Part IIA 3C5 2018

1(a) For the plate

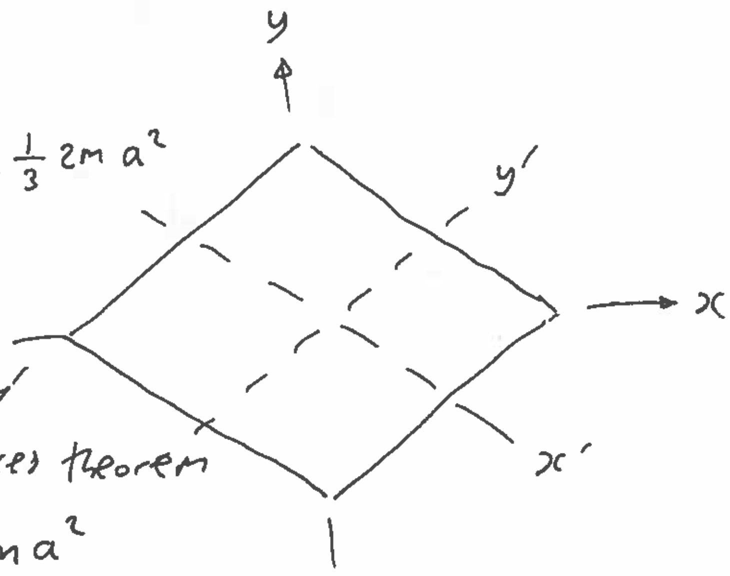
$I_{x'x'} = I_{y'y'} = \frac{1}{3} 2m a^2$

from Data Book

and $I_{zz} = I_{x'x'} + I_{y'y'}$

by perpendicular axes theorem

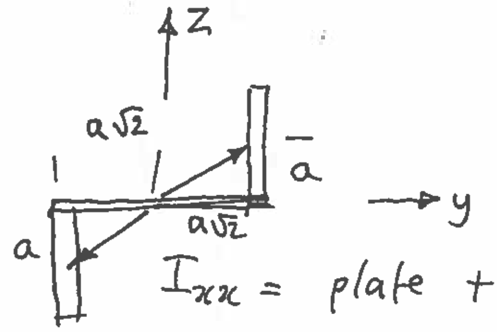
$I_{zz} = \frac{4}{3} m a^2$



The plate is like a disc (AAC body)

$\therefore \; I_{xx} = I_{yy} = \frac{2}{3} m a^2$

(b) View along x



I_{xx} = plate + 2 rods

$= \frac{2}{3} m a^2 + 2\left(\underbrace{\frac{1}{3} m a^2 + m a^2 (2+1)}_{\text{parallel axis theorem}}\right)$

$= \frac{2}{3} m a^2 + \frac{20}{3} m a^2$

$= \frac{22}{3} m a^2$

$I_{yz} = \int yz \, dm$

$= 2 a\sqrt{2} \int_0^{2a} z \, dz \, \frac{m}{2a}$

$= m\sqrt{2} \left[\frac{1}{2} z^2\right]_0^{2a}$

$= 2\sqrt{2} m a^2$

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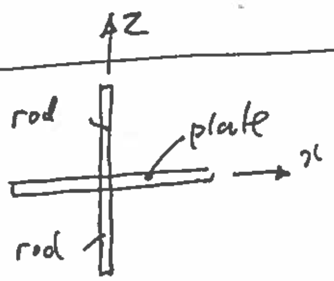
view along y



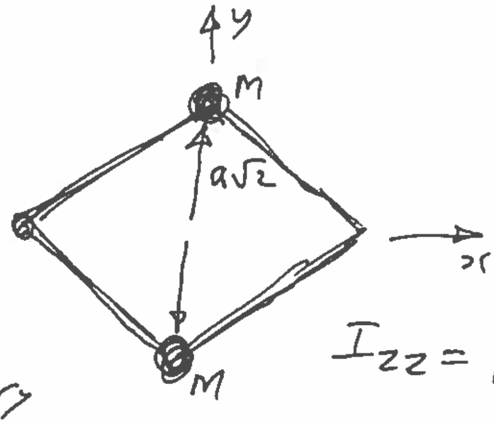
I_{yy} = plate + long rod

$= \frac{2}{3} m a^2 + \frac{1}{3} 2m (2a)^2$

$= \frac{10}{3} m a^2$

$I_{xz} = 0$ by symmetry

1 (b) cont view along z



$I_{xy} = 0$ by symmetry

I_{zz} = plate + 2 blobs

$= \frac{4}{3}Ma^2 + 2M(a\sqrt{2})^2$

$= \frac{16}{3}Ma^2$

$$\therefore I_G = \frac{2Ma^2}{3}\begin{bmatrix} 11 & 0 & 0 \\ 0 & 5 & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8 \end{bmatrix}$$

