Qu 1

(a) (i) fixed-fixed string: $y(0,t) = y(L,t) = 0$.

1D wave equation: $u_n = A_n \sin k_n x + B_n \cos k_n x$

BC's only satisfied by sin ...

$$u_n = A_n \sin k_n x.$$

$$\& \quad A_n \sin k_n L = 0$$

$$\text{so} \quad \sin k_n L = 0$$

$$k_n = \frac{n\pi}{L}$$

Gives $$u_n = A_n \sin \frac{n\pi x}{L}$$

$$\omega_n = c k_n = \frac{n\pi c}{L} = \frac{n\pi}{L}\sqrt{P/m}$$

[derivation not asked for so can quote results]

(ii) Step response: from datasheet:

$$y(x,t) = F_0 \underbrace{\sum_n \frac{u_n^2(a)}{\omega_n^2}\left[1 - e^{-\omega_n \zeta_n t} \cdot \cos \omega_n t\right]}_{\text{unit step response}}$$

Need to mass normalise u_n ...

Need $\int u_n^2(x)\, dx = 1$

$$\int_0^L A_n^2 \sin^2 k_n x \,.\, m\, dx = 1$$

$$\frac{A_n^2 m}{2} \int_0^L \left(1 - \cos 2k_n x\right) dx = 1$$

$$\frac{A_n^2 m L}{2} = 1$$

$$\Rightarrow A_n^2 = \frac{2}{mL}$$

Gives : $y(x,t) = \dfrac{2F_0}{mL} \sum_n \dfrac{\sin^2\left(\frac{n\pi a}{L}\right)}{\omega_n} \left[1 - e^{-\zeta_n \omega_n t} \cos \omega_n t\right]$

with $\omega_n = \dfrac{n\pi}{L}\sqrt{P/m}$.

(iii) Same as (ii) but -ve step & different ICs:

$$y(x,t) = C - \frac{2F_0}{mL} \sum_n \frac{\sin^2 \frac{n\pi a}{L}}{\omega_n} \left[1 - e^{-\zeta_n \omega_n t} \cos \omega_n t\right]$$

Note average response is zero (& $y \to 0$ as $t \to \infty$)

hence DC terms must cancel:

$$y = \frac{2F_0}{mL} \sum \frac{\sin^2 \frac{n\pi a}{L}}{\omega_n} e^{-\zeta_n \omega_n t} \cdot \cos \omega_n t$$

(b) (i) Fixed-fixed gives $u_n = A_n \sin \frac{n\pi x}{L}$ for all degrees of freedom.

transverse: $u_i = A_i \sin \frac{i\pi x}{L}$, $\omega_i = \frac{i\pi}{L}\sqrt{P/m}$

torsion: $u_j = A_j \sin \frac{j\pi x}{L}$, $\omega_j = \frac{j\pi}{L}\sqrt{G/\rho}$

axial: $u_k = A_k \sin \frac{k\pi x}{L}$, $\omega_k = \frac{k\pi}{L}\sqrt{E/\rho}$

(ii) $$G(a,a,\omega) = \sum_n \frac{u_n^2(a)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

- Need to separate degrees of freedom & mass normalise.
- Only transverse & torsion will be observed, as force orthogonal to axial motion.
- For torsion input torque & force related by $\tau = Fr$

  & output angle & disp related by $\theta = y/r$

  so $G_{torsion} \equiv \frac{\theta}{\tau} = \frac{1}{r^2}\frac{y}{F}$, so $\frac{y}{F} = r^2 G_{torsion}$

