

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 29 April 2026 9.30 to 11.10

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Attachment: 3C5 Dynamics and 3C6 Vibration datasheet 2023 (7 pages).

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A satellite is modelled as a solid cube of mass $2m$ and side $2a$ as shown in Fig. 1. The edges of the cube are parallel to a cartesian reference frame x, y, z with its origin at the centre of the cube. Four small particles A, B, C and D each of mass m are fixed to the cube at $(a, -a, -a)$, $(a, a, -a)$, (a, a, a) and $(-a, a, a)$ respectively as shown. The rigid assembly has total mass $6m$.

(a) Find the inertia matrix at the origin of the cartesian reference frame. [30%]

(b) Find the location of the centre of mass G. [10%]

(c) Show that the inertia matrix at G is

$$\frac{ma^2}{3} \begin{bmatrix} 26 & 2 & 6 \\ * & 26 & -6 \\ * & * & 24 \end{bmatrix}$$

and identify the missing elements *. [20%]

(d) A small rotor of negligible mass is fixed in the satellite with its spin axis parallel to the z axis. The moment of inertia of the rotor about its spin axis is J where $J \ll ma^2$. The rotor and the satellite are initially at rest. The rotor is quickly driven to a high spin speed ω using a small electric motor fixed to the satellite. Use conservation of angular momentum to show that the final angular velocity of the satellite is $\omega_s = \frac{J\omega}{28ma^2}(\mathbf{i} - \mathbf{j} - 4\mathbf{k})$. [40%]

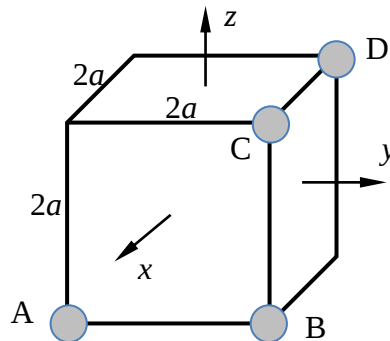


Fig. 1

2 A horizontal table is rotating at angular speed Ω about a fixed vertical shaft through its centre at O, as shown in Fig. 2. Rolling without slip on the surface of the table is a uniform hollow sphere of radius a and mass m . The components of the velocity of the centre of mass G of the sphere are $(\dot{x}, \dot{y}, \dot{z})$ in the fixed and *non-rotating* reference frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as shown in the figure. The shaft axis is parallel to $\mathbf{k}$. The angular velocity of the sphere is $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$.

Point P is the point of contact between sphere and table and the position vector OP is $x\mathbf{i} + y\mathbf{j}$.

- (a) Draw a free-body diagram showing all forces that act on the sphere and show that the moment of momentum about P of the sphere $\mathbf{h}_P$ is a constant. [15%]
- (b) Find an expression for $\mathbf{h}_P$ and hence show that $\frac{2}{3}a\omega_1 - \dot{y} = C_1$ and $\frac{2}{3}a\omega_2 + \dot{x} = C_2$ where C_1 and C_2 are constants. What can be said about ω_3 ? [25%]
- (c) Use a no-slip condition at P to show that $\dot{x} - a\omega_2 + y\Omega = 0$ and $\dot{y} + a\omega_1 - x\Omega = 0$. [25%]
- (d) Combine the results of (b) and (c) to show that the sphere moves in a space-fixed circle. Show that it completes one complete orbit in the same time that it takes the table to turn 2.5 times. [35%]

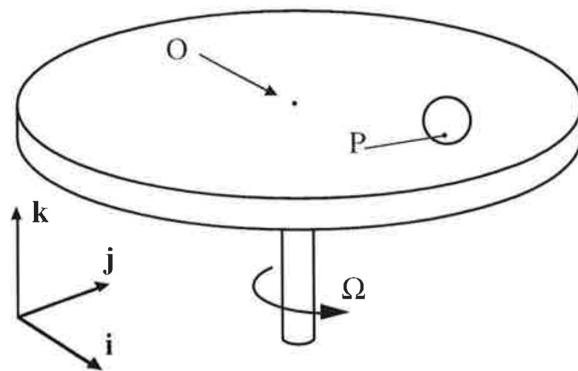


Fig. 2

3 (a) A rigid equilateral triangular plate of mass m and side a is spinning in free space at angular velocity ω about an axis perpendicular to its surface as shown in Fig. 3(a). At a certain instant a small impulse is delivered normal to the surface which induces small-amplitude nutation.