(c) For principal moments of inertia

$$\begin{vmatrix} 11-\lambda & 0 & 0 \\ 0 & 5-\lambda & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 8-\lambda \end{vmatrix} = 0$$

$\therefore (11-\lambda)\left[(5-\lambda)(8-\lambda) - (3\sqrt{2})^2\right] = 0$

$\therefore (11-\lambda)\left[\lambda^2 - 13\lambda + 40 - 18\right] = 0$

$\therefore (11-\lambda)(\lambda^2 - 13\lambda + 22) = 0$

$\therefore (11-\lambda)(11-\lambda)(2-\lambda) = 0$

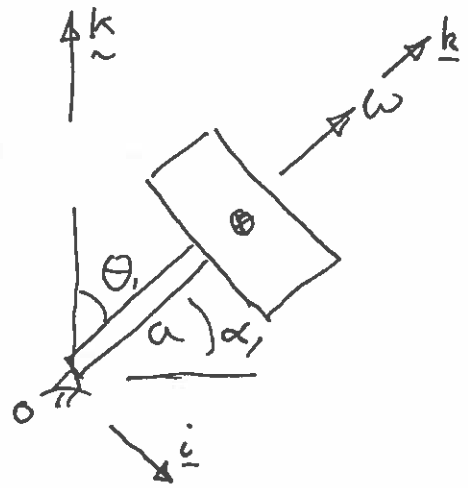
$\therefore$ principal moments of inertia are $\frac{22Ma^2}{3}, \frac{22Ma^2}{3}, \frac{4Ma^2}{3}$

The diagonal $B\text{-}D$ x-axis was principal before so it still is $\therefore I_{BD} = \frac{22Ma^2}{3}$

(d) This is an AAC body and cannot be distinguished, in inertial terms, from any other AAC body.

2.

(A)

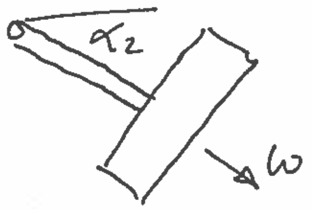


Gyro 2

$$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$

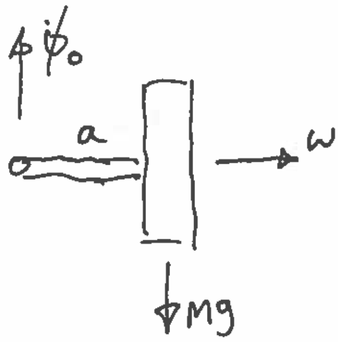
with
$$\Omega_1 = -\dot{\phi}\sin\theta$$
$$\Omega_2 = \dot{\theta}$$
$$\Omega_3 = \dot{\phi}\cos\theta$$
$$\omega_3 = \text{constant} = \omega$$

(B)



At the three positions (A) (B) and (C) $\dot{\theta}$ is zero. Also fast spin

(C)



(a) Steady state precession position (C) use Gyro 2 with $\theta = \frac{\pi}{2}$ and fast spin to give $mga = C\omega\dot{\phi}_0$

$$\therefore \boxed{\dot{\phi}_0 = \frac{mga}{C\omega}}$$

(b) throughout the motion $\underline{h} \cdot \underline{K}$ is conserved because no forces have a moment about the $\underline{K}$ axis

For position (C) $\underline{h} = A\dot{\phi}_0 \underline{K}$

so the conserved value of $\underline{h}\cdot\underline{K} = \underline{\underline{\frac{mgaA}{C\omega}}}$

2 (c) In position (A) just after release from rest

$\underline{\omega} = \omega \underline{k}$ so $\underline{h} = C\omega \underline{k}$

and $\underline{h} \cdot \underline{K} = C\omega \underline{k} \cdot \underline{K} = C\omega \cos\theta_1$

use $\theta_1 = \frac{\pi}{2} - \alpha_1$ $\therefore$ $C\omega \sin\alpha_1 = \frac{MgaA}{C\omega}$

and for small α $\underline{\underline{\alpha_1 = \frac{MgaA}{(C\omega)^2}}}$

(d) In motion from (A) to (B) energy is also conserved.

