Question 1

1 (a) A vertical uniform column of length L has Young's modulus E, density ρ, and cross-sectional area A. The column is fixed at the top ($z = 0$) and bottom ($z = L$) as shown in Figure 1a. The effect of the column's self-weight on its axial deflection can be neglected for the whole of this question.

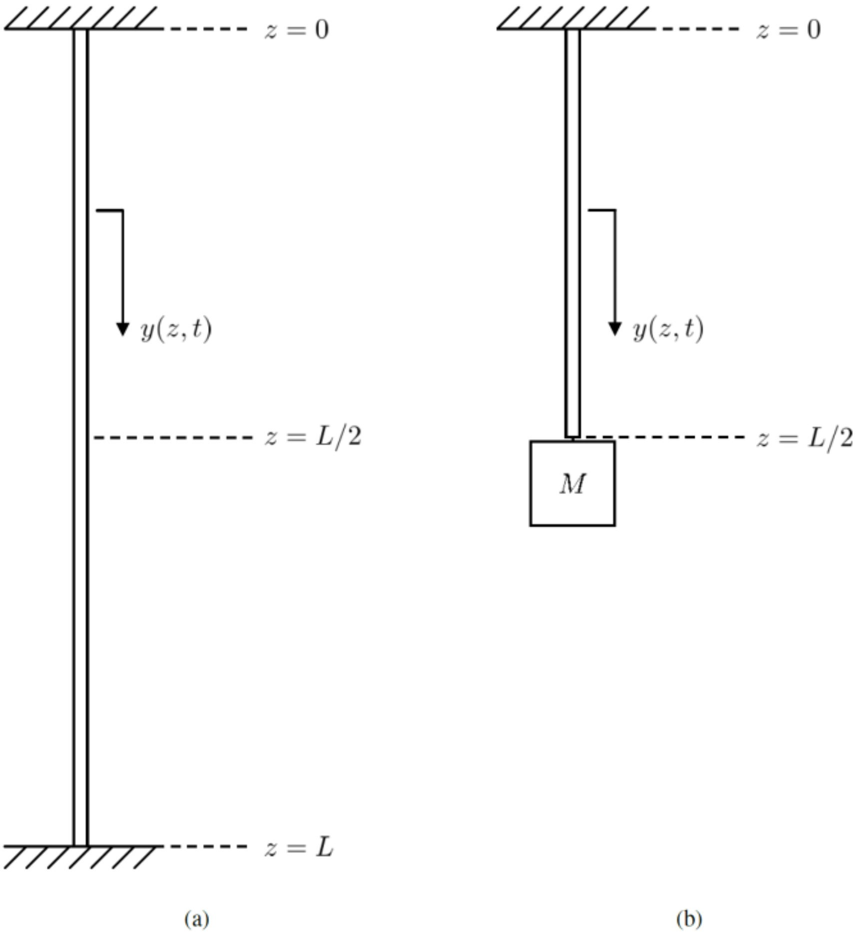
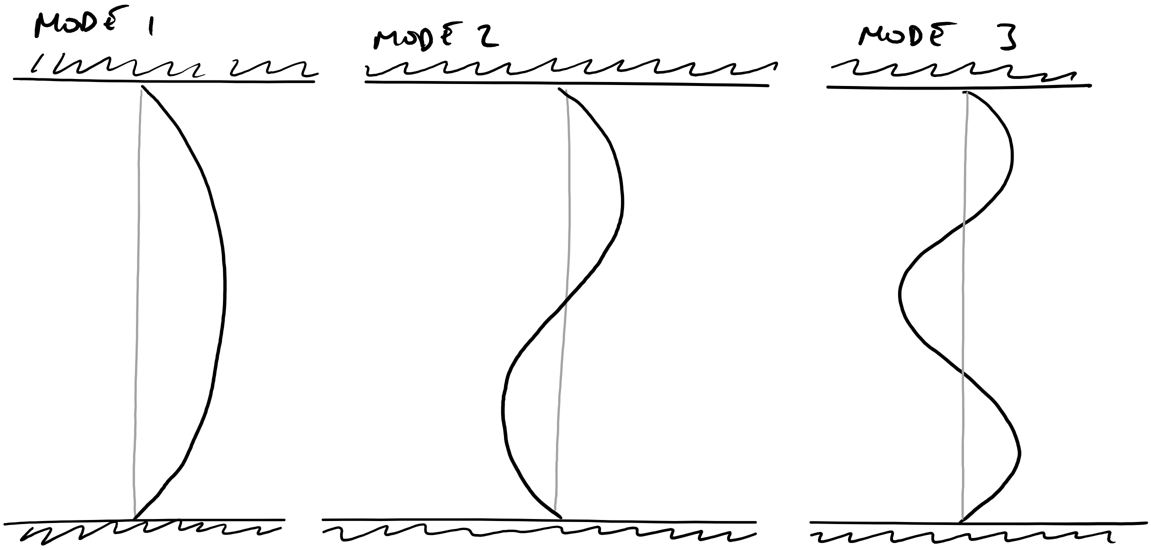


