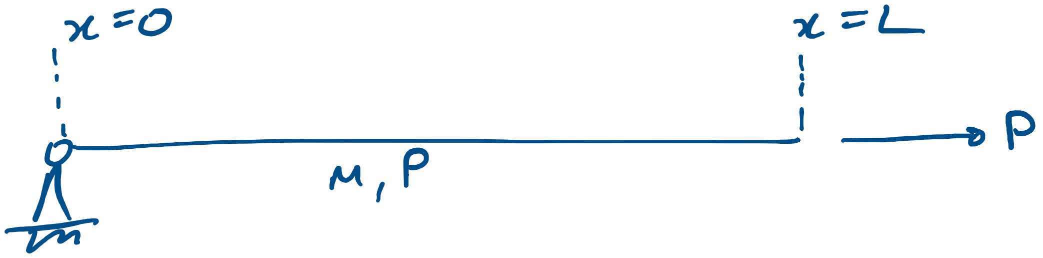
3C6 draft sols

Q1 (a)



(i) fixed-free $\Rightarrow$ $z_s(0) = 0$, $z_s'(L) = 0$.

so mode shapes: $u_n(x) = \beta \sin kx$

$$u_n' = \beta k \cos kx.$$

$$\Rightarrow \quad \cos kL = 0.$$

$$k_n = (n - \tfrac{1}{2})\pi / L.$$

$$u_n(x) = \beta_n \sin \frac{(n-\frac{1}{2})\pi x}{L} \qquad \omega_n = k_n c = \frac{(n-\frac{1}{2})\pi}{L}\sqrt{P/m}$$

(ii) Need to mass-normalise u_n:

$$\int_0^L \beta_n^2 \sin^2 k_n x \, . \underbrace{m\,dx}_{\text{"}dm\text{"}} = 1$$

$$\int_0^L \tfrac{1}{2}\beta_n^2 (1 - \cos 2k_n x)\, m\,dx = 1$$

$$\tfrac{1}{2}\beta_n^2 m \left[x + 2k_n \sin k_n x \right]_0^L = 1$$

$$\tfrac{1}{2}\beta_n^2 m L + \overbrace{0 - 0} = 1 \quad \Rightarrow \quad \beta_n^2 = \frac{2}{mL}$$

So $$G_s(x_1, x_2, \omega) = \frac{2}{mL} \sum_{n=1}^{\infty} \frac{\sin k_n x_1 \, . \, \sin k_n x_2}{\omega_n^2 + 2i\zeta_n \omega_n \omega - \omega^2}$$

solutions with/without damping accepted.

(b) Fixed-free axial vibration identical, just different mass normalisation:

$$\int_0^L C_n^2 \sin^2 k_n x \, . \underbrace{\rho A dx}_{\text{"}dm\text{"}} = 1 \quad \Longrightarrow \quad C_n^2 = \frac{2}{\rho A L}$$

$$\Longrightarrow \quad G_b(y_1, y_2, \omega) = \frac{2}{\rho A L} \sum_{n=1}^{\infty} \frac{\sin k_n y_1 \, . \, \sin k_n y_2}{\omega_n^2 + 2i\zeta_n \omega_n \omega - \omega^2}$$

with $k_n = (n - \tfrac{1}{2})\pi/L$, $\omega_n = k_n \sqrt{E/\rho}$

(c) For $G_s = G_b$ need ω_n & mass normalisation same:

$$\Longrightarrow \quad P/m = E/\rho$$ — condition ① (same wavespeed)

& $$\frac{2}{m\cancel{L}} = \frac{2}{\rho A \cancel{L}}$$

$$\Longrightarrow \quad m = \rho A$$ — condition ② (same mass per unit length)

(Combining this also gives $P = EA$)

