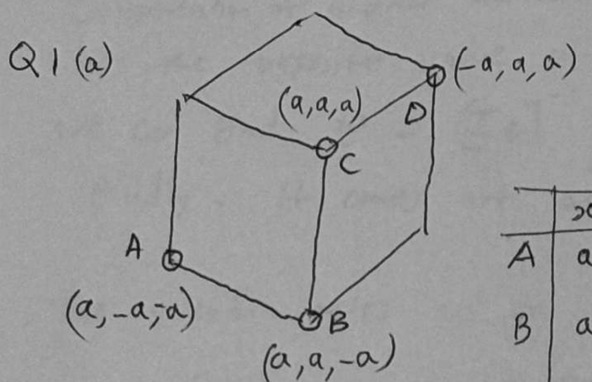
Part IIA 3C5 2026

Q1 (a)



Cube $I_{xx} = I_{yy} = I_{zz} = \frac{1}{6} 2m(2a)^2$

$= \frac{4}{3} m a^2$

$I_{xy} = I_{xz} = I_{yz} = 0$

	x_i	y_i	z_i	I_{xx}	I_{yy}	I_{zz}	I_{xy}	I_{xz}	I_{yz}
A	a	$-a$	$-a$	$2ma^2$	$2ma^2$	$2ma^2$	$-ma^2$	$-ma^2$	ma^2
B	a	a	$-a$	$2ma^2$	$2ma^2$	$2ma^2$	ma^2	$-ma^2$	$-ma^2$
C	a	a	a	$2ma^2$	$2ma^2$	$2ma^2$	ma^2	ma^2	ma^2
D	$-a$	a	a	$2ma^2$	$2ma^2$	$2ma^2$	$-ma^2$	$-ma^2$	ma^2
Σ				$8ma^2$	$8ma^2$	$8ma^2$	0	$-2ma^2$	$2ma^2$

Use $I_{xx} = \sum m_i (y_i^2 + z_i^2)$

$I_{yy} = \sum m_i (x_i^2 + z_i^2)$

$I_{zz} = \sum m_i (x_i^2 + y_i^2)$

$I_{xy} = \sum m_i x_i y_i$, $I_{xz} = \sum m_i x_i z_i$, $I_{yz} = \sum m_i y_i z_i$

Add it all up $\therefore$ $I_o = ma^2 \begin{bmatrix} \frac{4}{3}+8 & 0 & 2 \\ 0 & \frac{4}{3}+8 & -2 \\ 2 & -2 & \frac{4}{3}+8 \end{bmatrix} = \frac{ma^2}{3} \begin{bmatrix} 28 & 0 & 6 \\ 0 & 28 & -6 \\ 6 & -6 & 28 \end{bmatrix}$

(b) $\sum m_i x_i = 2m \times 0 + m(a+a+a-a) = 2ma$

$\sum m_i y_i = 2m \times 0 + m(-a+a+a+a) = 2ma$

$\sum m_i z_i = \underset{\text{(cube)}}{2m \times 0} + m(\underset{A}{-a} \underset{B}{-a} \underset{C}{+a} \underset{D}{+a}) = 0$

total mass $= 6m$

$\therefore x_G = \frac{a}{3}$

$y_G = \frac{a}{3}$

$z_G = 0$

(c) $I_o = I_G + 6m \begin{bmatrix} Y^2+Z^2 & -XY & -XZ \\ -XY & X^2+Z^2 & -\cancel{X}Z \\ -XZ & -YZ & X^2+Z^2 \end{bmatrix}$ with $X = -\frac{a}{3}$, $Y = -\frac{a}{3}$, $Z = 0$

$\therefore I_G = \frac{ma^2}{3} \begin{bmatrix} 28 & 0 & 6 \\ 0 & 28 & -6 \\ 6 & -6 & 28 \end{bmatrix} - \frac{6ma^2}{9} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{ma^2}{3} \begin{bmatrix} 26 & 2 & 6 \\ 2 & 26 & -6 \\ 6 & -6 & 24 \end{bmatrix}$

(note that keeping track of the signs is pretty important!)