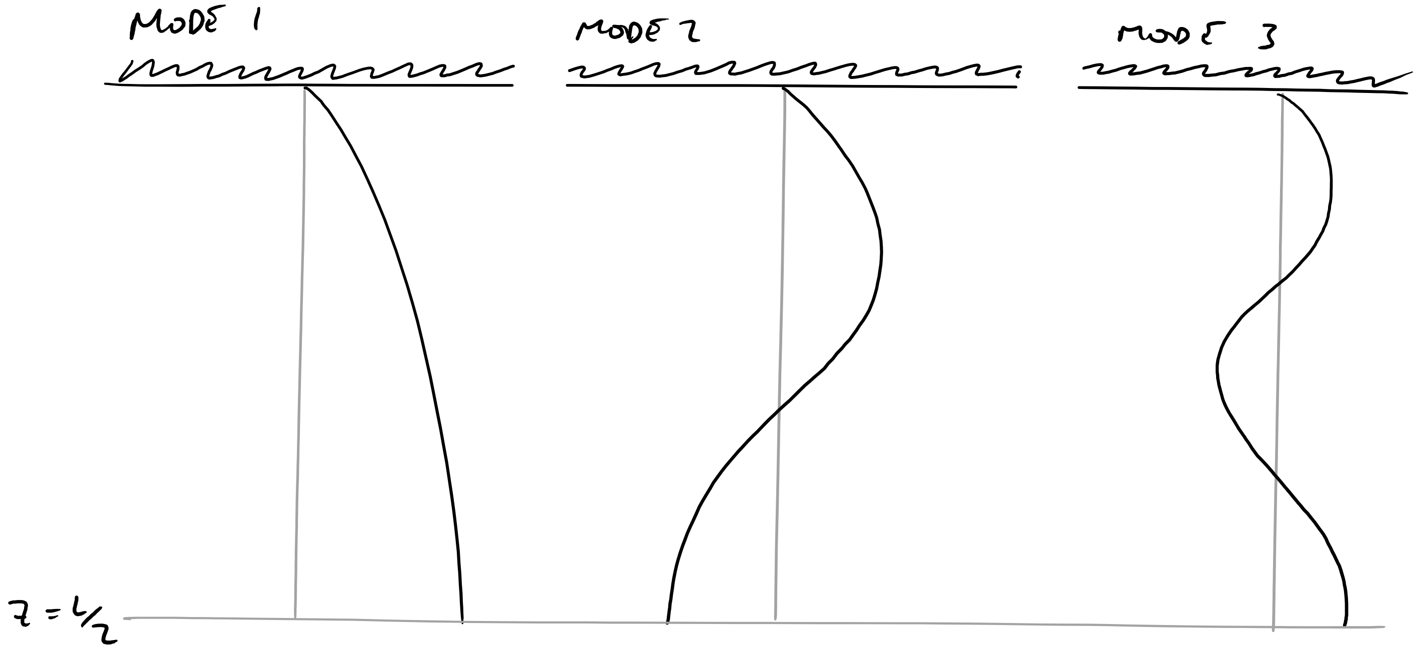
Fig. 1

(i) Write down an expression for the mode shapes of axial vibration of the column and sketch the first three mode shapes. [10%]

$$u_n(x) = A \sin \frac{n\pi x}{L}$$



(ii) A second column of half the length, but which is otherwise identical, is fixed at the top ($z = 0$) and is free at its end ($z = L/2$). Sketch the first three mode shapes of this column and identify how they relate to modes of the original system. [20%]



[note: $u_n(x) = \sin \dfrac{(n-\frac{1}{2})\pi x}{L}$ to match fixed-free boundary conditions]

Each mode of half-length bar has same frequency and mode shape for $0 < x < L/2$ as symmetric modes of full-length bar.

half-length	full-length
mode 1	mode 1
mode 2	mode 3
mode 3	mode 5

(b) The half-length column is used to support a mass M as shown in Figure 1b. The column and mass are initially at rest, when suddenly the connection between the column and the mass fails at time $t = 0$ such that the mass falls freely. The general solution $y(z,t)$ for axial vibration of a column can be written in terms of two components y_1 and y_2:

$$y(z,t) = y_1(z - ct) + y_2(z + ct)$$

(i) What is the physical interpretation of y_1 and y_2, and what is c in terms of the properties of the column? [10%]

y_1 : forward travelling wave

y_2 : backward travelling wave. //

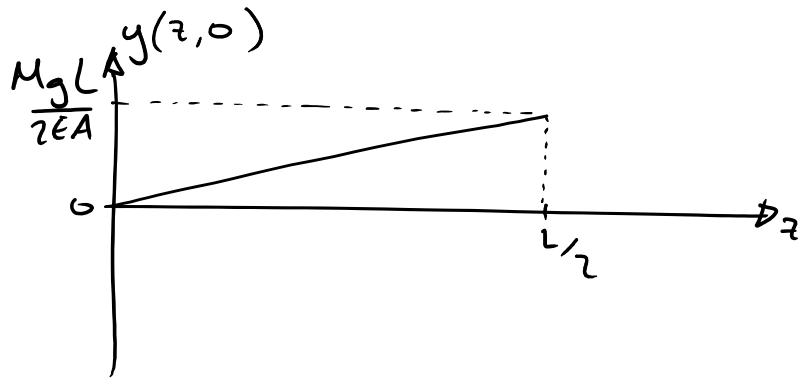
c = wavespeed, $c = \sqrt{E/\rho}$ //

(ii) What are the initial conditions $y(z,t)$ and $\dot{y}(z,t)$ of the half-length column at time $t = 0$? Note that the static axial stiffness of a column of length L is given by EA/L. [5%]

Statics: $F = kx$

$Mg = \frac{EA}{(L/2)} y(L/2, 0)$ so $y(L/2, 0) = \frac{MgL}{2EA}$

Initial displacement: $y_0(z,0) = \frac{Mg}{EA} z$ //



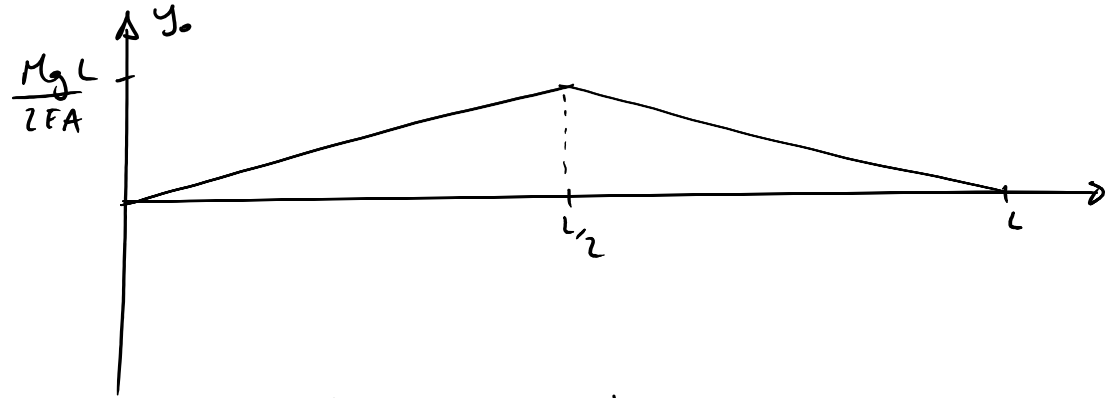
Initial velocity $\dot{y}_0(z,0) = 0$ /

(iii) If the initial conditions for the full-length column were identical to the half-length column for $0 < z < L/2$ and symmetrical about $z = L/2$, justify why the response of the two columns would be identical for $0 < z < L/2$. [10%]

- The mode shapes and natural frequencies for the half-length column are identical to the symmetric modes of the full-length column.
- Therefore the response of the full-length column to symmetric initial conditions will only involve symmetric modes.
- The contribution of these symmetric modes to the initial conditions will be identical to those of the half-length column.
- Therefore the response will be identical over $0 < z < L/2$.

(iv) By considering the equivalent initial conditions for the full-length column derive an expression for the initial functions $y_1(z)$ and $y_2(z)$ at time $t = 0$. [10%]

Full-length initial conditions:



$$y(z,t) = y_1(z-ct) + y_2(z+ct).$$

$$y(z,0) = y_1(z) + y_2(z) = y_0. \quad \text{———} \quad (1)$$

$$\dot{y}(z,0) = 0, \Rightarrow -cy_1' + cy_2' = 0.$$

$$\Rightarrow y_1' = y_2'$$

so $$y_1 = y_2 + K \quad \text{———} \quad (2).$$

(2) → (1) ⇒ $2y_2 + K = y_0$, so $y_2 = \dfrac{y_0 - K}{2}$

& $y_1 = \dfrac{y_0 + K}{2}$

can choose $K = 0$, so $y_1 = y_2 = y_0/2$

(v) Using the results above and by considering the boundary conditions of the equivalent full-length column at $z = 0$ and $z = L$ find conditions on y_1 and y_2 that determine the transient response. [20%]

Boundary Conditions: $y(0,t) = 0$
$y(L,t) = 0$.