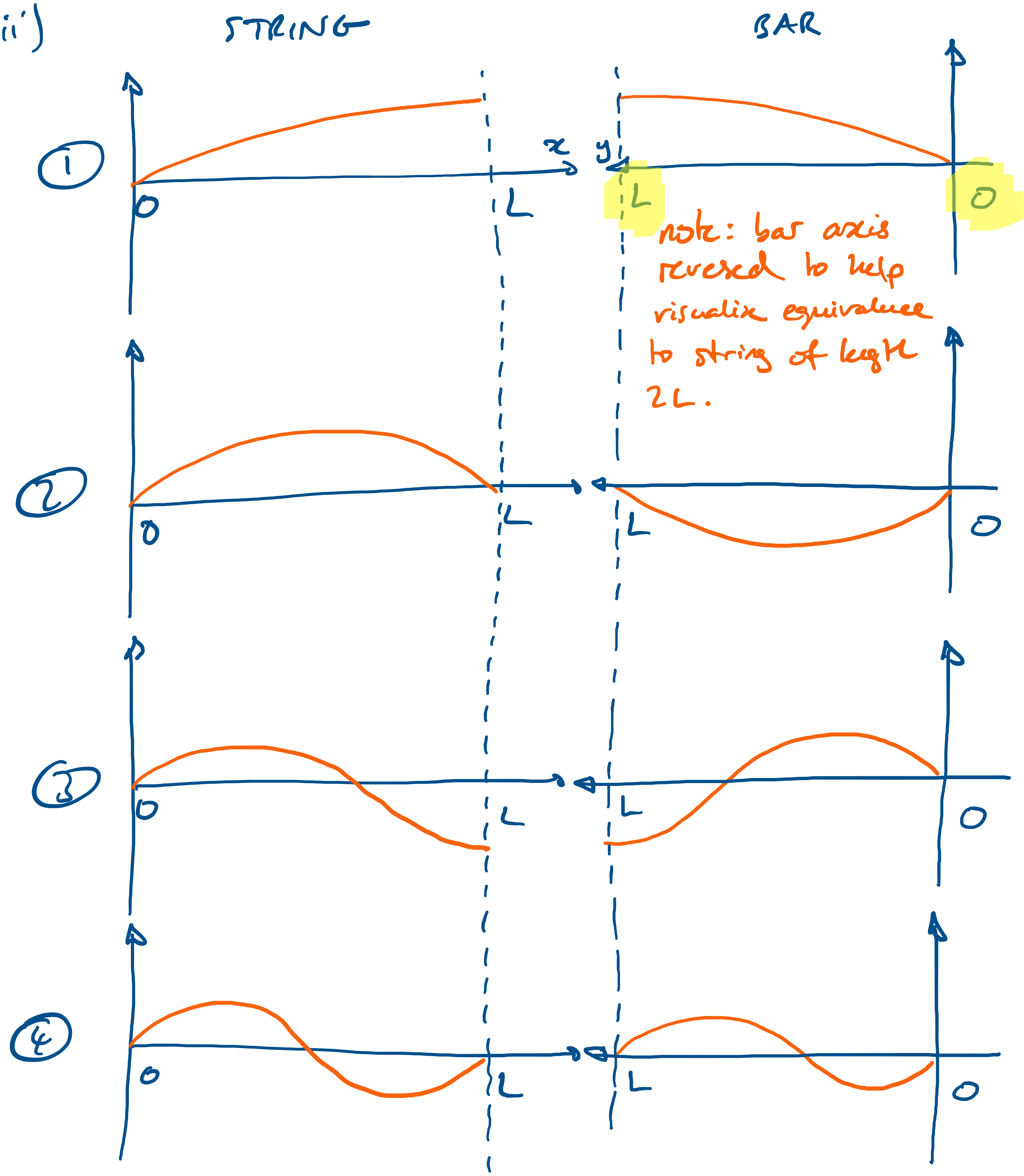
(d)(i) $$\frac{1}{G_c} = \frac{1}{G_s} + \frac{1}{G_b}$$

$$G_s = G_b = G \quad \Longrightarrow \quad G_c = G/2$$

so $$G_c = \underset{\substack{\| \\ \frac{1}{\rho A L}}}{\frac{1}{mL}} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 + 2i\zeta_n\omega_n - \omega^2}$$

, $\omega_n = \frac{(n-\frac{1}{2})\pi}{L} \sqrt{P/m}$

(ii)



Note: ① & ③ same as uncoupled ω_n's, visible in G_c

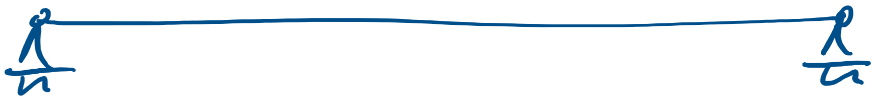
② & ④ node at $x = y = L$, not visible in G_c.

Sign of string & bar mode shapes opposite in ② & ④ for force balance.

Coupled system is equiv. to string of length 2L.

Hence $\omega_n = \frac{n\pi}{2L}\sqrt{P/m} = \frac{n\pi}{2L}\sqrt{E/\rho}$ //

Q2//



(a) $\rho A \ddot{y} + EI y'''' = 0$

by s.o.v. let $y = u(x) e^{i\omega t}$.

$\Rightarrow EI u'''' - \rho A \omega^2 u = 0$.

4^{th} order ODE has standard solution:

$$u(x) = D_1 \cos kx + D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx$$

$$u'(x) = k(-D_1 \sin kx + D_2 \cos kx + D_3 \sinh kx + D_4 \cosh kx)$$

$$u''(x) = k^2(-D_1 \cos kx - D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx)$$

— don't need u''', & $k^4 = \omega^2 \rho A / EI$

pinned @ $x = 0 \Rightarrow u(0) = u''(0) = 0$.

$\Rightarrow D_1 + D_3 = 0$.