(d) The little rotor produces $\underline{h} = J\omega \underline{k}$

Conservation of angular momentum means that the satellite has the opposite value of $\underline{h}$, with $\underline{h} = [I_G]\underline{\omega}_s$

we can find $\underline{\omega}_s = [I_G]^{-1} \begin{bmatrix} 0 \\ 0 \\ -J\omega \end{bmatrix}$ but it's a bit fiddly. It comes out as $\underline{\omega}_s = \dfrac{J\omega}{28Ma^2} \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$

if the question was about verifying the result, you could have used

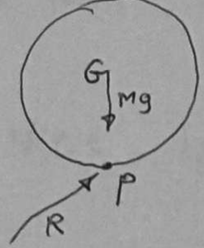
$$\underline{h} = [I_G]\underline{\omega}_s = \frac{Ma^2}{3} \begin{bmatrix} 26 & 2 & 6 \\ 2 & 26 & -6 \\ 6 & -6 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix} \times \frac{J\omega}{28Ma^2}$$

$$= \frac{J\omega}{84} \begin{bmatrix} 0 \\ 0 \\ -84 \end{bmatrix} = \underline{\underline{-J\omega \underline{k}}}$$

✓

so with this value of $\underline{\omega}_s$ the angular momentum is conserved

2 (a)



weight force passes through P

reaction at P

∴ net moment at P $= 0$

$Q_P = 0$

Use $\underline{Q}_P = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p}$ from data sheet

with $\underline{p} = m\dot{\underline{r}}_G$

and note that P is always below G

$\therefore \dot{r}_P = \dot{r}_G$ $\qquad \therefore \dot{r}_P \times \underline{p} = 0$

$\therefore \quad 0 = \dot{\underline{h}}_P + 0 \qquad \therefore \underline{h_P = \text{const}}$

(b) Data sheet $\underline{h}_P = \underline{h}_G + (\underline{r}_G - \underline{r}_P) \times \underline{p}$

$\underline{p} = m(\dot{x}\,\underline{i} + \dot{y}\,\underline{j}) \qquad \underline{r}_G - \underline{r}_P = a\,\underline{k}$

$\therefore \underline{h}_P = A(\omega_1\,\underline{i} + \omega_2\,\underline{j} + \omega_3\,\underline{k}) + ma\,\underline{k} \times (\dot{x}\,\underline{i} + \dot{y}\,\underline{j})$

with $A = \frac{2}{3}ma^2$ for a hollow sphere

$$\therefore \underline{h}_P = \left(\frac{2}{3}ma^2\omega_1 - ma\dot{y}\right)\underline{i} + \left(\frac{2}{3}ma^2\omega_2 + ma\dot{x}\right)\underline{j} + \frac{2}{3}ma^2\omega_3\,\underline{k}$$

$= \text{constant}$

$\therefore \frac{2}{3}a\omega_1 - \dot{y} = C_1$ ①

$\frac{2}{3}a\omega_2 + \dot{x} = C_2$ ②

and ω_3 is constant

(c) No slip at P means that the sphere and the table have the same velocity at P

For the table, $\underline{v}_P = \Omega \underline{k} \times (x\underline{i} + y\underline{j})$

$$= \Omega(x\underline{j} - y\underline{i}) \qquad (A)$$

For the sphere, $\underline{v}_P = \underline{v}_G + \underline{\omega} \times \underline{r}_{P/G}$

$$= \dot{x}\underline{i} + \dot{y}\underline{j} + (\omega_1\underline{i} + \omega_2\underline{j} + \omega_3\underline{k}) \times (-a\underline{k})$$

$$= (\dot{x} - a\omega_2)\underline{i} + (\dot{y} + a\omega_1)\underline{j} \qquad (B)$$

No slip: (A) = (B) $\therefore$ $\boxed{\begin{aligned} \dot{x} - a\omega_2 + \Omega y &= 0 \\ \dot{y} + a\omega_1 - \Omega x &= 0 \end{aligned}}$ $\begin{aligned} &(3) \\ &(4) \end{aligned}$

d/ from (3) $a\omega_2 = \dot{x} + \Omega y$

& (4) $a\omega_1 = -\dot{y} + \Omega x$

substitute into (1) $\frac{2}{3}(-\dot{y} + \Omega x) - \dot{y} = C_1$

& (2) $\frac{2}{3}(\dot{x} + \Omega y) + \dot{x} = C_2$

$$\therefore \left.\begin{aligned} -5\dot{y} + 2x\Omega &= 3C_1 \\ 5\dot{x} + 2y\Omega &= 3C_2 \end{aligned}\right\} \begin{aligned} &(1a) \\ &(2a) \end{aligned}$$