$y_1(0-ct) + y_2(0+ct) = 0 \Rightarrow y_1(-ct) = -y_2(ct)$ — ③

& because initially $y_1 = y_2$
then $y_1(\alpha) = -y_1(-\alpha)$
so initially odd function //

$y_1(L-ct) + y_2(L+ct) = 0$

hence $y_1(L-ct) = -y_2(L+ct)$

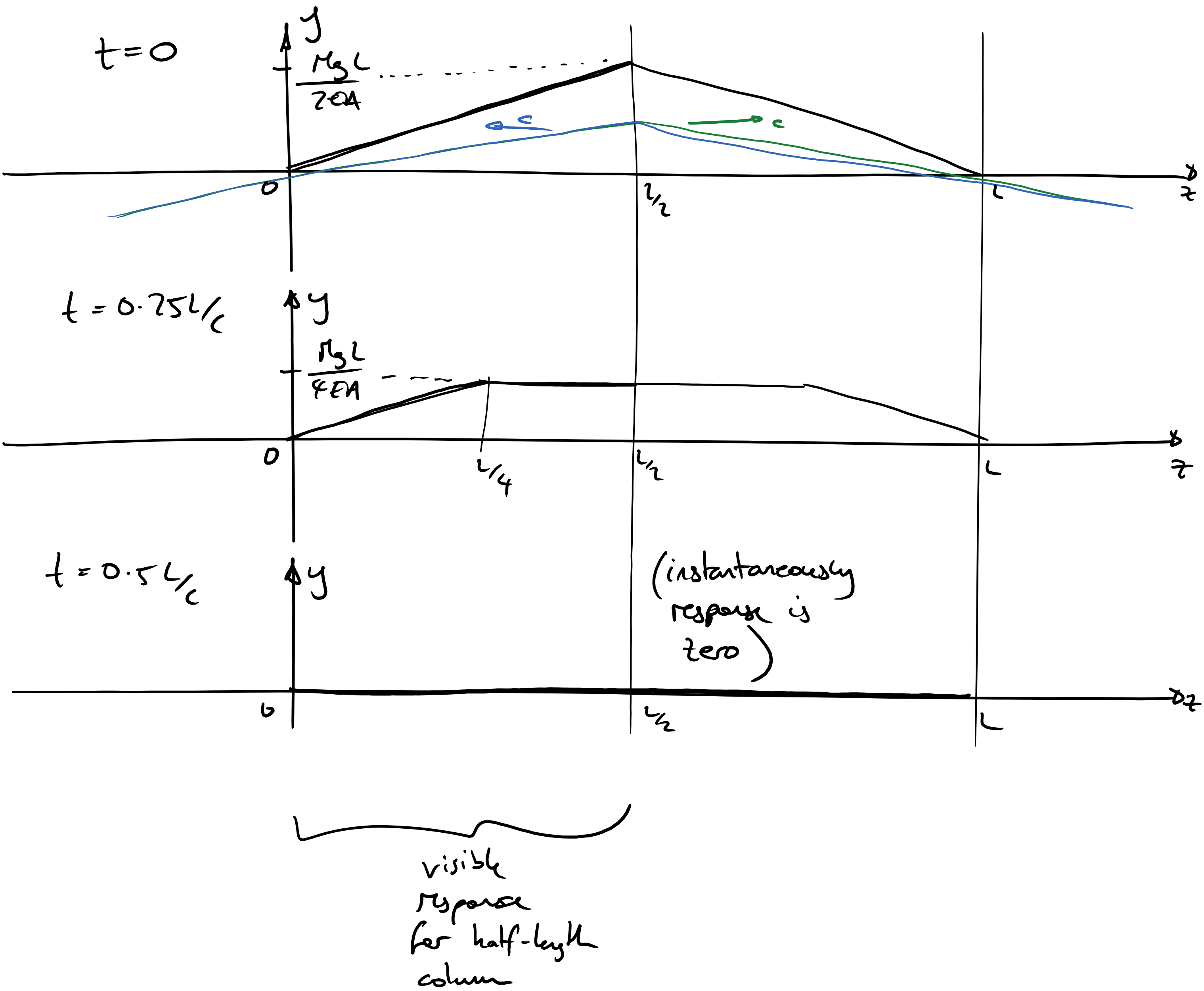
using ③ : $y_1(L-ct) = +y_1(-L-ct)$ so periodic over $2L$ //

(vi) Sketch the axial displacement $y(z)$ at times $t = 0$, $t = 0.25L/c$, and $t = 0.5L/c$ for the half-length column after the mass has been released. [15%]

$t = 0$: initial shape

$t = 0.25L/c$: time for axial wave to travel distance $0.25L$.

$t = 0.5L/c$: wave travels $0.5L$.


t=0
y
MgL / 2EA
0
L/2
L
z
c
c
t = 0.75L/c
MgL / 4EA
L/4
t = 0.5L/c
(instantaneously response is zero)
visible response for half-length column

Question 2

2 A beam of length L is shown in Figure 2(a). The beam is made from a material with Young's modulus E and density ρ, and the cross-section of the beam has a second moment of area I.

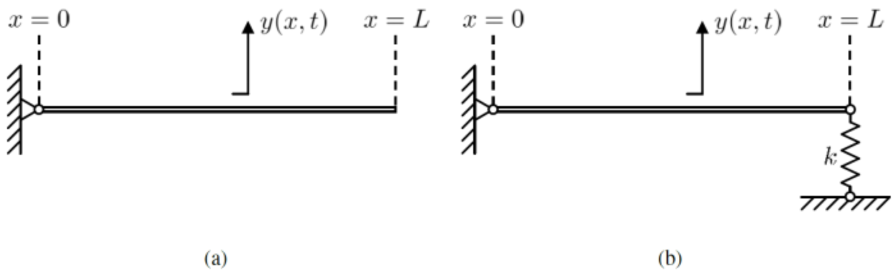


Fig. 2

(a) The beam is pinned at $x = 0$ and free at $x = L$, and the lateral deflection of the beam is $y = y(x,t)$.