& $-D_1 + D_3 = 0$ so $D_1 = D_3 = 0$.

pinned @ $x = L \Rightarrow u(L) = u''(L) = 0$.

$\rightarrow D_2 \sin kL + D_4 \cosh kL = 0$

& $-D_2 \sin kL + D_4 \cosh kL = 0$.

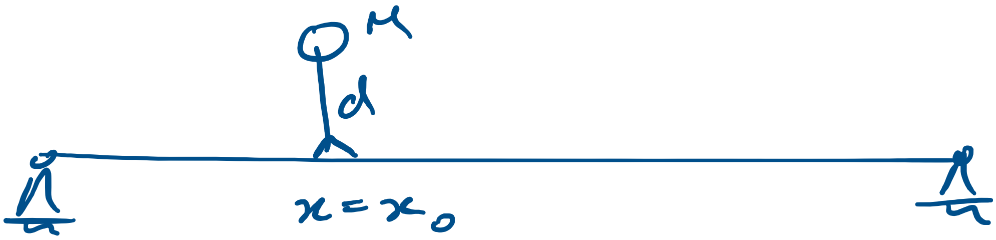
so $\underbrace{D_2 \sin kL = 0}_{\sin kL = 0}$ & $\underbrace{D_4 \cosh kL = 0}_{\cosh kL \neq 0}$

$k_n = n\pi / L$ // so $D_4 = 0$.

Giving $u_n(x) = D_2 \sin n\pi x / L$ //, same as fixed-fixed string.

& $\omega_n = k_n^2 \sqrt{EI/\rho A}$ //

(b)



Assume mode shapes unaffected.

$$R = \frac{V}{\tilde{T}} \simeq \omega_n'^2$$

V unchanged.

$$\tilde{T}' = \tilde{T}_0 + \delta\tilde{T} = \tilde{T}_0 \left(1 + \delta\tilde{T}/\tilde{T}_0\right)$$

so $$R \simeq \underbrace{\frac{V}{\tilde{T}_0}}_{=\omega_n^2}\left(1 - \underbrace{\delta\tilde{T}/\tilde{T}_0}_{\ll \text{ assume small.}}\right) = \omega_n'^2$$

hence $$\omega_n' \simeq \omega_n\left(1 - \underbrace{\tfrac{1}{2}\,\delta\tilde{T}/\tilde{T}_0}_{=\Delta}\right)$$

Need $\delta\tilde{T}$ & $\tilde{T}_0$.

$$\tilde{T}_0 = \tfrac{1}{2}\rho A \int_0^L u_n^2\,dx = \tfrac{1}{2}\rho A \int_0^L \sin^2\frac{n\pi x}{L}\,dx$$

$$= \tfrac{1}{4}\rho A \left[1 - \cos\frac{2n\pi x}{L}\right]_0^L$$

$$= \tfrac{1}{4}\rho A L$$

$$\delta\tilde{T} = \tfrac{1}{2}M u^2(x_0) + \tfrac{1}{2}M d^2 u'^2(x_0)$$

$$= \tfrac{1}{2}M \sin^2\frac{n\pi x_0}{L} + \tfrac{1}{2}M d^2\left(\frac{n\pi}{L}\right)^2 \cos\frac{n\pi x_0}{L}$$

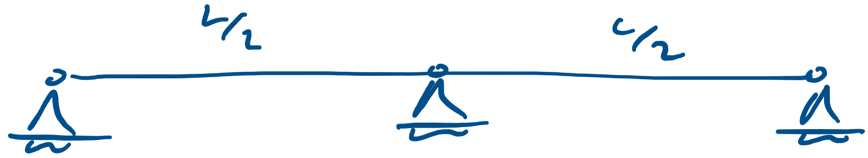
$$\delta\tilde{T} = \frac{1}{2}M\left(\sin^2\frac{n\pi x_0}{L} + \left(\frac{n\pi d}{L}\right)^2\cos^2\frac{n\pi x_0}{L}\right)$$

Here $\quad \omega_n' \simeq \omega_n(1+\Delta)$

& $\quad \Delta = -\frac{1}{2}\frac{\delta\tilde{T}}{T_0} = -\frac{1}{2}\cdot\frac{\frac{1}{2}M\left(\sin^2\frac{n\pi x_0}{L} + \left(\frac{n\pi d}{L}\right)^2\cos^2\frac{n\pi x_0}{L}\right)}{\frac{1}{4}\rho AL}$

$$= -\frac{M}{\rho AL}\cdot\left(\sin^2\frac{n\pi x_0}{L} + \left(\frac{n\pi d}{L}\right)^2\cos^2\frac{n\pi x_0}{L}\right)$$