(i) Use the Gyroscope Equations or otherwise to find an expression for the frequency of nutation. [40%]

(ii) How does the nutation frequency depend on the dimension a and the mass m ? [10%]

(b) A mass m is mounted with a pin joint through an inextensible link AB of length a , as shown in Fig. 3(b). Motion is described by the small angle θ .

(i) Evaluate the Lagrangian of the system. [10%]

(ii) Evaluate the Hamiltonian of the system. [10%]

(iii) Find Hamilton's equations. [10%]

(iv) An alternative generalised coordinate x is used to describe the translation of the mass. Find the Hamiltonian $K(P_x, x)$ of the system in the new coordinate where the motion is described by the small displacement x , by using a Type 2 canonical transformation:

$$G_2(q, P, t) = P_x \frac{x}{a}$$

and the corresponding expressions in the Datasheet table of Canonical Transforms. [20%]

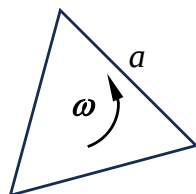


Fig. 3(a)

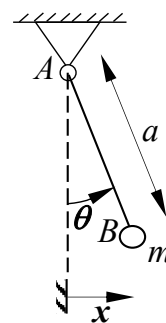


Fig. 3(b)

4 A simplified model of a jet engine is shown in Fig. 4. Three rigid discs with moments of inertia J_1 , J_2 and J_3 are mounted on two light elastic shafts each of torsional stiffness k . The angles of rotations of the discs from their equilibrium positions are θ_1 , θ_2 and θ_3 , as shown.

- (a) Evaluate the Lagrangian of the system. [10%]
- (b) Derive the three equations of motion of this system when each disc is subject to an external torque, such that the generalised force Q_j associated with each generalised coordinate q_j can be expressed as $Q_1 = -c_1\dot{\theta}_1$ and $Q_2 = -c_2\dot{\theta}_2$, and $Q_3 = 0$, with c_1 and c_2 being constant terms. [25%]
- (c) Find the Hamiltonian $H(p_{\theta_1}, p_{\theta_2}, p_{\theta_3}, \theta_1, \theta_2, \theta_3)$ of this system. [25%]
- (d) Use Poisson brackets to determine whether the the total momentum of the system is conserved. Justify your answer on physical grounds, considering the presence of the generalised forces. [20%]
- (e) Use Poisson brackets to determine whether the Hamiltonian of the system is conserved. Justify your answer on physical grounds, considering the presence of the generalised forces. [20%]

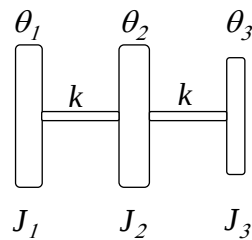


Fig. 4

END OF PAPER

THIS PAGE IS BLANK

Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i :

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}^{(e)}$$

where $\mathbf{p} = M\mathbf{v}_G$, M is the total mass, $\mathbf{v}_G$ is the velocity of the centre of mass and $\mathbf{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\begin{aligned}\mathbf{Q}^{(e)} &= (\mathbf{r}_G - \mathbf{r}_P) \times \dot{\mathbf{p}} + \dot{\mathbf{h}}_G \\ &= \dot{\mathbf{h}}_P + \dot{\mathbf{r}}_P \times \mathbf{p}\end{aligned}$$

where $\mathbf{Q}^{(e)}$ is the total moment of external forces about P. Here $\mathbf{h}_P$ and $\mathbf{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\begin{aligned}\mathbf{h}_P &= \sum_i (\mathbf{r}_i - \mathbf{r}_P) \times m_i \dot{\mathbf{r}}_i \\ &= \mathbf{h}_G + (\mathbf{r}_G - \mathbf{r}_P) \times \mathbf{p}\end{aligned}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\mathbf{h}_P = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dm = \mathbf{I}\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\mathbf{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned}\text{and } A &= \int (y^2 + z^2) dm & B &= \int (z^2 + x^2) dm & C &= \int (x^2 + y^2) dm \\ D &= \int yz \, dm & E &= \int zx \, dm & F &= \int xy \, dm\end{aligned}$$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity Ω