(i) Starting from the equation of motion for a beam, derive an expression whose solutions give the wavenumbers k_n for the modes of the beam. [30%]

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0 .$$

let $y = u(x) e^{i\omega t}$

$$\Rightarrow \rho A \omega^2 u + EI u^{iv} = 0 .$$

$$u = D_1 \sin kx + D_2 \cos kx + D_3 \sinh kx + D_4 \cosh kx$$

$$u' = k(D_1 \cos kx - D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx)$$

$$u'' = k^2(-D_1 \sin kx - D_2 \cos kx + D_3 \sinh kx + D_4 \cosh kx)$$

$$u''' = k^3(-D_1 \cos kx + D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx)$$

BC's : $x = 0$: disp. $u(0) = 0 \Rightarrow D_2 + D_4 = 0$.

moment $u''(0) = 0 \rightarrow -D_2 + D_4 = 0$

$$\Rightarrow D_2 = D_4 = 0 \checkmark$$

$x = L$: $u''(L) = 0 \Rightarrow -D_1 \sin kL + D_3 \sinh kL = 0$

moment

$u'''(L) = 0 \Rightarrow -D_1 \cos kL + D_3 \cosh kL = 0$.

shear

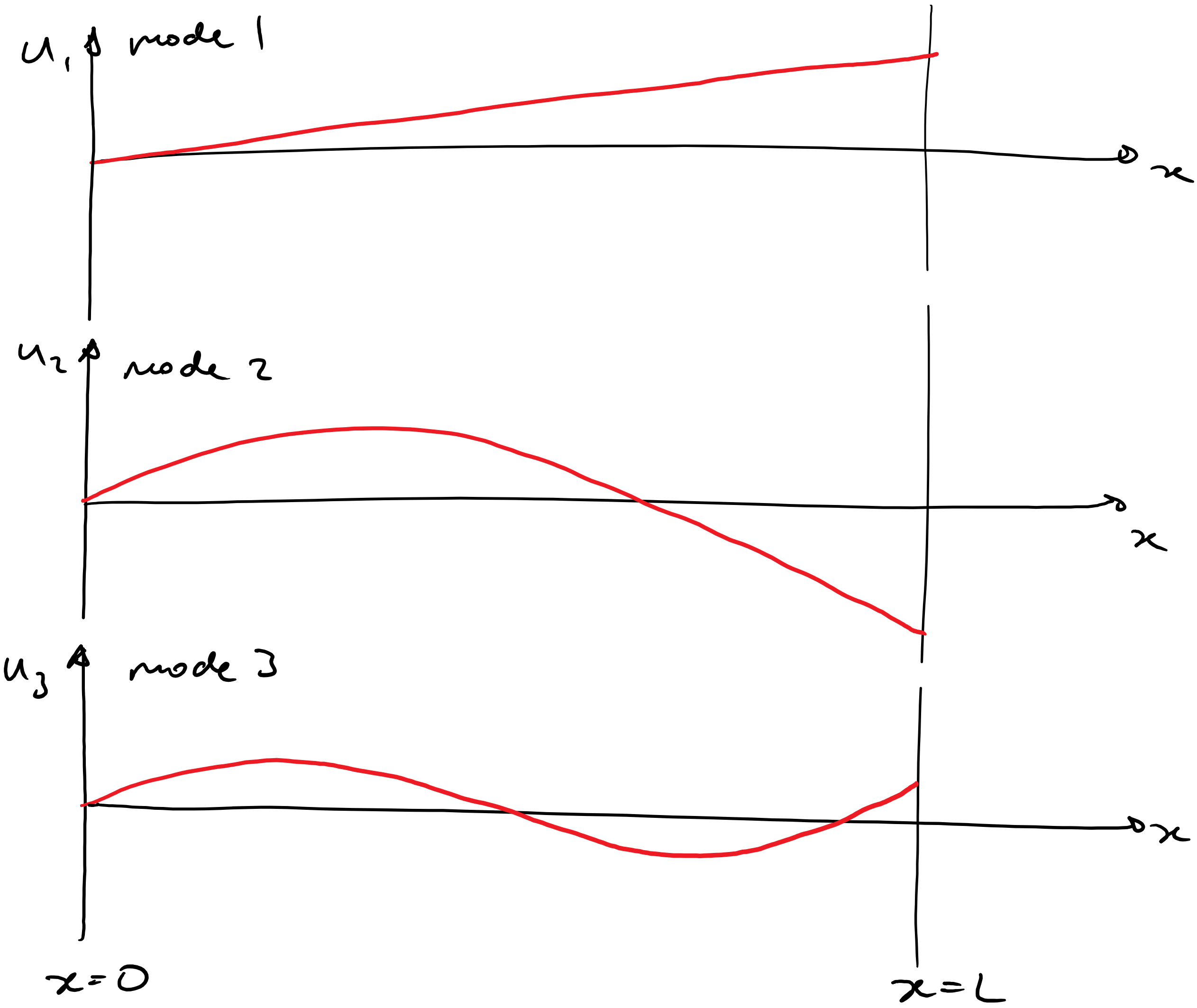
$$\begin{bmatrix} -\sin kL & \sinh kL \\ -\cos kL & \cosh kL \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = 0$$

non-trivial solution when $\begin{vmatrix} -\sin kL & \sinh kL \\ -\cos kL & \cosh kL \end{vmatrix} = 0$.

$$-\sin kL \cosh kL + \cos kL \sinh kL = 0.$$

$$\tan kL = \tanh kL$$

(ii) Sketch the mode shapes corresponding to the first three natural frequencies. [20%]



(b) A spring of stiffness k connects the same beam at $x = L$ to ground as shown in Figure 2(b).

(i) Using a transfer function approach derive an equation whose solutions give the natural frequencies of the modified system, in terms of the spring constant k, and the original mode shapes $u_n(x)$ and natural frequencies ω_n of the unmodified beam. [20%]

beam without spring :

$$G(L, L, \omega) = \sum_n \frac{u_n(L)\, u_n(L)}{\omega_n^2 - \omega^2}$$

$$Y_{coupled} = \frac{G/k}{1/k + G}$$

natural frequencies occur when $G + 1/k = 0$

ie when $\sum_n \frac{u_n(L) u_n(L)}{\omega_n^2 - \omega^2} + \frac{1}{k} = 0$

(ii) Using the function $u(x) = x$ as an estimate for the first mode shape, use Rayleigh's principle to derive an approximate expression for the first natural frequency of the combined system. Comment on the validity of the assumed mode shape function. [20%]

$u(x) = x$.

$$V = \frac{1}{2} EI \int_0^L \underbrace{[u''(x)]^2}_{0} dx + \frac{1}{2} k L^2 = \frac{1}{2} k L^2$$