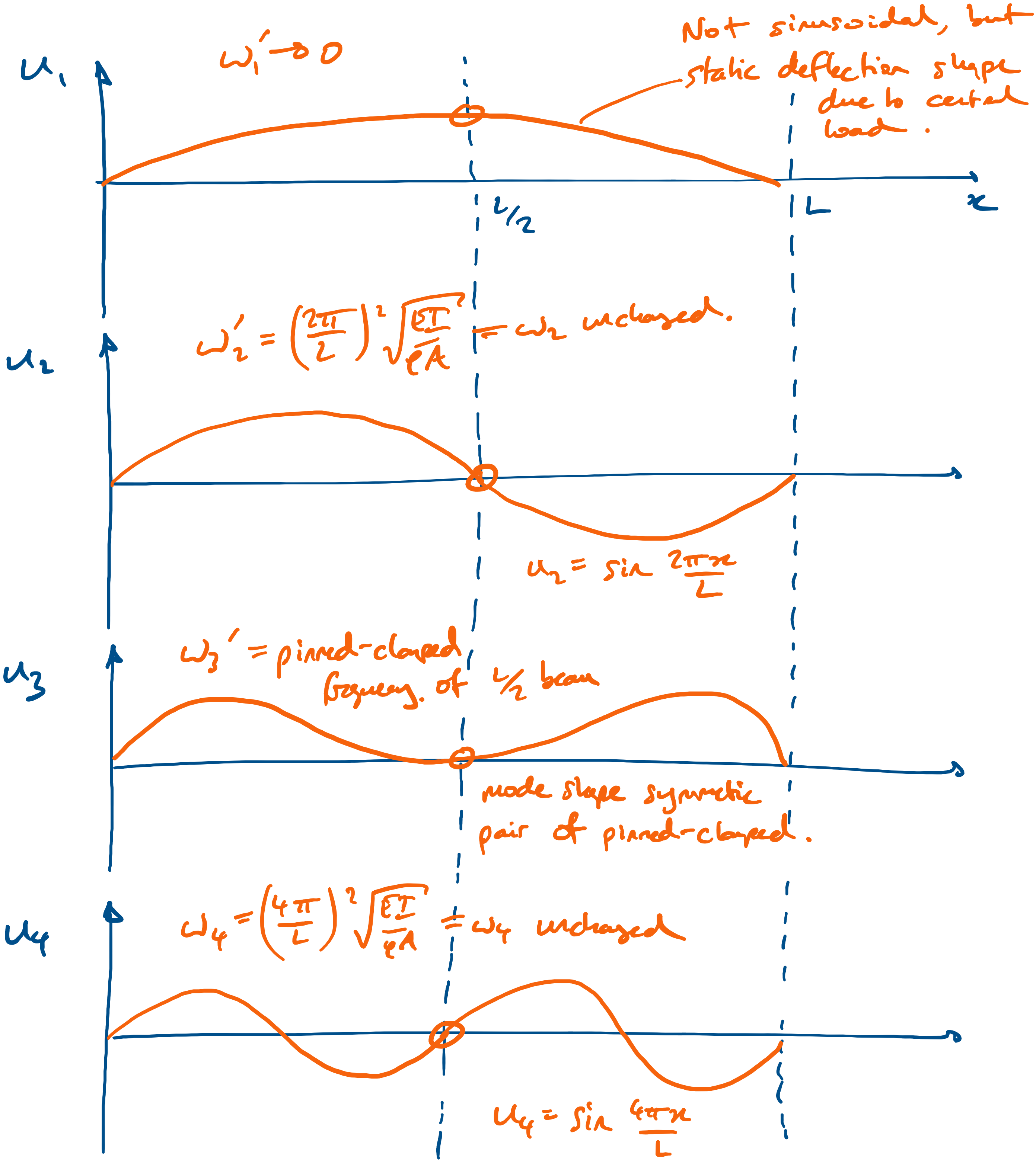
(ii) $\quad M\to\infty,\ d=0,\ x_0 = L/2 \implies$ Mass becomes pinned constraint except for first mode



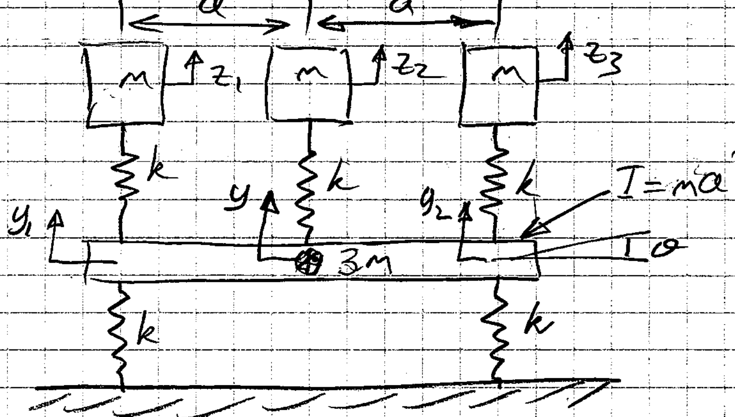
Expect symmetric & antisymmetric modes.