At (A) $KE = \frac{1}{2} C\omega^2$ $PE = Mga \sin\alpha_1$
$= Mga\,\alpha_1$

At (B) $KE = \frac{1}{2} C\omega^2 + \frac{1}{2} A(\dot{\phi} \sin\theta)^2$
$= \frac{1}{2} C\omega^2 + \frac{1}{2} A\dot{\phi}^2$ given $\theta = \frac{\pi}{2} - \alpha$ and small α

$PE = -Mga\,\alpha_2$

$\therefore (KE + PE)_{(A)} = (KE + PE)_{(B)}$

$\therefore \frac{1}{2} C\omega^2 + Mga\,\alpha_1 = \frac{1}{2} C\omega^2 + \frac{1}{2} A\dot{\phi}^2 - Mga\,\alpha_2$

$\therefore \alpha_1 + \alpha_2 = \frac{A}{2Mga} \dot{\phi}^2$ (1)

Moment of Momentum $\underline{h} \cdot \underline{K} = -C\omega\,\alpha_2 + A\dot{\phi} = \frac{MgaA}{C\omega}$

$\therefore \alpha_2 = \frac{A}{C\omega}\dot{\phi} - \frac{MgaA}{(C\omega)^2}$

$= \frac{A}{C\omega}\dot{\phi} - \alpha_1$ (2)

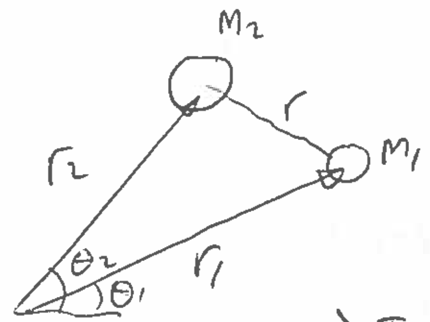
(1) and (2) give $\dot{\phi} = 0$ (position (A)) or $\underline{\underline{\dot{\phi} = \frac{2Mga}{C\omega} = 2\dot{\phi}_0}}$

and (1) gives $\alpha_1 + \alpha_2 = \frac{2MgaA}{(C\omega)^2} = 2\alpha_1$

$\therefore \underline{\underline{\alpha_2 = \alpha_1}}$ $v_G = a\dot{\phi} = \frac{2Mga^2}{C\omega}$

3/

$$\underline{r} = r\underline{e}_r \, , \quad \dot{\underline{r}} = \dot r \underline{e}_r + r\dot\theta \underline{e}_\theta$$



$$\therefore v^2 = \dot r^2 + (r\dot\theta)^2$$

$$\therefore T = \tfrac{1}{2} m_1 (\dot r_1^2 + r_1^2 \dot\theta_1^2) + \tfrac{1}{2} m_2 (\dot r_2^2 + r_2^2 \dot\theta_2^2)$$

$$\frac{\partial T}{\partial \dot r_1} = m_1 \dot r_1 \, , \quad \frac{\partial T}{\partial \dot r_2} = m_2 \dot r_2 \, , \quad \frac{\partial T}{\partial \dot\theta_1} = m_1 r_1^2 \dot\theta_1$$

$$\frac{\partial T}{\partial \dot\theta_2} = m_2 r_2^2 \dot\theta_2$$

$$\frac{\partial T}{\partial r_1} = m_1 r_1 \dot\theta_1^2 \qquad \frac{\partial T}{\partial r_2} = m_2 r_2 \dot\theta_2^2 \qquad \frac{\partial T}{\partial \theta_1} = \frac{\partial T}{\partial \theta_2} = 0$$

$$V = -\frac{G m_1 m_2}{r}$$

with $r^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_2 - \theta_1)$ (cosine rule)

$$\therefore \frac{\partial V}{\partial r_1} = \frac{G m_1 m_2}{r^2} \frac{r_1 - r_2 \cos(\theta_2 - \theta_1)}{r}$$

$$\frac{\partial V}{\partial r_2} = \frac{G m_1 m_2}{r^2} \frac{r_2 - r_1 \cos(\theta_2 - \theta_1)}{r}$$

$$\frac{\partial V}{\partial \theta_1} = -\frac{G m_1 m_2}{r^2} \frac{r_1 r_2 \sin(\theta_2 - \theta_1)}{r}$$

$$\frac{\partial V}{\partial \theta_2} = +\frac{G m_1 m_2}{r^2} \frac{r_1 r_2 \sin(\theta_2 - \theta_1)}{r}$$