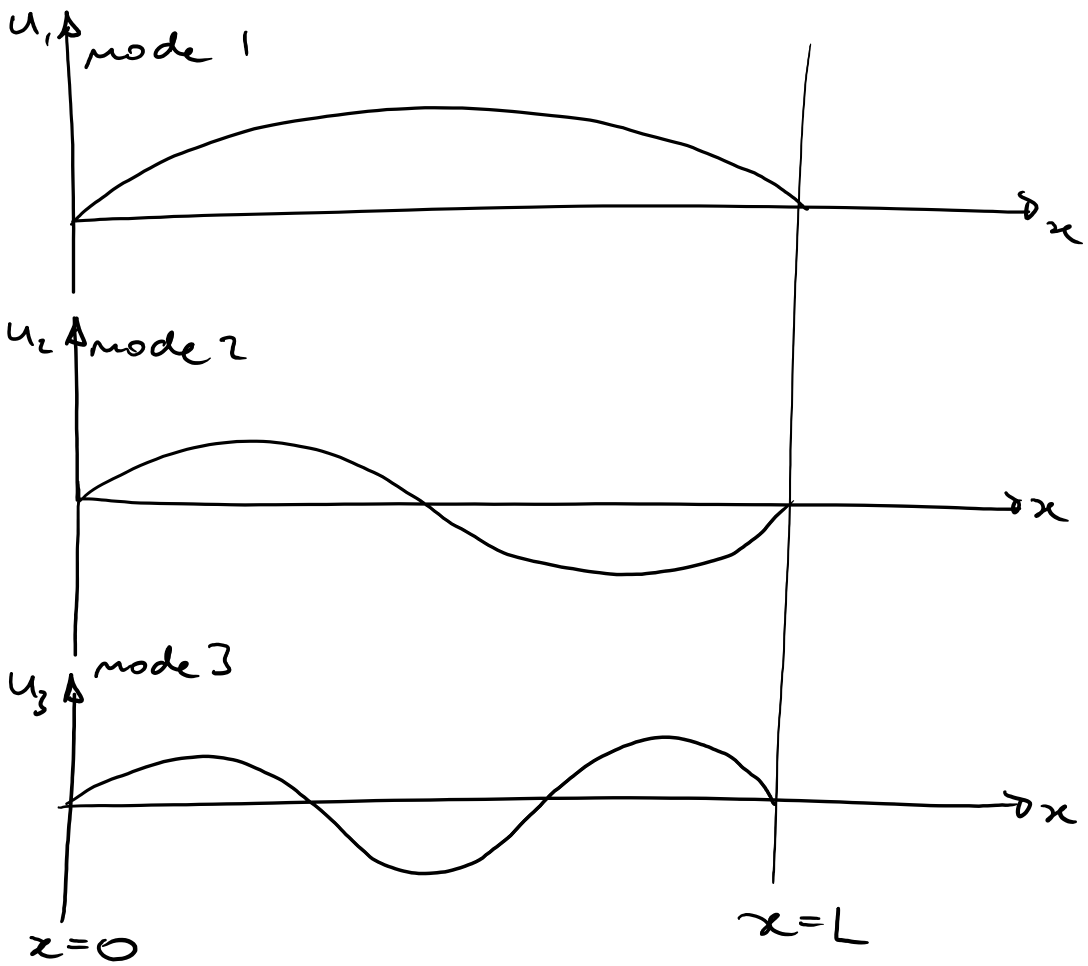
$$\tilde{T} = \frac{1}{2} \rho A \int_0^L [u(x)]^2 dx$$

$$= \frac{1}{2} \rho A \int_0^L x^2 dx = \frac{1}{2} \rho A \left[x^3/3 \right]_0^L = \rho A L^3 / 6$$

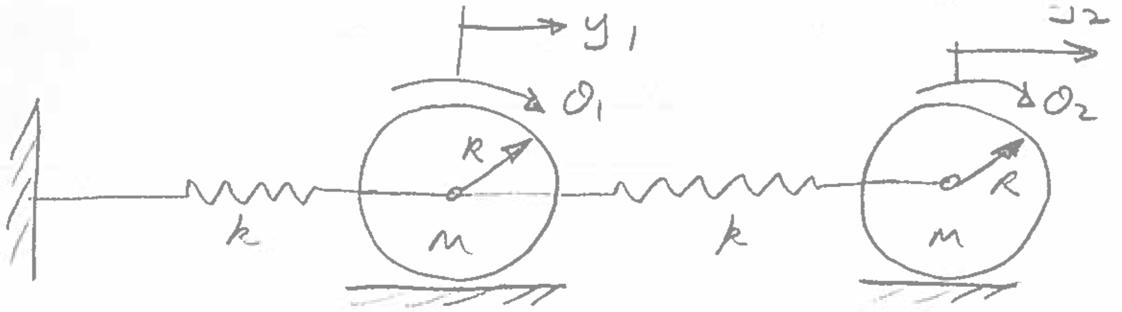
$$\omega_1^2 \simeq \frac{\frac{1}{2} k L^2}{\frac{1}{6} \rho A L^3} = \frac{3k}{\rho A L}$$

Accurate for small k, as $u(x) = x$ is rigid body mode shape.

(iii) Sketch the mode shapes for the first three modes of the modified system for the case when the stiffness is large, i.e. $k \to \infty$. [10%]



Q3



(a) $T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} J \dot{\theta}_1^2 + \frac{1}{2} m \dot{y}_2^2 + \frac{1}{2} J \dot{\theta}_2^2$

$J = mR^2/2$ & $\dot{y}_1 = R\dot{\theta}_1$, $\dot{y}_2 = R\dot{\theta}_2$

So $T = \frac{1}{2} m\dot{y}_1^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{\dot{y}_1}{R}\right)^2 + \frac{1}{2} m \dot{y}_2^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{\dot{y}_2}{R}\right)^2$

$T = \frac{1}{2}\left(\frac{3}{2} m \dot{y}_1^2 + \frac{3}{2} m \dot{y}_2^2\right) \Rightarrow [M] = \frac{3m}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$V = \frac{1}{2} k y_1^2 + \frac{1}{2} k (y_2 - y_1)^2$
$= \frac{1}{2} k (y_1^2 + y_2^2 - 2 y_2 y_1 + y_1^2)$
$= \frac{1}{2} k (2y_1^2 - 2y_1 y_2 + y_2^2)$ $\quad [K] = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(b) Eigenvalue problem is $\left([K] - \omega^2 [m]\right)\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = 0$

$$\begin{bmatrix} 2k - \omega^2 3m/2 & -k \\ -k & k - \omega^2 3m/2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = 0$$

$\underline{CE} \Rightarrow (2k - \omega^2 3m/2)(k - \omega^2 3m/2) - k^2 = 0$

$$k^2 - \omega^2 3m/2\,(2k + k) + \omega^4\, 9m^2/4 = 0$$

ie $\frac{9m^2}{4}\omega^4 - \frac{9k}{2} m\omega^2 + k^2 = 0$

$$\omega^2 = \frac{\frac{9km}{2} \pm \sqrt{\frac{81}{4} k^2 m^2 - 9m^2k^2}}{9m^2/2}$$

$$\omega^2 = \frac{\frac{3km}{2} \pm \frac{km\sqrt{5}}{2}}{3m^2/2} = \frac{k}{m} \pm \frac{\sqrt{5}}{3} k/m = 0.255\, k/m,\ 1.74$$