$\Downarrow$ equiv to pinned-clamped

$\Downarrow$ equiv to pinned-pinned.


u_1
$\omega_1' \to 0$
Not sinusoidal, but static deflection shape due to central load.
$L/2$
L
x
u_2
$\omega_2' = \left(\frac{2\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} = \omega_2$ unchanged.
$u_2 = \sin \frac{2\pi x}{L}$
u_3
$\omega_3' =$ pinned-clamped frequency of $L/2$ beam
mode shape symmetric pair of pinned-clamped.
u_4
$\omega_4 = \left(\frac{4\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} = \omega_4$ unchanged
$u_4 = \sin \frac{4\pi x}{L}$

3/(a)



KE

$$T = \tfrac{1}{2} m\dot{z}_1^2 + \tfrac{1}{2} m\dot{z}_2^2 + \tfrac{1}{2} m\dot{z}_3^2 + \tfrac{1}{2}(3m)\dot{y}^2 + \tfrac{1}{2}(ma^2)\dot{\theta}^2$$

$$= \tfrac{1}{2} m\left(\dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2 + 3\dot{y}^2 + a^2\dot{\theta}^2\right)$$

$$= \tfrac{1}{2} m [\dot{z}_1\ \dot{z}_2\ \dot{z}_3\ \dot{y}\ \dot{\theta}]\, m \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 3 & \\ & & & & a^2 \end{bmatrix}}_{[M]} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}$$

PE

$$y_1 = y - a\theta, \quad y_2 = y + a\theta$$

$$V = \tfrac{1}{2} k y_1^2 + \tfrac{1}{2} k y_2^2 + \tfrac{1}{2} k (z_3 - y_2)^2 + \tfrac{1}{2} k (z_1 - y_1)^2 + \tfrac{1}{2} k (z_2 - y)^2$$

$$= \tfrac{1}{2} k \left[(y - a\theta)^2 + (y + a\theta)^2 + \{z_3 - (y + a\theta)\}^2 + \{z_1 - (y - a\theta)\}^2 + (z_2 - y)^2 \right]$$

$$= \tfrac{1}{2} k \Big[y^2 - \cancel{2ay\theta} + a^2\theta^2 + y^2 + \cancel{2ay\theta} + a^2\theta^2 + z_3^2 + y^2 + a^2\theta^2$$
$$- 2z_3 y - 2a z_3\theta + \cancel{2a\theta y} + z_1^2 + y^2 + a^2\theta^2 - 2z_1 y + 2a z_1\theta - \cancel{2ay\theta}$$
$$+ z_2^2 - 2z_2 y + y^2$$

$$= \tfrac{1}{2} k \{ z_1^2 + z_2^2 + z_3^2 + 5y^2 + 4a^2\theta^2 - 2z_3 y - 2a z_3\theta$$
$$- 2z_1 y + 2a z_1\theta - 2z_2 y \}$$

$$= \tfrac{1}{2} [z_1\ z_2\ z_3\ y\ \theta]\, k \underbrace{\begin{array}{c} \begin{array}{ccccc} z_1 & z_2 & z_3 & y & \theta \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & a \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -a \\ -1 & -1 & -1 & 5 & 0 \\ a & 0 & -a & -0 & 4a^2 \end{bmatrix} \end{array}}_{[K]} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ y \\ \theta \end{Bmatrix}$$

(b) Method 1 $([K] - \omega^2[M])\underline{u} = 0$ with $\underline{u} = [1\ 1\ 1\ \beta\ 0]^T$

$$\begin{bmatrix} k - \omega^2 m & 0 & 0 & -k & ka \\ 0 & k - \omega^2 m & 0 & -k & 0 \\ 0 & 0 & k - \omega^2 m & -k & -ka \\ -k & -k & -k & 5k - \omega^2 3m & 0 \\ ka & 0 & -ka & 0 & 4ka^2 - \omega^2 ma^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ \beta \\ 0 \end{Bmatrix} = \underline{0}$$

First row: $k - \omega^2 m - k\beta = 0 \Rightarrow \beta = 1 - \omega^2 m/k$ —(1)

4th row: $-3k + (5k - 3\omega^2 m)\beta = 0 \quad \beta = +3k/(5k - 3\omega^2 m)$ —(2)