$\therefore 2r \frac{\partial r}{\partial r_1} = 2(r_1 - r_2 \cos(\theta_2 - \theta_1))$

& $2r \frac{\partial r}{\partial r_2} = 2(r_2 - r_1 \cos(\theta_2 - \theta_1))$

& $2r \frac{\partial r}{\partial \theta_1} = -2 r_1 r_2 \sin(\theta_2 - \theta_1)$

& $2r \frac{\partial r}{\partial \theta_2} = +2 r_1 r_2 \sin(\theta_2 - \theta_1)$

Lagrange:

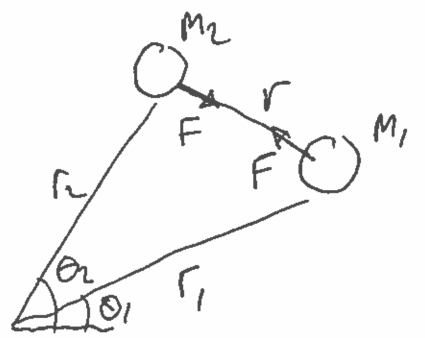
r_1 : $\displaystyle m_1 \ddot r_1 - m_1 r_1 \dot\theta_1^2 + \frac{G m_1 m_2}{r^2} \frac{r_1 - r_2 \cos(\theta_2 - \theta_1)}{r} = 0$

r_2 : $\displaystyle m_2 \ddot r_2 - m_2 r_2 \dot\theta_2^2 + \frac{G m_1 m_2}{r^2} \frac{r_2 - r_1 \cos(\theta_2 - \theta_1)}{r} = 0$

θ_1 : $\displaystyle \frac{d}{dt}(m_1 r_1^2 \dot\theta_1) - \frac{G m_1 m_2}{r^2} \frac{r_1 r_2 \sin(\theta_2 - \theta_1)}{r} = 0$

θ_2 : $\displaystyle \frac{d}{dt}(m_2 r_2^2 \dot\theta_2) + \frac{G m_1 m_2}{r^2} \frac{r_1 r_2 \sin(\theta_2 - \theta_1)}{r} = 0$

The same can be obtained by using Generalized force:



$$dW = -F\,dr$$

$$r^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$2r\,dr = 2r_1\,dr_1 + 2r_2\,dr_2$$
$$- 2r_1 \cos(\theta_2 - \theta_1)\,dr_2$$
$$- 2r_2 \cos(\theta_2 - \theta_1)\,dr_1$$
$$- 2r_1 r_2 \sin(\theta_2 - \theta_1)\,d\theta_1$$
$$+ 2r_1 r_2 \sin(\theta_2 - \theta_1)\,d\theta_2$$

$$\therefore \quad Q_{\theta_2} = -F \frac{r_1 r_2 \sin(\theta_2 - \theta_1)}{r}$$

exactly as $\frac{\partial V}{\partial \theta_2}$ gave before

b/ if O is at C of M then $\theta_2 - \theta_1 = \pi$

$$\therefore \quad M r_1^2 \dot{\theta}_1 = \text{const}$$
$$\text{and} \quad M_1 r_2^2 \dot{\theta}_2 = \text{const}$$

c/ from r_1

$$M_1 \ddot{a} - M_1 a \dot{\theta}^2 + \frac{GM_1M_2}{r^2}\frac{r_1 + r_2}{r} = 0$$

