frac{(\Delta \mathrm{t})^3}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_{N,i}$$

And

$$F_i^{N-1} = F_i^N - \Delta \mathrm{t}\left.\frac{dF}{dt}\right|_{N,i} + \frac{(\Delta \mathrm{t})^2}{2!}\left.\frac{d^2 F}{dt^2}\right|_{N,i} + \mathrm{O}(\Delta t^3)$$

Substituting into the time step rule,

$$\Delta \mathrm{t}\left.\frac{\partial u}{\partial t}\right|_{N,i} + \frac{(\Delta \mathrm{t})^2}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{N,i} + \mathrm{O}(\Delta t^3)$$
$$= -\Delta t\left[\alpha F_i^n + \beta\left(F_i^N - \Delta \mathrm{t}\left.\frac{\partial F}{\partial t}\right|_{N,i} + \frac{(\Delta \mathrm{t})^2}{2!}\left.\frac{\partial^2 F}{\partial t^2}\right|_{N,i} + O(\Delta t^3)\right)\right]$$

$$\left.\frac{\partial u}{\partial t}\right|_{N,i} + \frac{\Delta \mathrm{t}}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{N,i} + \mathrm{O}(\Delta t^2) = -\left[\alpha F_i^n + \beta\left(F_i^N - \Delta \mathrm{t}\left.\frac{\partial F}{\partial t}\right|_{N,i} + \frac{(\Delta \mathrm{t})^2}{2!}\left.\frac{\partial^2 F}{\partial t^2}\right|_{N,i} + O(\Delta t^3)\right)\right]$$

From the original equation

$$\frac{\partial^2 \mathrm{u}}{\partial t^2} + \frac{\partial F}{\partial t} = 0$$

$$\left.\frac{\partial u}{\partial t}\right|_{N,i} + \frac{\Delta \mathrm{t}}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{N,i} + \mathrm{O}(\Delta t^2) = -\left[\alpha F_i^n + \beta\left(F_i^N + \Delta \mathrm{t}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta \mathrm{t})^2}{2!}\left.\frac{\partial^2 F}{\partial t^2}\right|_{N,i} + O(\Delta t^3)\right)\right]$$

$$\left.\frac{\partial u}{\partial t}\right|_{N,i} + \mathrm{O}(\Delta t^2) = -\frac{\Delta \mathrm{t}}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_{N,i} - \beta\Delta t\frac{\partial^2 u}{\partial t^2} - \left[\alpha F_i^n + \beta F_i^N - \beta\left.\frac{\partial^2 F}{\partial t^2}\right|_{N,i}\frac{(\Delta \mathrm{t})^2}{2!} + O(\Delta t^3)\right]$$

Thus, to eliminate the $\mathrm{O}(\Delta \mathrm{t})$ terms,

$$\beta = -\frac{1}{2}$$

And comparing with the original equation, $\alpha + \beta = 1$ so $\boldsymbol{\alpha} = \mathbf{3/2}$

Giving

$$\left.\frac{\partial u}{\partial t}\right|_{N,i} + F_i^{\ n} = O(\Delta t^2)$$

(b) (i)

$$F\,(u, x, t) = 2u\frac{\partial u}{\partial x} - x$$

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} - x = 0$$

Applying the rule from part (a)

$$u_i^{n+1} = u_i^n - \Delta t\left[\frac{3}{2}2u_i^n\frac{(u_i^n - u_{i-1}^n)}{\Delta x} - \frac{3}{2}x_i \quad - \frac{1}{2}2u_i^{n-1}\frac{(u_i^{n-1} - u_{i-1}^{n-1})}{\Delta x} + \frac{1}{2}x_i\right]$$

(b)(ii)