3(b) Cont Combine (1) & (2)

$$\frac{1-\omega^2 m}{k} - \frac{3k}{5k-3\omega^2 m} = 0$$

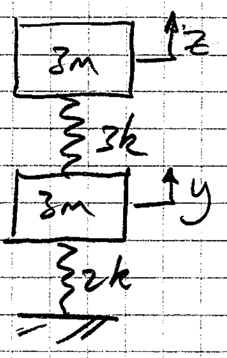
$$(5k-3\omega^2 m)(k-\omega^2 m) - 3k^2 = 0 \qquad \text{———} \quad (3)$$

$$3m^2\omega^4 - 8km\omega^2 + 2k^2 = 0 \Rightarrow \omega^2 = \left(\frac{4}{3} \pm \frac{\sqrt{40}}{6}\right)\frac{k}{m}$$

$$\text{So} \quad \omega_{1,2}^2 = 0.2792\,k/m,\ 2.3874\,k/m$$

$$(1) \Rightarrow \text{with} \quad \beta_{12} = 1 - \omega^2 m/k = 0.721,\ -1.387$$

Method 2 with this mode shape, system is equivalent to:



$$\Rightarrow \quad T = \tfrac{1}{2}3m(\dot{z}^2 + \dot{y}^2)$$

$$V = \tfrac{1}{2}2ky^2 + \tfrac{1}{2}3k(z-y)^2$$

$$= \tfrac{1}{2}k\left[2y^2 + 3(z^2 - 2yz + y^2)\right]$$

$$= \tfrac{1}{2}k\left[5y^2 - 6zy + 3z^2\right]$$

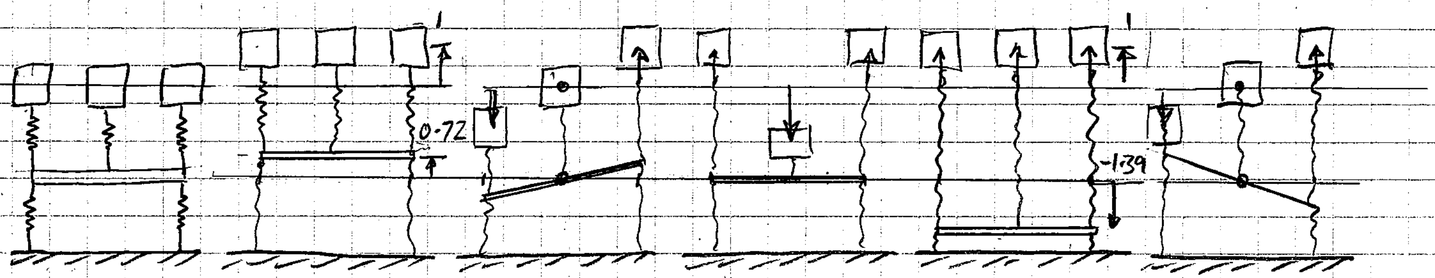
$$= \tfrac{1}{2}\begin{bmatrix} z & y \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix}\begin{Bmatrix} z \\ y \end{Bmatrix}$$

So the E.V.P. is $\begin{bmatrix} 3k-\omega^2 3m & -3k \\ -3k & 5k-\omega^2 3m \end{bmatrix}\begin{Bmatrix} z \\ y \end{Bmatrix} = \underline{0}$

from which the characteristic equation is

$$(\cancel{3}k - \omega^2 \cancel{3}m)(5k-\omega^2 3m) - \cancel{9}_3 k^2 = 0 \qquad \underline{\text{as before (3)}}$$

(c)



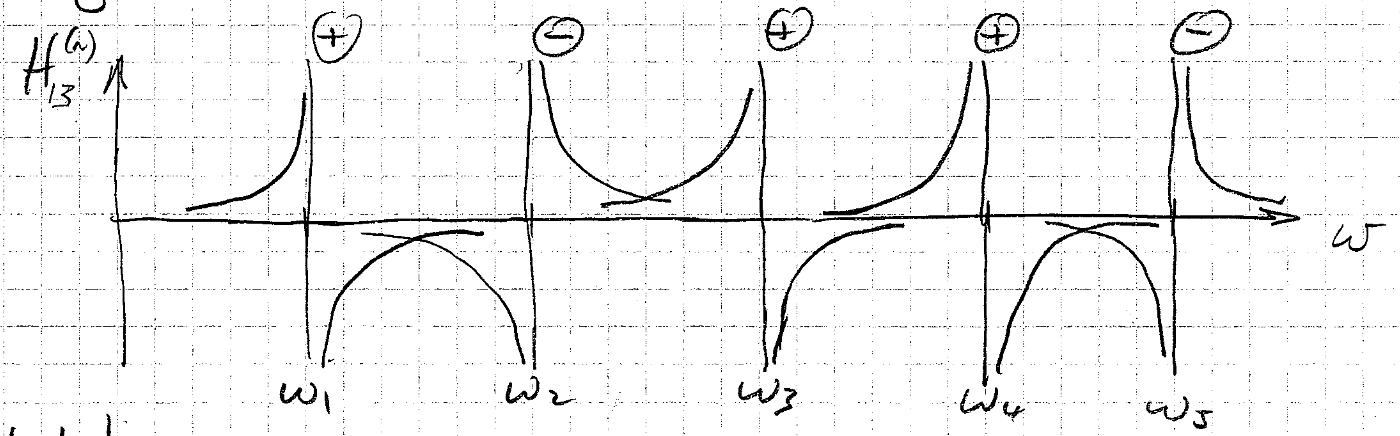
$\omega_1^2 = 0.2792k/m$ $\qquad$ $\omega^2 = 2.387k/m$