$r_1 = a$

$r_2 = \frac{M_1}{M_2} a$

$r = r_1 + r_2$

$= a\left(1 + \frac{M_1}{M_2}\right)$

$$\therefore \quad \ddot{a} - a\dot{\theta}^2 + \frac{GM_2}{a^2\left(1+\frac{M_1}{M_2}\right)^2} = 0$$

from θ_1

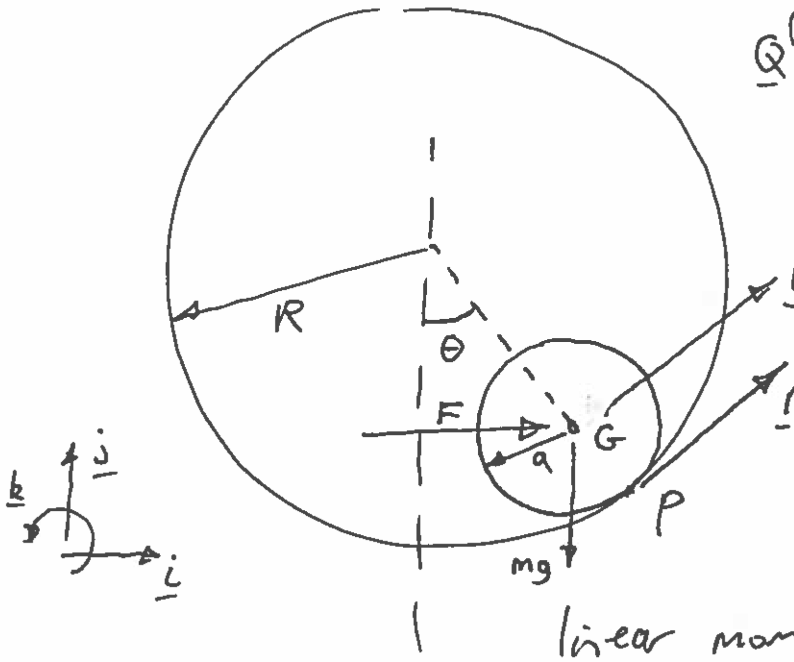
$$\frac{d}{dt}\left(M_1 a^2 \dot{\theta}\right) = 0 \qquad \therefore \quad a^2\dot{\theta} = \text{const}$$

d/ $a = A = \text{const}$ $\therefore$ $\dot{\theta} = \text{const} = \frac{2\pi}{T}$

$$\therefore \quad A\left(\frac{2\pi}{T}\right)^2 = \frac{GM_2}{A^2\left(1+\frac{M_1}{M_2}\right)^2} \qquad \therefore \quad \left(\frac{2\pi}{T}\right)^2 = \frac{GM_2}{A^3\left(1+\frac{M_1}{M_2}\right)^2}$$

$$\therefore \quad T = 2\pi\left(1+\frac{M_1}{M_2}\right)\sqrt{\frac{A^3}{GM_2}}$$

4 (a)



$$\underline{Q}^{(e)} = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p}$$

$\dot{\underline{r}}_G$ = velocity of point G $= (R-a)\dot{\theta} \nearrow$

$\dot{\underline{r}}_P$ = velocity of point P $= R\dot{\theta} \nearrow$

linear momentum $\underline{p} = m\dot{\underline{r}}_G = m(R-a)\dot{\theta} \nearrow$

So $\dot{\underline{r}}_P \times \underline{p} = 0$

$$\underline{Q}^{(e)} = mga\sin\theta\,\underline{k} - Fa\cos\theta\,\underline{k}$$

$$\underline{h}_P = \underline{h}_G + (\underline{r}_G - \underline{r}_P) \times \underline{p} \qquad \text{from data sheet}$$

$$= I\omega\,\underline{k} + (a \nwarrow) \times (m(R-a)\dot{\theta} \nearrow)$$

$$= I\omega\,\underline{k} - ma(R-a)\dot{\theta}\,\underline{k} \qquad I = ma^2 \text{ for hoop}$$

No slip at P $\therefore \dot{\underline{r}}_G + \omega\underline{k} \times \vec{GP} = 0$