At steady state the N, N-1 and N+1 value are equal

$$u_i^{n+1} = u_i^n - \Delta t\left[\frac{3}{2}2u_i^n\frac{(u_i^n - u_{i-1}^n)}{\Delta x} - \frac{3}{2}x_i \quad - \frac{1}{2}2u_i^{n-1}\frac{(u_i^{n-1} - u_{i-1}^{n-1})}{\Delta x} + \frac{1}{2}x_i\right]$$

$$= -\Delta t\left[3u_i^n\frac{(u_i^n - u_{i-1}^n)}{\Delta x} - x_i \quad - u_i^{n-1}\frac{(u_i^{n-1} - u_{i-1}^{n-1})}{\Delta x}\right]$$

$$= [3u_i^n(u_i^n - u_{i-1}^n) - x_i\Delta x \;\; - u_i^{n-1}(u_i^{n-1} - u_{i-1}^{n-1})]$$

$$= \left[3u_i\;\left(u_i\;\; - u_{i-1}\right) - x_i\Delta x \;\; - u_i\;\left(u_i\;\; - u_{i-1}\right)\right]$$

$$0 = 2u_i^2 - 2u_i\;\, u_{i-1} - x_i\,\Delta x$$

Grid points $x = 0,\ x = 2, x = 4,\ \Delta x = 2$

At $x = 0,\ i = 0$

$$u_0 \quad = 0$$

At $x = 2,\ i = 1$

$$0 = 2u_1^2 - 2u_1\;\, u_0 \quad - 4 \Rightarrow 2u_1^2 = 4 \Rightarrow u_1 = \sqrt{2}$$

At $x = 4,\ i = 2$

$$0 = 2u_2^2 - 2u_2\, u_1 \;\, - 8$$

$$0 = u_2^2 - \sqrt{2}u_2 \ - 4$$

$$u_2 = \frac{\sqrt{2}}{2} \pm \ \frac{1}{2}\sqrt{18} = \frac{\sqrt{2}}{2} \pm \ \frac{3}{2}\sqrt{2} = 2\sqrt{2}$$

(b)(iii)

From

$$\frac{\partial\, u}{\partial t} + 2\mathrm{u}\frac{\partial u}{\partial x} - \mathrm{x} = 0$$

The steady state is

$$2\mathrm{u}\frac{\partial u}{\partial x} - \mathrm{x} = 0$$

$$\frac{\partial u^2}{\partial x} = \mathrm{x}$$

$$u^2 = \frac{x^2}{2} + \mathrm{c}$$

At x = 0 u = 0 so c= 0, so the steady state solution is

$$\mathrm{u} = \frac{x}{\sqrt{(2)}}$$

i.e. $\mathrm{u}(0) = \ 0, \ \mathrm{u}(2) = \sqrt{2}, \mathrm{u}(4) = 2\,\sqrt{2}$

Thus, there is no error in the approximate solution, despite the low spatial resolution. This is because the true solution is linear, so the first order approximations made for the spatial derivatives is exact.

Q7

a) $$\frac{\partial u}{\partial t} + \underbrace{\left(x - \frac{1}{2}\right)}_{\text{Characteristic velocity}} \frac{\partial u}{\partial x} = 0$$

The characteristic velocity changes sign at $x = \frac{1}{2}$.
For $x < \frac{1}{2}$ the velocity is negative, so information flows leftwards and leaves the domain. Therefore, no boundary condition needed at $x = 0$.
Similarly for $x > \frac{1}{2}$ the velocity is positive so information leaves the domain at $x = 1$. No boundary condition needed.

b) $$\frac{\partial u}{\partial t} + \left(x - \frac{1}{2}\right) \frac{\partial u}{\partial x} = 0$$

compare with

$$\frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx = dU$$

$$\Rightarrow \quad \frac{dU}{dt} = 0 \quad \text{along lines of} \quad \frac{dx}{dt} = x - \frac{1}{2}$$