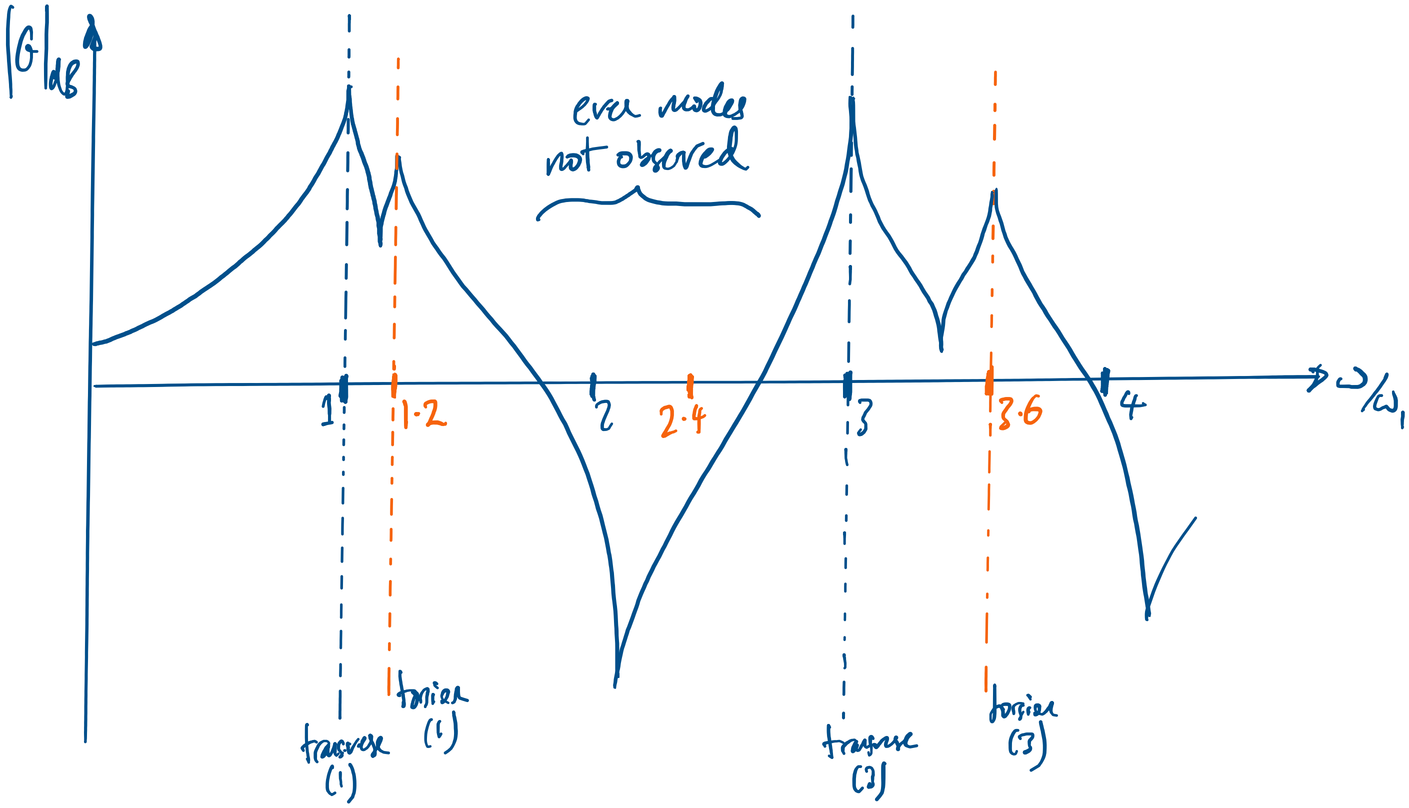
$$G = \underbrace{\frac{2}{mL}\sum_n \frac{\sin^2 \frac{n\pi a}{L}}{\omega_n^2 + 2i\zeta_n\omega_n - \omega^2}}_{\text{transverse}} + \underbrace{\frac{2r^2}{\rho JL}\sum_m \frac{\sin^2\frac{m\pi a}{L}}{\omega_m^2 + 2i\zeta_m\omega_m - \omega^2}}_{\text{torsion}}$$

where $\omega_n = \frac{n\pi}{L}\sqrt{P/m}$ & $\omega_m = \frac{m\pi}{L}\sqrt{G/\rho}$

(iii) $a = L/2$, $\sqrt{G/\rho} = 1.2\sqrt{P/m}$

$\Rightarrow \omega_m = 1.2\,\omega_n$ for each $m = n$.