Differentiate (2a) and substitute $\dot{x}$ from (2b)

$$-5\ddot{y} + 2\dot{x}\Omega = 0 \qquad \text{with } \dot{x} = -\frac{2}{5}y\Omega + \frac{3}{5}C_2$$

$$\therefore \ddot{y} + \left(\frac{2}{5}\Omega\right)^2 y = \text{constant}, \qquad (5)$$

Similarly differentiate (2b) and substitute $\dot{y}$ from (2a)

$$\therefore \ddot{x} + \left(\frac{2}{5}\Omega\right)^2 x = \text{constant}_2 \qquad (6)$$

(5) gives (say) $y = R\sin\frac{2}{5}\Omega t + y_0$

and (6) gives $x = R\cos\frac{2}{5}\Omega t + x_0$

note that from (1a) & (1b) y and x must be sin and cos

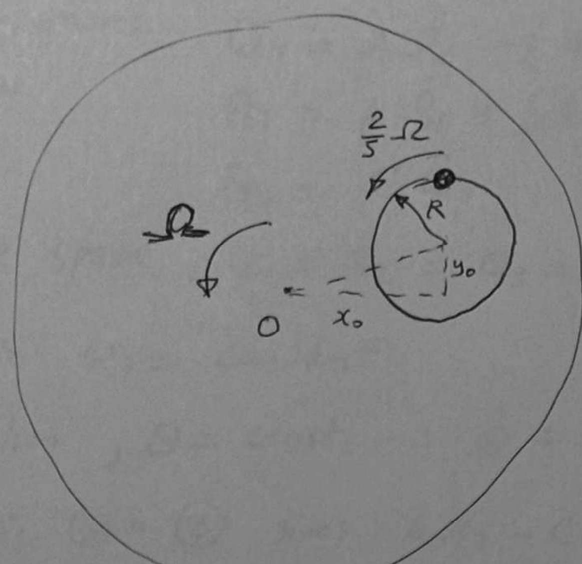
~~Now $x^2+y^2=$~~

Now $(x-x_0)^2+(y-y_0)^2 = R^2$

So the motion is a circle of radius R centred on (x_0, y_0)

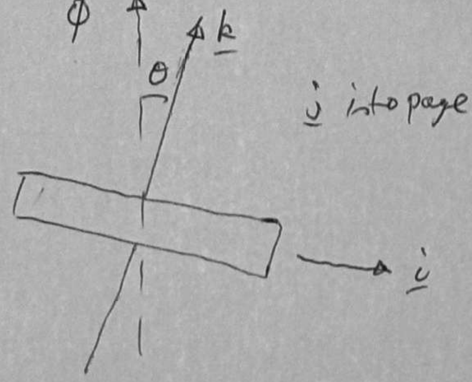
The frequency of the motion is $\frac{2}{5}\Omega$

so the period is $\frac{5}{2}$ times the ~~orbital~~ rotation period of the table



3 (a) The triangle is an AAC body with

$C = 2A$ by perpendicular axis Theorem



Euler angles

$\omega_1 = \Omega_1 = -\dot{\phi}\sin\theta$

$\omega_2 = \Omega_2 = \dot{\theta}$

$\Omega_3 = \dot{\phi}\cos\theta$

$\omega_3 = \text{spin}$

Gyro equations
Data Sheet

$Q_1 = A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2$ ①

$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$ ②

$Q_3 = C\dot{\omega}_3$ ③

In free space, $Q_1 = Q_2 = Q_3 = 0$

③ gives $\omega_3 =$ constant

For nutation, $\theta =$ const $\therefore \dot{\theta} = 0 \therefore \Omega_2 = 0$

$\therefore$ Gyro ② gives $A\Omega_3 - C\omega_3 = 0$

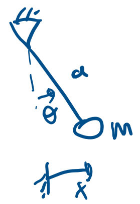
$\therefore A\dot{\phi}\cos\theta - C\omega_3 = 0$

For small θ $\dot{\phi} = \frac{C}{A}\omega_3 = 2\omega_3$

This is the wobble rate and is nutation

Note that this result holds for any AAC lamina so the mass and size of the triangle don't matter