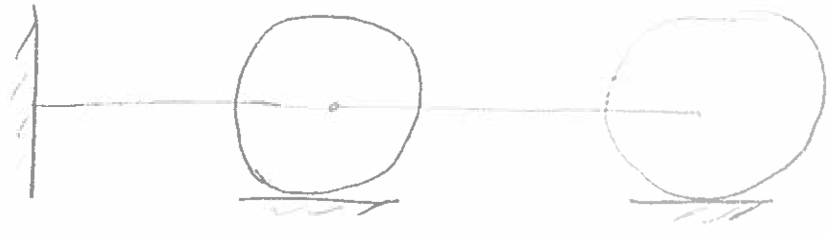
3 (Cont) Mode shapes

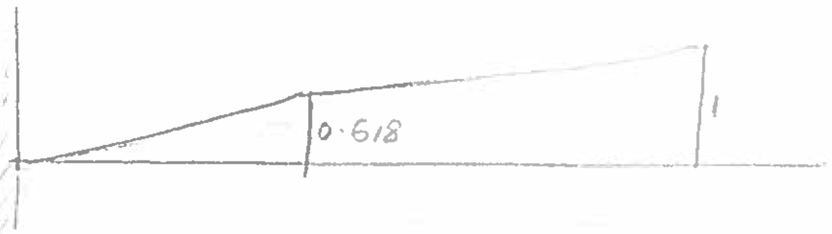
$$\left(2k - \omega^2 \frac{3m}{2}\right) y_1 - k y_2 = 0$$

$$\Rightarrow \frac{y_1}{y_2} = \frac{k}{2k - \omega^2 3m/2}$$

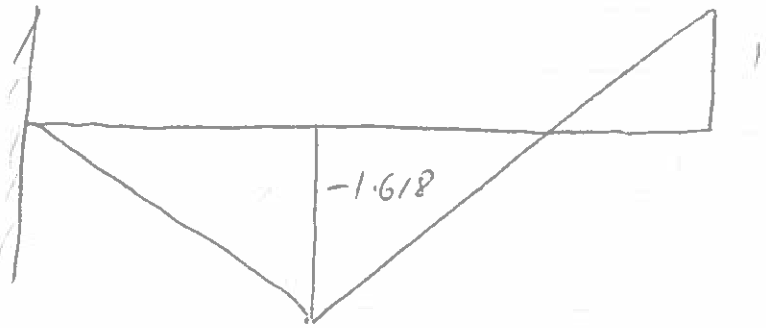
$$@\ \omega_1^2 = 0.255\ k/m, \quad y_1/y_2^{(1)} = \frac{1}{2 - \frac{3}{2}(0.255)} = 0.618$$

$$@\ \omega_2^2 = 1.745\ k/m \quad y_1/y_2^{(2)} = \frac{1}{2 - 3/2\,(1.745)} = -1.618$$





$@\ \omega_1^2 = 0.255\,k/m$



$@\ \omega_2^2 = 1.745\,k/m$

(c) Transient response

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix} (A \cos \omega_1 t + B \sin \omega_1 t) + \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} (C \cos \omega_2 t + D \sin \omega_2 t)$$

Initial Conds $\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} R\pi/4 \\ 0 \end{Bmatrix}$ @ $t = 0$.

Initial displacement $\Rightarrow$ Sin terms are zero $B = D = 0$.

3 (cont)

$$\begin{Bmatrix} R\pi/4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} C$$

$$\Rightarrow \quad R\pi/4 = 0.618A - 1.618C$$

$$0 = A + C \quad \Rightarrow \quad A = -C$$

So $R\pi/4 = 0.618A + 1.618A \Rightarrow A = 0.351R$

$C = -0.351R$

so $y_1 = 0.217R\cos\omega_1 t + 0.568R\cos\omega_2 t$

& $y_2 = 0.351R\cos\omega_1 t - 0.351R\cos\omega_2 t$

check $t=0$, $y_1 = 0.785R = \pi R/4$ ✓

$y_2 = 0$ ✓

At $t = \sqrt{\frac{m}{k}}$, $y_1 = 0.217R\cos\sqrt{0.255}$

$+ 0.568R\cos\sqrt{1.745}$

& $\theta_1 = y_1/R = \underline{\underline{0.33}}$ radians $(18.9°)$

(d) k_2 is increased by 20%
Calculate new ω's using Rayleigh's quotient with old mode shapes.

Rayleigh: $\omega^2 = \dfrac{V_{max}}{T^*} = \dfrac{\cancel{\tfrac{1}{2}} k y_1^2 + \cancel{\tfrac{1}{2}}(1.2k)(y_2 - y_1)^2}{\cancel{\tfrac{1}{2}} \tfrac{3}{2}M(y_1^2 + y_2^2)}$

$$= \frac{2}{3} k/m \left[\frac{y_1^2 + 1.2(y_2 - y_1)^2}{y_1^2 + y_2^2}\right]$$

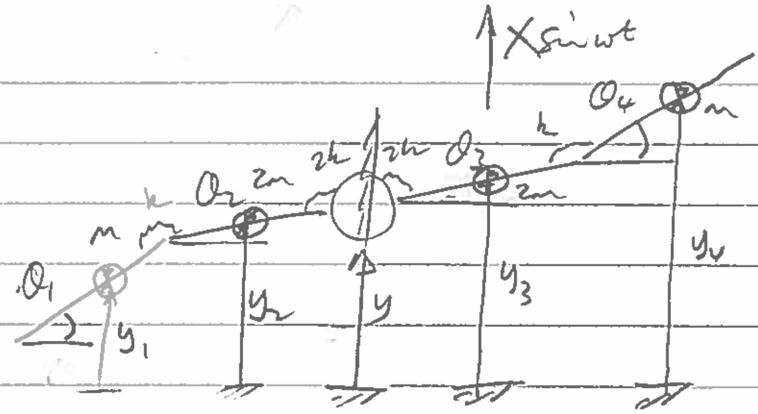
$\begin{Bmatrix} y_1^{(1)} \\ y_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix} \Rightarrow \omega_1^2 \simeq \frac{2}{3} k/m \left(\frac{0.618^2 + 1.2(0.382)^2}{0.618^2 + 1^2}\right) = \underline{\underline{0.269}}$ (higher as expected)

3 Cont $\begin{Bmatrix} y_1^{(2)} \\ y_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} \Rightarrow \omega_2^2 \simeq \frac{2}{3}\frac{k}{m}\left(\frac{1.618^2 + 1.2\,(2.618)^2}{1.618^2 + 1^2}\right)$

$$\underline{\underline{\omega_2^2 = 2.0\,k/m}}$$

At $t = \sqrt{m/k}$, $\theta_1 \simeq 0.217 \cos\sqrt{0.269} + 0.568 \cos\sqrt{2.0}$
$= 0.277$ $\underline{\text{rads}}$