Time derivatives of vectors must be replaced by the “rotating frame” form, so that for example

$$\dot{\mathbf{p}} + \Omega \times \mathbf{p} = \mathbf{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\mathbf{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler’s dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = Q_1$$

$$B\dot{\omega}_2 - (C - A)\omega_3\omega_1 = Q_2$$

$$C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = Q_3$$

where A , B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\mathbf{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the “Gyroscope equations”):

$$A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 = Q_1$$

$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2$$

$$C\dot{\omega}_3 = Q_3$$

where A , A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\mathbf{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange’s equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_i} \right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalised momenta p_i and the Hamiltonian $H(\mathbf{p}, \mathbf{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \quad H(\mathbf{p}, \mathbf{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\mathbf{p}, \mathbf{q})$ must be expressed as a function of the generalised momenta and the generalised displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

Special case when the kinetic energy is expressible using a mass matrix $\mathbf{M}$:

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} = \frac{1}{2} \mathbf{p}^T \mathbf{M}^{-1} \mathbf{p} \quad \text{and} \quad H = T + V$$

(b) Extension topics

The total time derivative of some function $f(\mathbf{p}, \mathbf{q}, t)$ can be expressed in terms of the Poisson bracket $\{f, H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}, \quad \{f, H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\mathbf{q}, \mathbf{Q}, t)$	$\mathbf{p} = \frac{\partial G_1}{\partial \mathbf{q}}$	$\mathbf{P} = -\frac{\partial G_1}{\partial \mathbf{Q}}$	$K = H + \frac{\partial G_1}{\partial t}$
2	$G_2(\mathbf{q}, \mathbf{P}, t)$	$\mathbf{p} = \frac{\partial G_2}{\partial \mathbf{q}}$	$\mathbf{Q} = \frac{\partial G_2}{\partial \mathbf{P}}$	$K = H + \frac{\partial G_2}{\partial t}$
3	$G_3(\mathbf{p}, \mathbf{Q}, t)$	$\mathbf{q} = -\frac{\partial G_3}{\partial \mathbf{p}}$	$\mathbf{P} = -\frac{\partial G_3}{\partial \mathbf{Q}}$	$K = H + \frac{\partial G_3}{\partial t}$
4	$G_4(\mathbf{p}, \mathbf{P}, t)$	$\mathbf{q} = -\frac{\partial G_4}{\partial \mathbf{p}}$	$\mathbf{Q} = \frac{\partial G_4}{\partial \mathbf{P}}$	$K = H + \frac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems

1. Equation of motion

The forced vibration of an N -degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

2. Kinetic Energy

$$T = \frac{1}{2} \dot{\mathbf{y}}^T \mathbf{M} \dot{\mathbf{y}}$$

3. Potential Energy

$$V = \frac{1}{2} \mathbf{y}^T \mathbf{K} \mathbf{y}$$

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2 \mathbf{M}\mathbf{u}^{(n)}$$

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)T} \mathbf{M} \mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)T} \mathbf{K} \mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^N q_j(t) \mathbf{u}^{(j)} = \mathbf{U} \mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the ‘quantity’ of the j th mode.

Continuous Systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

$$T = \frac{1}{2} \int \dot{y}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See Section 3 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x) u_k(x) dm = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses:

$$y(x, t) = \sum_j q_j(t) u_j(x)$$

where $y(x, t)$ is the displacement and q_j can be thought of as the ‘quantity’ of the j th mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + [\text{diag}(\omega_j^2)] \mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T \mathbf{f}$.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k , the transfer function is

$$H(j, k, \omega) = \frac{y_k}{f_j} = \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j, k, \omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j, k, t) = y_k(t) = \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j, k, t) \approx \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x, t) u_j(x) dm$ and $f(x, t)$ is the external applied force distribution.