AC685, Q3b



a) $q = \theta$

$$T = \frac{1}{2} m (a\dot\theta)^2 = \frac{1}{2} J \dot\theta^2$$

with $J = ma^2$

$$V = mga(1-\cos\theta)$$
$$\simeq mga\,\frac{1}{2}\theta^2$$

$$L = T - V = \frac{1}{2} ma^2\dot\theta^2 - mga\,\frac{1}{2}\theta^2$$

b) $H = p_\theta \dot\theta - T + V$

$$p_\theta = \frac{\partial T}{\partial \dot\theta} = \frac{1}{\cancel{2}} ma^2 \cancel{2}\dot\theta \Rightarrow \dot\theta = \frac{p_\theta}{ma^2}$$

$$H = \frac{p_\theta^2}{ma^2} - \frac{1}{2}\frac{p_\theta^2}{ma^2} + mga\frac{1}{2}\theta^2$$
$$= \frac{1}{2}\frac{p_\theta^2}{ma^2} + mga\frac{1}{2}\theta^2$$

c) $\frac{\partial H}{\partial p_\theta} = \dot\theta \rightarrow \frac{p_\theta}{ma^2} = \dot\theta$

$\frac{\partial H}{\partial \theta} = -\dot p_\theta \rightarrow mga\theta = -\dot p_\theta$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}\theta \\ p_\theta\end{bmatrix} = \begin{bmatrix}\frac{p_\theta}{ma^2} \\ -mga\theta\end{bmatrix}$$

d) $\theta \rightarrow x$

$x = a\sin\theta \xrightarrow{\text{small angle}} x = a\theta$

From $(\theta, p_\theta) \rightarrow$ to (X, P_X)

$\theta = \frac{x}{a}$ the simplest transformation

$G_2(q,p,t) = G_2(\theta, P_x, t) = P_x\left(\frac{x}{a}\right) = P_x\theta \Rightarrow$ given

Convert T in new coordinate

$$T = \frac{1}{2} ma^2\left(\frac{\dot x}{a}\right)^2 = \frac{1}{2} m\dot x^2$$

$$P_x = \frac{\partial T}{\partial \dot x} = m\dot x$$

to find $\rightarrow P_x$

by using the expressions in the datasheet

$$p_\theta = \frac{\partial G_2}{\partial \theta} = P_x$$
$$Q = \frac{\partial G_2}{\partial P_x} = x/a = \theta$$

these confirm that Hamilton's Equation will hold

$\Rightarrow H = K + \cancel{\frac{\partial H}{\partial t}} = 0 \Rightarrow K = \frac{1}{2}\frac{P_x^2}{m} + \frac{1}{2} m g \cancel{a} \frac{x^2}{a^{\cancel{2}}}$

Hamiltonian in the new coordinate system

APPROACH (A)

$$H = \frac{1}{2}\frac{P_\theta^2}{ma^2} + mga\,\frac{1}{2}\theta^2$$

$$P_\theta = ma^2\dot\theta$$

$$H = \frac{1}{2}\frac{(ma^2\dot\theta)^2}{ma^2} + mga\,\frac{1}{2}\theta^2$$

$$\theta = x/a$$

$$\Rightarrow K = \frac{1}{2}ma^2\left(\frac{\dot x}{a}\right)^2 + mga\frac{1}{2}\left(\frac{x}{a}\right)^2$$

$$K = \frac{1}{2}m\dot x^2 + mg\,a\frac{1}{2}\left(\frac{x}{a}\right)^2$$

$$\Rightarrow K = \frac{1}{2}\frac{p_x^2}{m} + mga\frac{1}{2}\left(\frac{x}{a}\right)^2$$

APPROACH (B)

or form T in new coordinate

$$T = \frac{1}{2}ma^2\dot\theta^2 = \frac{1}{2}m\cancel{a^2}\frac{\dot x^2}{\cancel{a^2}} = \frac{1}{2}m\dot x^2$$