$\therefore (R-a)\dot{\theta} + \omega a = 0$

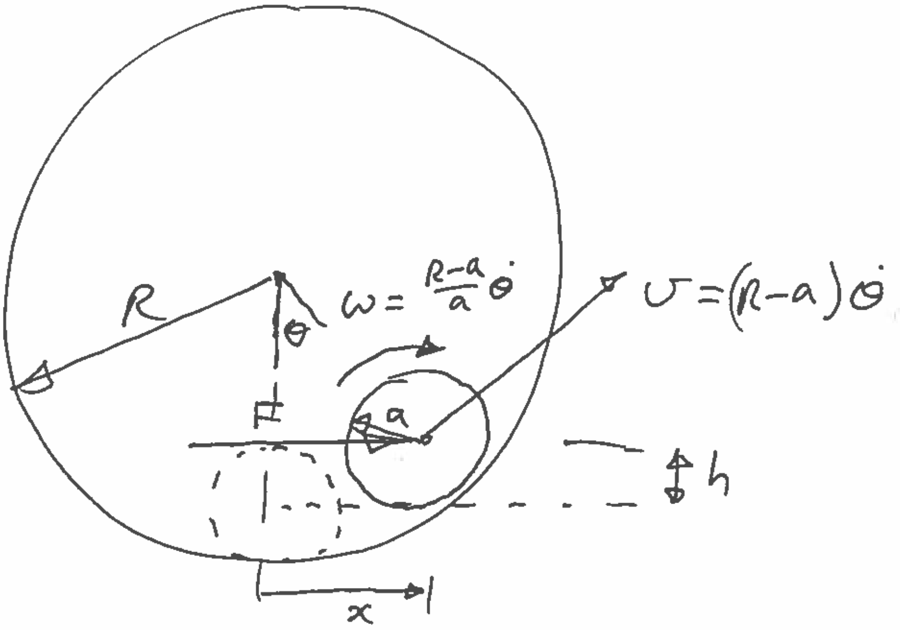
$\therefore \omega = -\frac{R-a}{a}\dot{\theta}$

$$\therefore \underline{h}_P = \left(-ma^2\frac{R-a}{a}\dot{\theta} - ma(R-a)\dot{\theta}\right)\underline{k}$$

$$= -2ma(R-a)\dot{\theta}\,\underline{k}$$

$\underline{Q}^{(e)} = \dot{\underline{h}}_P \quad \therefore \quad \boxed{2(R-a)\ddot{\theta} + g\sin\theta = \frac{F}{m}\cos\theta}$

4 (b)



$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m \left((R-a)\dot{\theta}\right)^2 + \frac{1}{2} m a^2 \left(\frac{R-a}{a}\dot{\theta}\right)^2$$

$$= m (R-a)^2 \dot{\theta}^2$$

$$V = mgh = mg(R-a)(1-\cos\theta)$$

Generalized force: $dw = F\,dx$

with $x = (R-a)\sin\theta \quad \therefore \quad dx = (R-a)\cos\theta\, d\theta$

$\therefore \quad dw = F(R-a)\cos\theta\, d\theta$

$\therefore \quad Q_\theta = F(R-a)\cos\theta$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

$$\therefore \quad 2m(R-a)^2\ddot{\theta} - 0 + mg(R-a)\sin\theta = F(R-a)\cos\theta$$

$$\therefore \quad \boxed{2(R-a)\ddot{\theta} + g\sin\theta = \frac{F}{m}\cos\theta}$$ as before

4 (c) $\quad F = mg \quad \therefore$ equilibrium position Θ_0

$$g \sin\Theta_0 = \frac{mg}{m} \cos\Theta_0 \qquad \therefore \quad \Theta_0 = \frac{\pi}{4}$$

put $x = \Theta - \Theta_0 \qquad \ddot{x} = \ddot{\Theta}$, small x

$$\therefore \quad 2(R-a)\ddot{x} + g\sin\left(x + \frac{\pi}{4}\right) = g\cos\left(x + \frac{\pi}{4}\right)$$

$$\therefore \quad 2(R-a)\ddot{x} + g\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)$$

$$= g\left(\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}\right)$$

$$\therefore \quad 2(R-a)\ddot{x} + g x \frac{\sqrt{2}}{2} = -g x \frac{\sqrt{2}}{2}$$

$$\therefore \quad \ddot{x} + \frac{g\sqrt{2}}{2(R-a)} x = 0$$

SHM with frequency $\omega_n = \sqrt{\dfrac{g\sqrt{2}}{2(R-a)}}$

Alternatively (and quicker) note that

$\underline{F} + m\underline{g} = \sqrt{2}mg$ at angle $\frac{\pi}{4}$

So put $\sqrt{2}g$ in place of g in the EOM to get $2(R-a)\ddot{\Theta} + \sqrt{2}g\,\Theta = 0$

and thus $\omega_n = \sqrt{\dfrac{g\sqrt{2}}{2(R-a)}}$