4(a)



$$\dot y_3 = \dot y + \tfrac{L}{2}\dot\theta_3 \qquad \dot y_4 = \dot y + L\dot\theta_3 + \tfrac{L}{2}\dot\theta_4$$

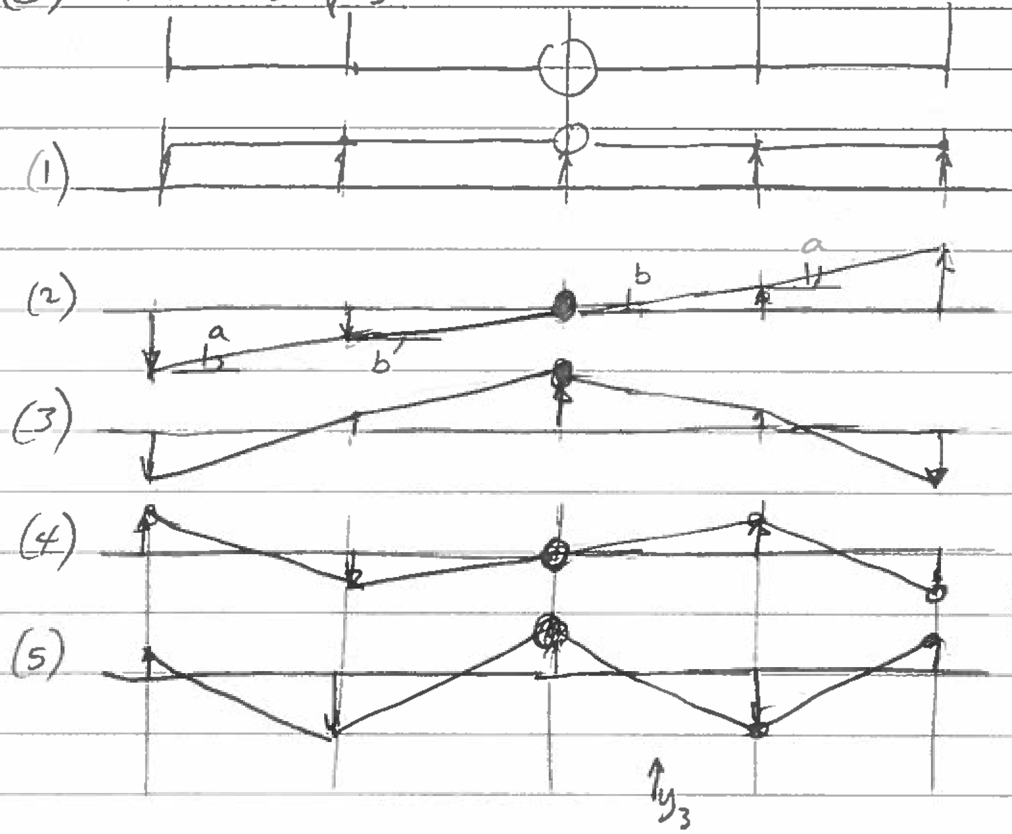
$$\dot y_2 = \dot y - \tfrac{L}{2}\dot\theta_2 \qquad \dot y_1 = \dot y - L\dot\theta_2 - \tfrac{L}{2}\dot\theta_1$$

$$T = \tfrac{1}{2}4m\dot y^2 + \tfrac{1}{2}m\dot y_1^2 + \tfrac{1}{2}2m\dot y_2^2 + \tfrac{1}{2}2m\dot y_3^2 + \tfrac{1}{2}m\dot y_4^2$$
$$+ \tfrac{1}{2}I_1\dot\theta_1^2 + \tfrac{1}{2}I_2\dot\theta_2^2 + \tfrac{1}{2}I_3\dot\theta_3^2 + \tfrac{1}{2}I_4\dot\theta_4^2$$
$$= \tfrac{1}{2}4m\dot y^2 + \tfrac{1}{2}m(\dot y - L\dot\theta_2 - \tfrac{L}{2}\dot\theta_1)^2 + \tfrac{1}{2}2m(\dot y - \tfrac{L}{2}\dot\theta_2)^2$$
$$+ \tfrac{1}{2}m(\dot y + L\dot\theta_3 + \tfrac{L}{2}\dot\theta_4)^2 + \tfrac{1}{2}2m(\dot y + \tfrac{L}{2}\dot\theta_3)^2$$
$$+ \tfrac{1}{2}\frac{mL^2}{12}\dot\theta_1^2 + \tfrac{1}{2}\frac{mL^2}{6}\dot\theta_2^2 + \tfrac{1}{2}\frac{mL^2}{6}\dot\theta_3^2 + \tfrac{1}{2}\frac{mL^2}{12}\dot\theta_4^2$$

$$V = \tfrac{1}{2}k(\theta_2-\theta_1)^2 + \tfrac{1}{2}2k\,\theta_2^2 + \tfrac{1}{2}2k\,\theta_3^2 + \tfrac{1}{2}k(\theta_4-\theta_3)^2$$