$$T = \frac{1}{2}m\dot x^2 \qquad V = \frac{1}{2}mag\left(\frac{x}{a}\right)^2$$

$$K = T + V \qquad P_x = m\dot x$$

$$K = \frac{1}{2}\frac{P_x^2}{m} + \frac{1}{2}mag\left(\frac{x}{a}\right)^2$$

APPROACH C

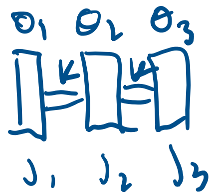
Alternatively, considering

$$P_x = P_\theta \Rightarrow ma^2\dot\theta = m\dot x$$ only true if

$$\frac{1}{2}\frac{P_\theta^2}{ma^2} \Rightarrow \frac{1}{2}\frac{P_x^2}{m}$$

$$K = \frac{1}{2}\frac{P_x^2}{m} + \frac{1}{2}amg\left(\frac{x}{a}\right)^2$$

AC685, Q4



a) $q_1 = \theta_1 \quad q_2 = \theta_2 \quad q_3 = \theta_3$

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} J_3 \dot{\theta}_3^2$$

$$V = \frac{1}{2} K (\theta_1 - \theta_2)^2 + \frac{1}{2} (\theta_2 - \theta_3)^2$$

$$L = T - V$$

$$L = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} J_3 \dot{\theta}_3^2 + - \frac{1}{2} K (\theta_1 - \theta_2)^2 - \frac{1}{2} (\theta_2 - \theta_3)^2$$

b) $$T = \frac{1}{2} \underline{\dot{q}}^T \underline{\underline{M}} \underline{\dot{q}}$$

$$\underline{\underline{M}} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

$$V = \frac{1}{2} \underline{q}^T \underline{\underline{K}} \underline{q}$$

$$\underline{\underline{K}} = \begin{bmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{bmatrix}$$

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} +$$

$$+ \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

c) $H(\theta_1, \theta_2, \theta_3, p_{\theta_1}, p_{\theta_2}, p_{\theta_3}) =$

$$= \sum_{j=1}^{3} p_{\theta_j} \dot{\theta}_j - T + V$$

$$p_{\theta_i} = \frac{\partial T}{\partial \dot{\theta}_i} = J_i \dot{\theta}_i \Rightarrow \dot{\theta}_i = \frac{p_{\theta_i}}{J_i}$$

$$H = \frac{1}{2}\frac{p_{\theta_1}^2}{J_1} + \frac{1}{2}\frac{p_{\theta_2}^2}{J_2} + \frac{1}{2}\frac{p_{\theta_3}^2}{J_3} + \frac{1}{2}k(\theta_1-\theta_2)^2 + \frac{1}{2}k(\theta_2-\theta_3)^2$$

d)

$$P_{TOT} = P_1 + P_2 + P_3 = J_1\dot{\theta}_1 + J_2\dot{\theta}_2 + J_3\dot{\theta}_3$$

$$\{P_{TOT}, H\} = \underbrace{\frac{\partial P_{TOT}}{\partial \theta_1}}_{=0}\frac{\partial H}{\partial p_{\theta_1}} - \underbrace{\frac{\partial P_{TOT}}{\partial p_{\theta_1}}}_{=1}\underbrace{\frac{\partial H}{\partial \theta_1}}_{=k(\theta_1-\theta_2)} +$$

$$+ \underbrace{\frac{\partial P_{TOT}}{\partial \theta_2}}_{=0}\frac{\partial H}{\partial p_{\theta_2}} - \underbrace{\frac{\partial P_{TOT}}{\partial p_{\theta_2}}}_{=1}\underbrace{\frac{\partial H}{\partial \theta_2}}_{=k(2\theta_2-\theta_1-\theta_3)}$$

$$+ \underbrace{\frac{\partial P_{TOT}}{\partial \theta_3}}_{=0}\frac{\partial H}{\partial p_{\theta_3}} - \underbrace{\frac{\partial P_{TOT}}{\partial p_{\theta_3}}}_{=1}\underbrace{\frac{\partial H}{\partial \theta_3}}_{=k(\theta_3-\theta_2)}$$

$$= K(\theta_1 - \theta_2) + K(2\theta_2 - \theta_1 - \theta_3) + K(\theta_3 - \theta_2) = 0$$

However, the

Total momentum is <u>NOT</u> conserved.

This is because there are two external generalized damping torques applied to the system

e)

$$\frac{d}{dt} f(\underline{p}, \underline{q}, t) = \frac{\partial f}{\partial t} + \{f, H\}$$

$$\rightarrow \frac{d}{dt} H = \frac{\partial H}{\partial t} + \{H, H\}$$

$$\{H, H\} = 0 \quad \text{and} \quad \frac{\partial H}{\partial t} = 0$$

since in H there is <u>NOT</u> an explicit time-dependence.

H is conserved.

However, this <u>CANNOT</u> be used as a statement of Energy conservation.

This is because of the generalized damping forces applied on the system