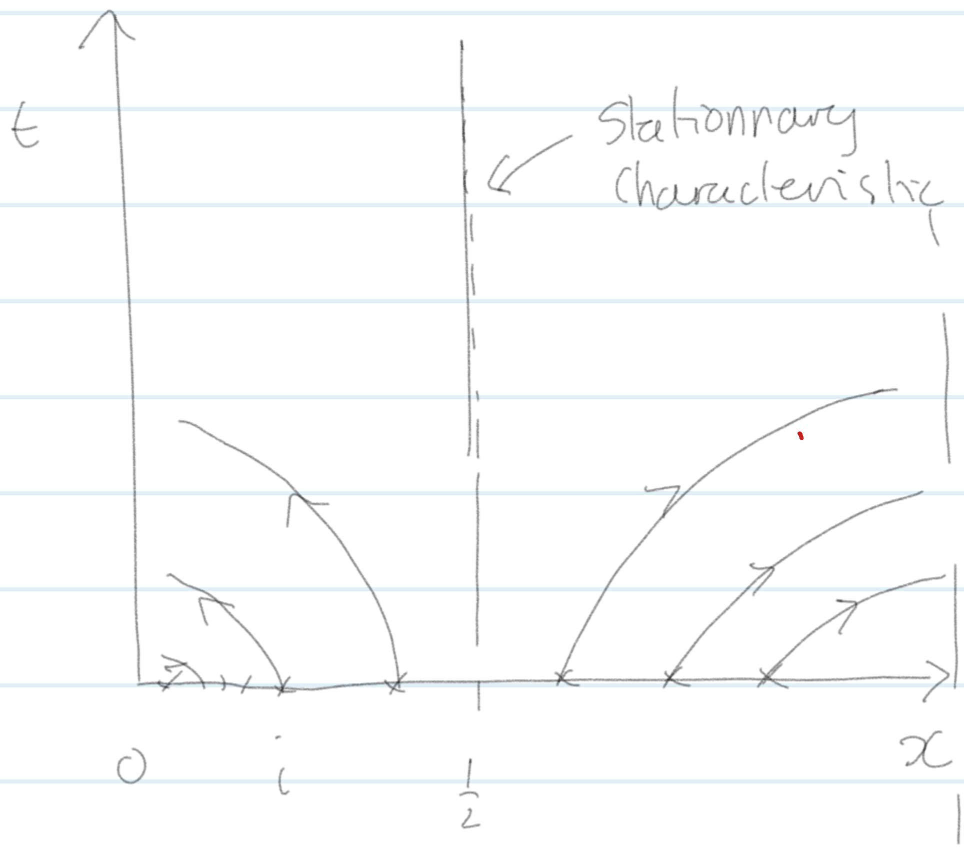
$$\frac{dx}{dt} = x - \frac{1}{2}$$

$$\int \frac{1}{(x - \frac{1}{2})} dx = \int dt$$

$$\ln\left[\frac{(x - \frac{1}{2})}{(x_0 - \frac{1}{2})}\right] = t$$

$$x = \frac{1}{2} + \left(x_0 - \frac{1}{2}\right) e^{t}$$

There is also the special case at $x = \frac{1}{2}$ when $\frac{dU}{dt} = 0$, ie. a stationary characteristic



c) Owing to the change in sign of the characteristic velocity, the upwinding direction needs to reverse at $x = \frac{1}{2}$.

So for $x > \frac{1}{2}$

$$\frac{U_i^{t+\Delta t} - U_i^t}{\Delta t} = -\left(x_i - \frac{1}{2}\right)\frac{\left(U_i^t - U_{i-1}^t\right)}{\Delta x}$$

$$U_i^{t+\Delta t} = U_i^t - \frac{\Delta t}{\Delta x}\left(x_i - \frac{1}{2}\right)\left(U_i^t - U_{i-1}^t\right)$$

and for $x < \frac{1}{2}$

$$U_i^{t+\Delta t} = U_i^t - \frac{\Delta t}{\Delta x}\left(x_i - \frac{1}{2}\right)\left(U_{i+1}^t - U_i^t\right)$$