(b) Mode shapes

$\underline{y} = [y\ \theta_1\ \theta_2\ \theta_3\ \theta_4$



$\omega_1 = 0$
$y^{(1)} = [1\ 0\ 0\ 0\ 0]$

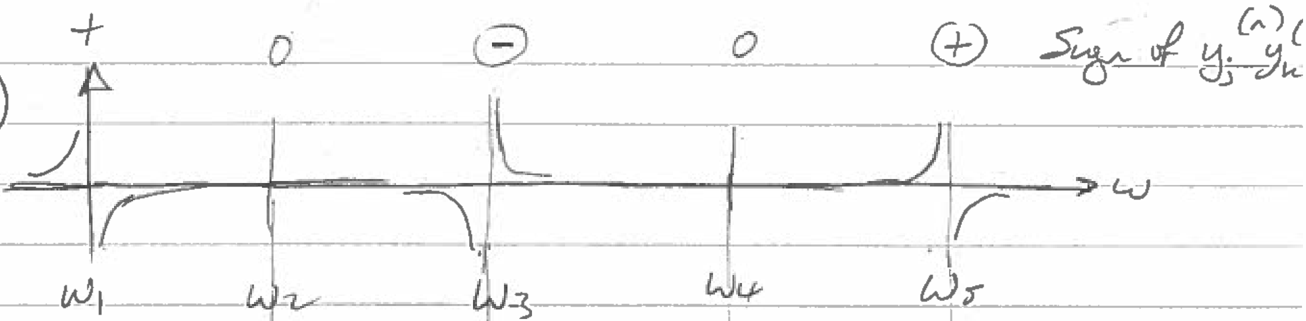
ω_2
$y^{(2)} = [0\ a\ b\ b\ a$

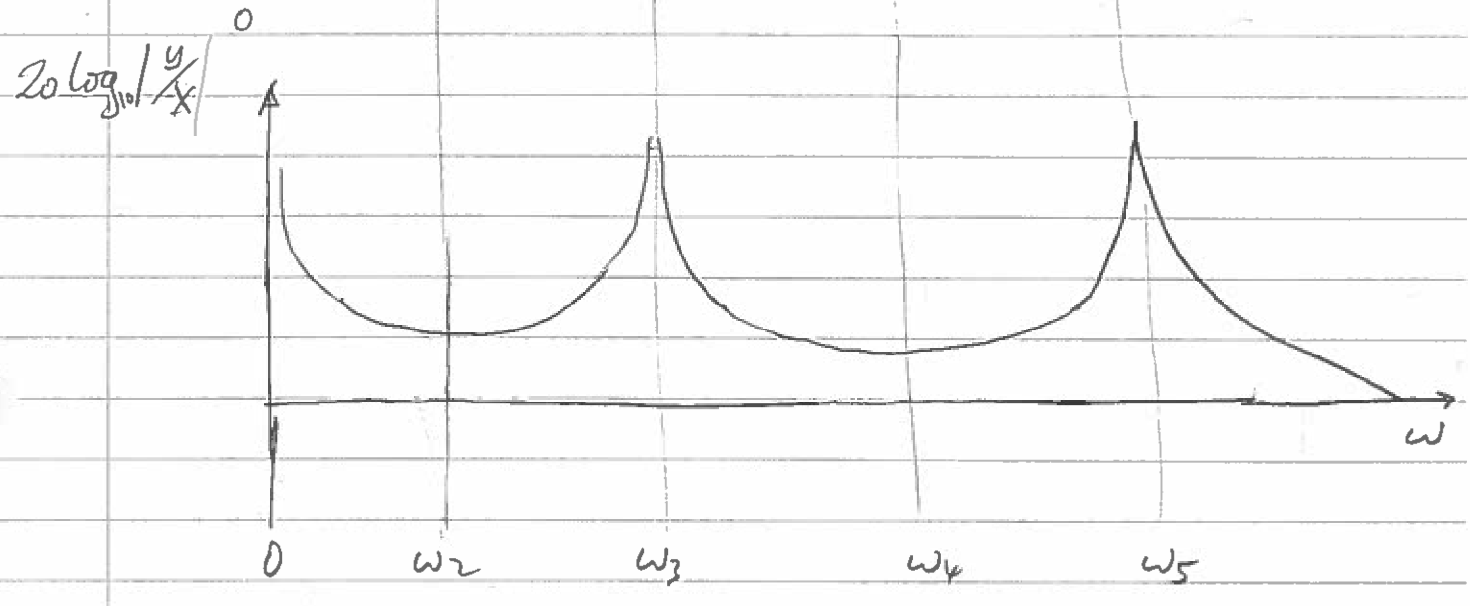
ω_3
$y^{(3)} = [1\ a\ b\ -a\ -b$

ω_4
$y^{(4)} = [0\ -a\ b\ b\ -a$

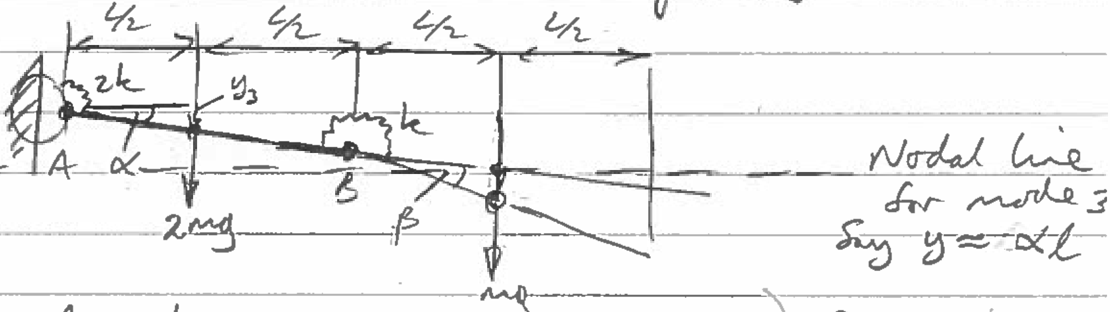
ω_5
$y^{(5)} = [1\ -a\ b\ -b\ a]$

(d)





(e) Need an approximate mode shape
Calculate deflection due to gravity:



$\circlearrowright \Sigma M_B$ $\quad mg\frac{l}{2} - k\beta = 0$

$\rightarrow \beta = \frac{mgl}{2k}$

$\circlearrowright \Sigma M_A$ $\quad mg\frac{3l}{2} + 2mg\frac{l}{2} - 2k\alpha = 0$

$\rightarrow \alpha = \frac{5mgl}{4k}$

$\theta_3 = \alpha = \frac{5}{4}\frac{mgl}{k}$

$\theta_4 = \alpha + \beta$
$= \frac{7}{4}\frac{mgl}{k}$

So mode shapes are $\underset{\sim}{y}^{(2)} \approx [0 \;\; 7/4 \;\; 5/4 \;\; 5/4 \;\; 7/4]^T$

& $\underset{\sim}{y}^{(3)} \approx [5/4\,l \;\; 7/4 \;\; 5/4 \;\; -5/4 \;\; -7/4]^T = [l \;\; 7/5 \;\; 1 \;\; -1 \;\; -7/5]^T$

$\theta_1 = 7/4$
$\theta_2 = -5/4$
$\theta_3 = -5/4$
$\theta_4 = 7/4$

Rayleigh $\omega_2^2 \approx \frac{V}{T^*}$; $V = \frac{1}{2}\left[\frac{1}{2}k(7/4 - 5/4)^2 + \frac{1}{2}2k(5/4)^2\right]$

$\underline{\underline{\omega_2}}$

$= k\left(1/4 + 25/8\right) = k\left(27/8\right)$

$$\frac{2T^*}{mL^2} = 2\left(5/4 + \frac{7}{8}\right)^2 + 2.2(5/8)^2 + \frac{2}{12}(7/4)^2 + \frac{2}{6}(5/4)^2$$

$$= \frac{289}{64} + \frac{50}{64} + 49/192 + \frac{25}{96} = 5.81$$

$$\underline{\text{So}} \quad \omega_2^2 = \frac{V}{T^*} = \frac{k\,(27/8)}{mL^2\ 5.81} = \frac{0.58}{l^2}\frac{k}{m}$$

$$\text{So} \quad \underline{\underline{\omega_2 = 0.76\sqrt{\frac{k}{ml^2}}}}$$

(c) If the fuselage mass was zero, ω_3 would be significantly higher. If the fuselage mass was then increased gradually, ω_3 would decrease.
As fuselage mass $\to \infty$, modes 2 and 3 would have the same frequency.
So ω_3 must be higher than ω_2

$\Rightarrow \omega_2$ is lowest
Modes are in the order shown.