For force F at frequency ω applied at point x_1 , and displacement y measured at point x_2 , the transfer function is

$$H(x_1, x_2, \omega) = \frac{y}{F} = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1, x_2, \omega) = \frac{y}{F} \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1) u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

For a unit impulse applied at $t = 0$ at point x_1 , the response at point x_2 is

$$g(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j, k, t) = y_k(t) = \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} [1 - \cos \omega_n t]$$

for $t \geq 0$ (with no damping), or

$$h(j, k, t) \approx \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} [1 - e^{-\omega_n \zeta_n t} \cos \omega_n t]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1 , the response at point x_2 is

$$h(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} [1 - \cos \omega_n t]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} [1 - e^{-\omega_n \zeta_n t} \cos \omega_n t]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The “Rayleigh quotient” for a discrete system is

$$\frac{V}{\tilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be

- (1) $\geq$ the smallest squared natural frequency;
- (2) $\leq$ the largest squared natural frequency;
- (3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\frac{V}{\tilde{T}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P , mass per unit length m , transverse displacement $y(x, t)$, applied lateral force $f(x, t)$ per unit length.

Equation of motion

Potential energy

Kinetic energy

$$m \frac{\partial^2 y}{\partial t^2} - P \frac{\partial^2 y}{\partial x^2} = f(x, t)$$

$$V = \frac{1}{2} P \int \left(\frac{\partial y}{\partial x} \right)^2 dx$$

$$T = \frac{1}{2} m \int \left(\frac{\partial y}{\partial t} \right)^2 dx$$

3.2 Torsional vibration of a circular shaft

Shear modulus G , density ρ , external radius a , internal radius b if shaft is hollow, angular displacement $\theta(x, t)$, applied torque $\tau(x, t)$ per unit length. The polar moment of area is given by $J = (\pi/2)(a^4 - b^4)$.

Equation of motion

Potential energy

Kinetic energy

$$\rho J \frac{\partial^2 \theta}{\partial t^2} - G J \frac{\partial^2 \theta}{\partial x^2} = \tau(x, t)$$

$$V = \frac{1}{2} G J \int \left(\frac{\partial \theta}{\partial x} \right)^2 dx$$

$$T = \frac{1}{2} \rho J \int \left(\frac{\partial \theta}{\partial t} \right)^2 dx$$

3.3 Axial vibration of a rod or column

Young's modulus E , density ρ , cross-sectional area A , axial displacement $y(x, t)$, applied axial force $f(x, t)$ per unit length.

Equation of motion

Potential energy

Kinetic energy

$$\rho A \frac{\partial^2 y}{\partial t^2} - E A \frac{\partial^2 y}{\partial x^2} = f(x, t)$$

$$V = \frac{1}{2} E A \int \left(\frac{\partial y}{\partial x} \right)^2 dx$$

$$T = \frac{1}{2} \rho A \int \left(\frac{\partial y}{\partial t} \right)^2 dx$$

3.4 Bending vibration of an Euler beam

Young's modulus E , density ρ , cross-sectional area A , second moment of area of cross-section I , transverse displacement $y(x, t)$, applied transverse force $f(x, t)$ per unit length.

Equation of motion

Potential energy

Kinetic energy

$$\rho A \frac{\partial^2 y}{\partial t^2} + E I \frac{\partial^4 y}{\partial x^4} = f(x, t)$$

$$V = \frac{1}{2} E I \int \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx$$

$$T = \frac{1}{2} \rho A \int \left(\frac{\partial y}{\partial t} \right)^2 dx$$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned} \cos \alpha \cosh \alpha + 1 &= 0, & \alpha_1 &= 1.8751 \\ \cos \alpha \cosh \alpha - 1 &= 0, & \alpha_1 &= 4.7300 \\ \tan \alpha - \tanh \alpha &= 0, & \alpha_1 &= 3.9266 \end{aligned}$$