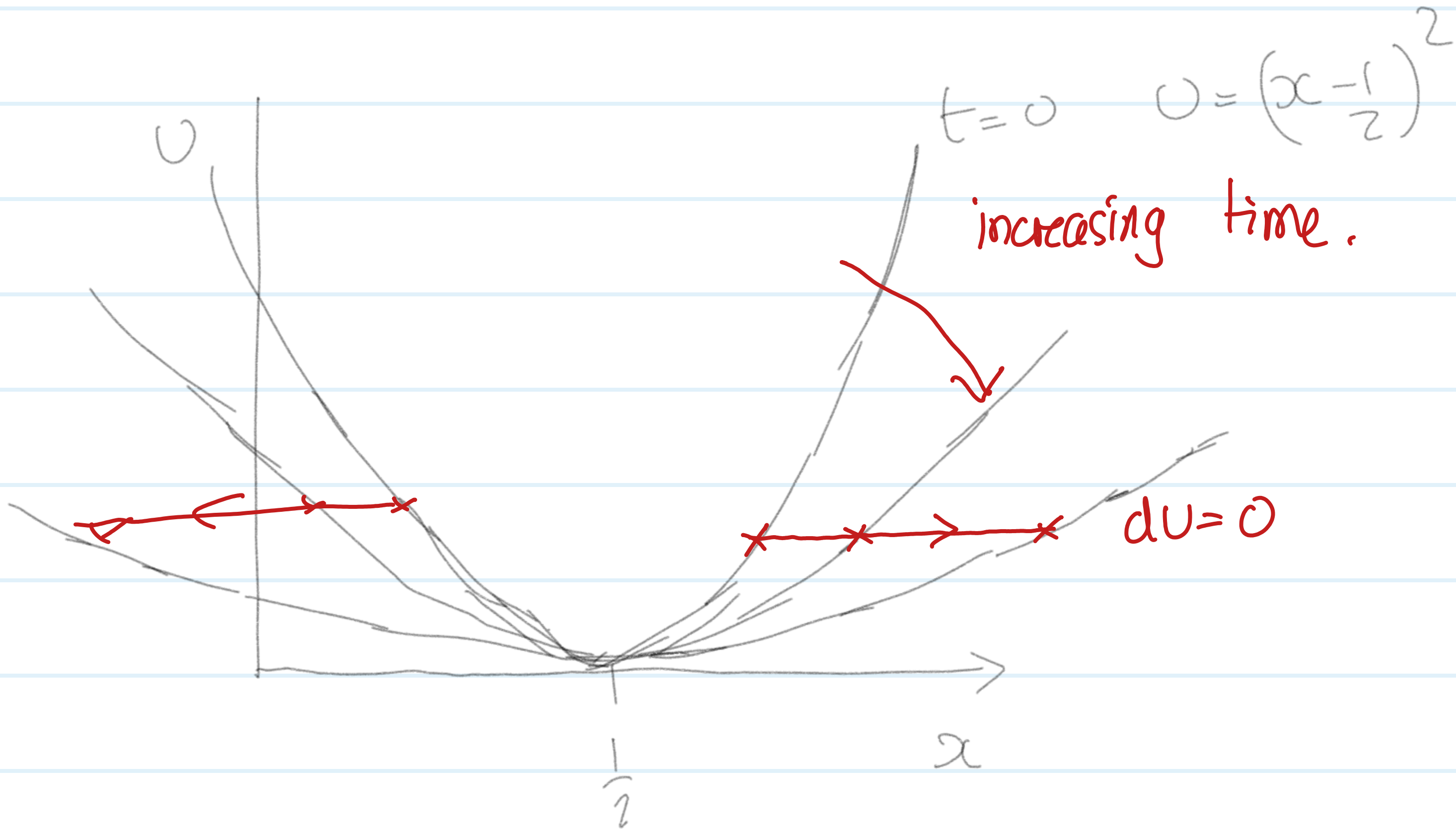
for stability, need to satisfy CFL

$$\frac{\Delta t}{\Delta x} \max\left(\left|x_i - \frac{1}{2}\right|\right) < 1$$

This occurs at the edges of the domain when the velocity is highest and $x = 1$ or 0

$$\frac{\Delta t}{2\Delta x} < 1 \Rightarrow \frac{\Delta t}{\Delta x} < \frac{1}{2}$$

d) The characteristic can be used to infer the evolution of the solution



Each value of u is convected with increasing velocity away from $x=\frac{1}{2}$

7 b) i)

$\phi = 0.4$

$\psi = 1.8$

$\alpha_1 = -45^\circ$

From Euler: $\psi = \dfrac{u(V_{t2} - V_{t3})}{u^2}$ note $V_{t1} = V_{t3}$ repeating stage