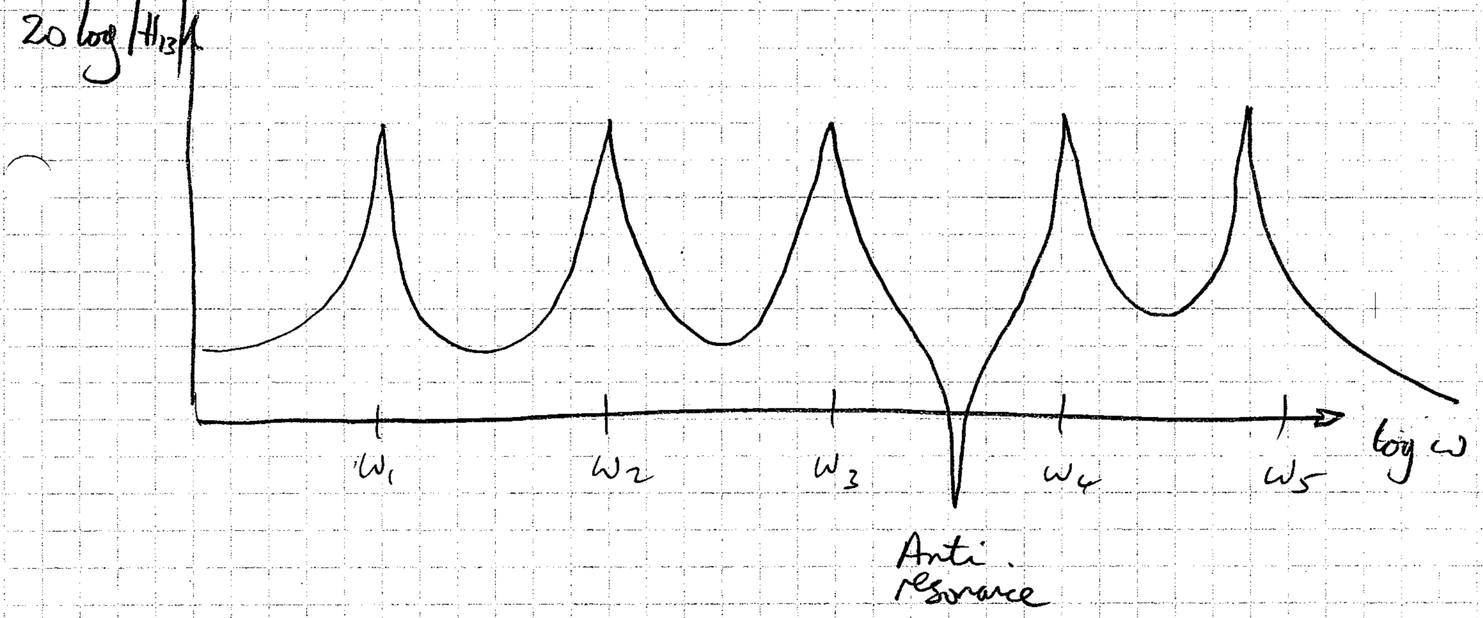
$\beta = 0.721$ $\qquad$ $\beta = -1.387$

(1) (2) (3) (4) (5)

3(d) Transfer function: $H_{13} = \sum_{n=1}^{5} \frac{u_1^{(n)} u_3^{(n)}}{\omega^2 - \omega_n^2} = z_1/f_3$

Signs of Modal constants $u_1^{(n)} u_3^{(n)}$ are: + − + + −





4(a) $T=\frac{1}{2}J\dot{\theta}_1^2+\frac{1}{2}J\dot{\theta}_2^2+\frac{1}{2}J\dot{\theta}_3^2+\frac{1}{2}J\dot{\theta}_4^2+\frac{1}{2}J\dot{\theta}_5^2$

$V=\frac{1}{2}k(\theta_2-\theta_1)^2+\frac{1}{2}k(\theta_3-\theta_2)^2+\frac{1}{2}k(\theta_4-\theta_3)^2+\frac{1}{2}s(\theta_5-\theta_4)^2$

(b) $$\omega^2\approx R=\frac{V}{T^*}=\frac{\frac{1}{2}k[(\theta_2-\theta_1)^2+(\theta_3-\theta_2)^2+(\theta_4-\theta_3)^2]+\frac{1}{2}s(\theta_5-\theta_4)^2}{\frac{1}{2}J[\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_5^2]} \quad \text{— (1)}$$

Lowest frequency is for rigid body mode $\underline{q}=[1\ 1\ 1\ 1\ 1]^T$

(1) ⇒ $$\omega^2=\frac{\frac{1}{2}k[0^2]+\frac{1}{2}s[0^2]}{\frac{1}{2}J[5]}=\underline{\underline{0}} \quad ✓$$

(c) When $s=k$, the system is symmetric.