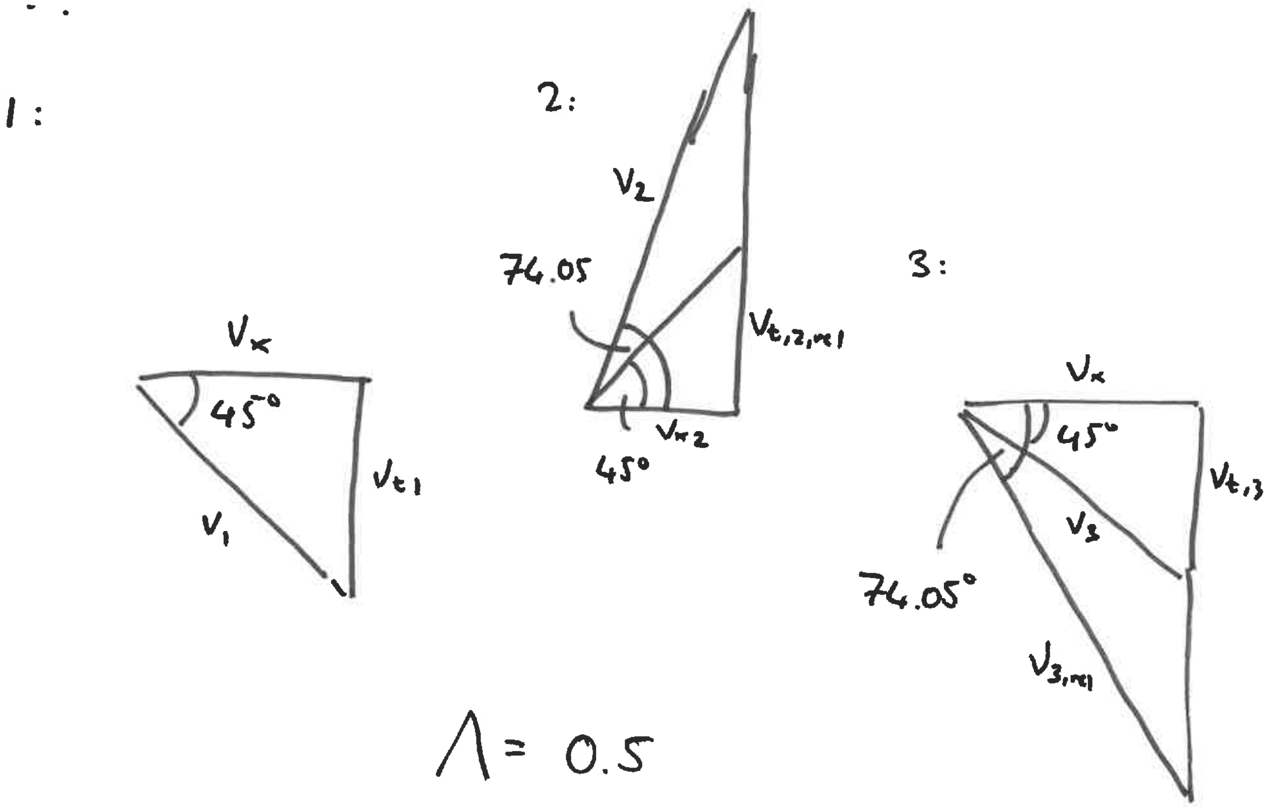
$= (\tan\alpha_2 - \tan\alpha_1)\,\phi$

$\therefore \quad \alpha_2 = 74.05^\circ$

Reference shift: $\tan\alpha_2 - \dfrac{1}{\phi} = \tan\beta_2 \longrightarrow \underline{\underline{\beta_2 = 45^\circ}}$

note: we see this is a symmetric or 50% Reaction turbine

$\therefore$



$\Lambda = 0.5$

$\alpha_1 = \alpha_3 = -\beta_2 = 45^\circ$

$\alpha_2 = -\beta_3 = 74.05^\circ$

b) ii)

$\phi = 0.4$, $\psi = 1.8$, $\Lambda = 0 \therefore \beta_2 = -\beta_3$

From Euler: $\psi = \dfrac{\Delta V_t}{u} = \phi(\tan\beta_2 - \tan\beta_3)$

$$= 2\phi(\tan\beta_2)$$

$\therefore \beta_2 = -\beta_3 = 66.04$

Reference shift: $\tan\alpha_2 = \tan\beta_2 + \dfrac{1}{\phi} \rightarrow \alpha_2 = 78.11°$

$\tan\alpha_3 = \tan\beta_3 + \dfrac{1}{\phi} \rightarrow \alpha_3 = \alpha_1 = 14.04°$

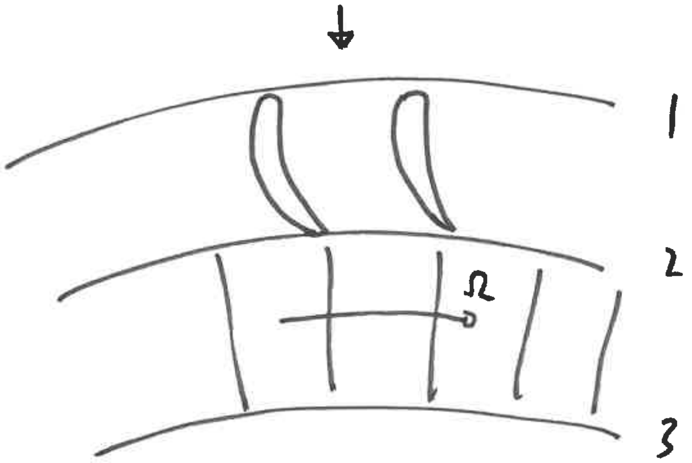
iii)

negative: No acceleration over rotor reduces efficiency

Positive: - No axial force on rotor; easier Mechanics

- No pressure drop over rotor $\therefore$ easier sealing

8



$r_1 = 0.05\,m$

$r_2 = 0.04\,m$

$r_3 = 0.03\,m$

$h = 0.01\,m$

$P_{01} = 200\,kPa$

$T_{01} = 800\,K$

$M_1 = 0.1$

a) i) $\dfrac{\dot{m}\sqrt{c_p T_{01}}}{A_1 P_{01}} = f(M_1) = 0.22$ (Tables)

$A_1 = 2\pi r_1 h_1 = 0.003141\,m^2$

$\underline{\dot{m} = 0.1538\ kg/s}$

ii) @ Stator exit $\alpha_2 = 76.5^\circ$, $T_{02} = T_{01}$, $P_{02} \neq P_{01}$

$A_2 = 2\pi r_2 h_2 \cos\alpha_2 = 0.0005867$

$\cancel{\dfrac{\dot{m}\sqrt{c_p T_{02}}}{A_2 P_{02}}}$ ← cant use as P_{02} not known

Zero incidence ∴ $V_{t2} = \Omega r_2 = 381.18\,m/s$

$V_2 = V_{t2} / \sin\alpha_2 = 392\,m/s$

$\dfrac{V_2}{\sqrt{c_p T_{02}}} = 0.4361 \rightarrow \underline{M_2 = 0.725}$

Very low as $M_2/M_1 \simeq 7.25$!

iii) $\dfrac{\dot{m}\sqrt{c_p T_{02}}}{A_2 P_{02}} = 1.189$ ∴ $P_{02} = 1.982 \times 10^5\,Pa$

$P_2/P_{02} = 0.705$ ∴ $P_2 = 1.397 \times 10^5\,Pa$. $Y_p = \dfrac{P_{01} - P_{02}}{P_{02} - P_2} = 0.031$

8 b)

Euler : $\Delta h_0 = U_2 V_{t2} - U_3 V_{t3}$

Zero incidence $\therefore$ $V_{t2} = U_2$

Zero deviation $\therefore$ $V_{t3} = U_3$

$$\therefore \quad \Delta h_0 = U_2^2 - U_3^2 = \Omega^2 (r_2^2 - r_3^2)$$

$$\psi = \frac{\Delta h_0}{U_2^2} = 1 - \left(\frac{r_3}{r_2}\right)^2$$

c) $P_{0,3} = 146 \text{ kPa}$

$$T_{0,3,s} = T_{01} \left(\frac{P_{03}}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}} = 731.2 \text{ K}$$

$$T_{03} = T_{01} - \left(1 - \left(\frac{r_3}{r_2}\right)^2\right) \frac{r_2^2 \omega^2}{C_p} = 737.06 \text{ K}$$

$$\eta_{TT} = \frac{T_{01} - T_{03}}{T_{01} - T_{03,s}} = \frac{800 - 737.06}{800 - 731.2} = 91.5\,\%$$

d) $T_3 = T_{03} - \frac{1}{2C_p} V_3^2$, $V_3^2 = U_3^2 + V_{r3}^2$

$= 81730.2 + 20164$

$= 101894.2$

$= 686.62 \text{ K}$

$$\eta_{TS} = \frac{T_{01} - T_{03}}{T_{01} - T_{3,s}}$$ $\therefore$ need $T_{3,s}$, $\therefore$ need P_3

$$\frac{P_3}{P_{03}} = \left(\frac{T_3}{T_{03}}\right)^{\gamma/\gamma-1} \Rightarrow P_3 = 1.1393 \times 10^5 \text{ Pa}$$

$$T_{3,s} = T_{01} \left(\frac{P_3}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}} = 681.176 \text{ K}$$