For $q=[1\ \alpha\ 0\ -\alpha\ -1]^T$:

$$\omega^2\approx R=\frac{\cancel{\frac{1}{2}}k[(\alpha-1)^2+(0-\alpha)^2+(-\alpha-0)^2+(-1+\alpha)^2]}{\cancel{\frac{1}{2}}J[1^2+\alpha^2+0^2+\alpha^2+1^2]}$$

$$=\frac{k}{J}\frac{\cancel{2}[(1-\alpha)^2+\alpha^2]}{\cancel{2}[1+\alpha^2]}=\frac{k}{J}\frac{(2\alpha^2-2\alpha+1)}{1+\alpha^2} \quad \text{— (2)}$$

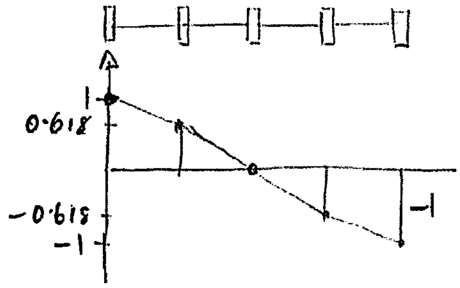
The exact mode shape is found by minimizing R:

$$\frac{dR}{d\alpha}=\frac{(1+\alpha^2)(4\alpha-2)-(2\alpha^2-2\alpha+1)(2\alpha)}{(1+\alpha^2)^2}=0$$

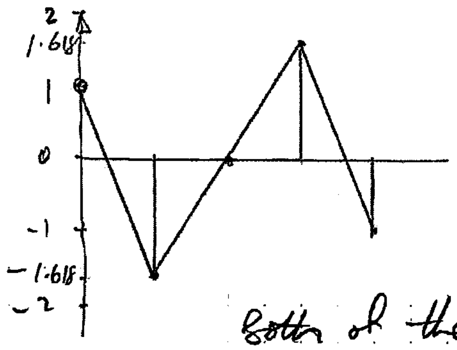
$\Rightarrow \quad \alpha^2+\alpha-1=0$

$\Rightarrow \quad \alpha=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}=0.618 \ \& \ -1.618$

& (2) $\Rightarrow \quad \omega_1^2=0.3819\,k/J \quad \& \quad \omega_2^2=2.618\,k/J$



Mode 1, $\alpha=0.618$, $\omega_1=0.618\sqrt{k/J}$



Mode 2, $\alpha=-1.618$, $\omega_2=1.618\sqrt{k/J}$

Both of these modes satisfy the mode shape $[1\ \alpha\ 0\ -\alpha\ -1]^T$ and they are therefore both $\underline{\text{exact}}$

4 (cont) (d)

For a small change in S, ie $S = 1{\cdot}1k$, the original mode shape is still a reasonable approximation to the correct mode. Rayleigh will therefore give a reasonable estimate of the frequency:

$$(1) \Rightarrow \quad \omega^2 \simeq \frac{\frac{1}{2}k\left[(\alpha-1)^2 + (0-\alpha)^2 + (-\alpha-0)^2\right] + \frac{1}{2}(1{\cdot}1k)(-1+\alpha)^2}{\frac{1}{2}J\left[2(1+\alpha^2)\right]}$$

$$= \frac{\frac{1}{2}k\left[(\alpha-1)^2 + 2\alpha^2 + (\alpha-1)^2\right]}{\frac{1}{2}J\left[2(1+\alpha^2)\right]} + \frac{\cancel{\tfrac{1}{2}}(0{\cdot}1k)(\alpha-1)^2}{\cancel{\tfrac{1}{2}}J\left[2(1+\alpha^2)\right]}$$

$$= \underbrace{\qquad\qquad}_{\omega_1^2} + \underbrace{\frac{0{\cdot}1k(\alpha-1)^2}{2J(1+\alpha^2)}}_{\Delta\omega^2}$$

for $\alpha = 0{\cdot}618$, $\quad \Delta\omega^2 = 0{\cdot}00528\ k/J$

So $\omega^2 = (0{\cdot}3819 + 0{\cdot}00528)\,k/J \doteq 0{\cdot}3872\ k/J$

Hence % increase in ω_1 is $\dfrac{\sqrt{0{\cdot}3872} - 0{\cdot}618}{0{\cdot}618} = \underline{\underline{0{\cdot}686\%}}$