$\eta_{TS} = 53.0\,\%$

8 e) If exhausting then Exit KE is useless

$\therefore \eta_{TS}$ is correct

- aims to reduce exit KE

1. swirl at rotor exit : add turning to blades

or reduce r_3

note reducing r_3 will work until V_r^2 increases faster than V_t^2 decreases

$V_t \propto r$ $\qquad$ $V_r \propto \frac{1}{r}$

Question 1:

Attempts: 85/85, raw average mark: 16.1/20, Mean: 80.9%, StDev: 14%

- Convergent-divergent nozzle.
- Mostly well answered, the most common mistakes were:
    - Omission of the graph of the Mach number distribution through the nozzle.
    - Some confusion about whether the flow is over or under expanded within the nozzle.
    - Incorrect setup of the control volume to determine the force on the nozzle, often missing out the change in momentum.

Question 2

Attempts: 85/85, raw average mark: 17.6/20, Mean: 87.5%, StDev: 16%

- Rayleigh flow.
- Many candidates achieved full marks.
- Mistakes were few and limited to missing items on the T-s diagram, calculation of the exit velocity using the stagnation temperature rather than static temperature and failing to successfully describe the process of cooling to accelerate the flow past Mach 1.

Question 3

Attempts: 59/85, raw average mark: 11.2/20, Mean: 56.1%, StDev: 23%

- Unsteady flow venting from pressure vessel.
- While many could successfully derive the Riemann invariant, few could use it successfully.
- Only four students managed to successfully complete the final part of the question, most incorrectly assumed steady choked flow at the pipe outlet to use the compressible mass flow function, yielding the incorrect area.

Question 4:

Attempts: 79/85, raw average mark: 13.8/20, Mean: 69.1%, StDev: 14.8%

- Supersonic flow around a diamond section
- Candidates performed well on this question, many accurate diagrams and calculations with the wave method on expansion fans and oblique shocks.

- Few made it all the way to the end to calculate the slip angle, although many could recall that static pressures are equal across the slip line.

Question 5:

Attempts: 2/85, raw average mark: 7.0/20, Mean: 35%, StDev: 35%

- Prandtl-Meyer functions
- The least popular question and no one made it to the end.
- Both parts (a) and (b) caused difficulty although both candidates were able to successfully describe the physical significance of the derived angle in (c).

Question 6:

Attempts: 33/85, raw average mark: 8.8/20, Mean: 43%, StDev: 28%

- Numerical integration
- Some candidates answered this question very well and most had a uniform performance throughout the question.
- The two most challenging parts were the exact answers for alpha and beta, a lot of algebra could result in dropped values in the first part of the question, and the approximate solution in (b)(ii).

Question 7:

Attempts: 47/85, raw average mark: 10.8/20, Mean: 54%, StDev: 24%

- Compressible turbomachinery through a radial inflow turbine
- The question was answered well by many candidates, in particular the mass flow and the derivation of stage loading were well answered. The mean is dragged down by some very partial answers.
- Many candidates assumed constant density through the stator leading to errors, there was also confusion on obtaining P02, this being fully prescribed by the velocity at this station – given by the zero incidence condition and continuity. The efficiencies were often excluded from answers, either time constraints or uncertainty on how to tackle. The final part was well attempted by almost all candidates.

Question 8:

Attempts: 36/85, raw average mark: 5.7/20, Mean: 28.6%, StDev: 15.8%

- Question 8 is a split question, the first half tested numerical methods, and identification of equation characteristics followed by numerical solutions. The second half tested velocity triangles and differences between reaction and impulse turbines.
- The question as a whole did not hit the target mean, this is heavily due to partial attempts to the question. Only 36 students attempted it, and of those a majority provided partial answers. Perhaps due to the split nature, or perhaps due to being the last question on the paper.
- The numerical methods part was generally poorly answered, the justification for not needing boundary conditions was only fully explained by a couple of candidates. The plotting of characteristics and solutions with time was completed by a handful of candidates successfully. In the numerical approximation, candidates often formed a second order scheme in space, the question asking for first order accuracy.
- The Turbomachinery component was moderately answered for the first 5 marks, the subject of reaction was not well understood, and this led to the remainder falling short. In the velocity triangles a common mistake was flipping the addition/subtraction of blade speed, when moving between reference frames.