(ie torsion modes are $1.2\times$ freq of transverse)



Qu2

(a) Pinned – pinned $\Rightarrow$ same mode shapes as fixed-fixed string

$$u_n = A_n \sin\frac{n\pi x}{L}, \quad \omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

(b) $$G(x_1, x_2, \omega) = \frac{V}{F} = \frac{i\omega Y}{F} = \sum_n \frac{i\omega u_n(x_1) u_n(x_2)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

mass normalise mode shapes:

$$\int u_n^2 dm = 1$$

$$\int_0^L B_n^2 \sin^2\frac{n\pi x}{L} \cdot \rho A\, dx = 1$$

$$\frac{\rho A B_n^2}{2} \int_0^L \left(1 - \cos\frac{2n\pi x}{L}\right) = 1$$

$$\frac{\rho A L . B_n^2}{2} = 1 \quad \text{so} \quad B_n^2 = \frac{2}{\rho A L}$$

Hence $$G = \frac{2}{\rho A L} \sum_n \frac{i\omega \sin\frac{n\pi x_1}{L} \cdot \sin\frac{n\pi x_2}{L}}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

(c) $$R = \frac{V}{\tilde{T}} = \frac{V_0 + \Delta V}{\tilde{T}_0 + \Delta\tilde{T}} = \frac{V_0\left(1 + \Delta V/V_0\right)}{\tilde{T}_0\left(1 + \Delta\tilde{T}/\tilde{T}_0\right)}$$

$$\simeq \frac{V_0}{\tilde{T}_0}\left(1 + \frac{\Delta V}{V_0}\right)\left(1 - \Delta\tilde{T}/\tilde{T}\right)$$

$$\simeq \frac{V_0}{\tilde{T}_0}\left(1 + \frac{\Delta V}{V_0} - \frac{\Delta T}{\tilde{T}_0}\right)$$

so $$\omega_n'^2 \simeq \omega_n^2\left(1 + \frac{\Delta V}{V_0} - \frac{\Delta T}{\tilde{T}_0}\right)$$

& $$\omega_n' \simeq \omega_n\left(1 + \frac{\Delta V}{2V_0} - \frac{\Delta T}{2\tilde{T}_0}\right)$$

ie $\alpha_n = \dfrac{1}{2V_0}$ & $\beta_n = -\dfrac{1}{2\tilde{T}_0}$

(d)(i) Symmetric structure ⇒ sym or antisym modes.
In trial u_n, C_n is coeff. for sym component.
D_n is coeff for antisym component.
Hence one of C_n or D_n will always be zero

(ii) Odd n ⇒ symmetric modes
⇒ $D_n = 0$

$$u_n = C_n + \sin\frac{n\pi x}{L}.$$

$$V = \underbrace{\frac{1}{2}EI\int\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\,dx}_{V_0} + \underbrace{\frac{1}{2}KC_n^2 \times 2}_{\Delta V}$$

$$\tilde{T} = \frac{1}{2}\rho A\int_0^L\left(C_n + \sin\frac{n\pi x}{L}\right)^2 dx + \frac{1}{2}M(C_n+1)^2$$

$$= \frac{1}{2}\rho A\int_0^L\left(C_n^2 + \sin^2\frac{n\pi x}{L}\right)dx + \frac{1}{2}M(C_n+1)^2$$

(note: $\int_0^L C_n \sin\frac{n\pi x}{L}\,dx = 0$)

$$= \underbrace{\frac{1}{2}\rho A\int_0^L \sin^2\frac{n\pi x}{L}\,dx}_{T_0} + \underbrace{\frac{1}{2}\rho A C_n^2 L + \frac{1}{2}M(C_n+1)^2}_{\Delta T}$$

(iii) $$\omega_n' \simeq \omega_n\left(1 + \frac{KC_n^2}{2V_0} - \frac{\frac{1}{2}\rho A L C_n^2 + \frac{1}{2}M(C_n+1)^2}{2\tilde{T}_0}\right)$$

$$\omega_n' \simeq \omega_n\left(1 + \frac{KC_n^2}{2V_0} - \frac{\rho A L C_n^2 + \frac{1}{2}M(C_n+1)^2}{4\tilde{T}_0}\right)$$

$$V = \frac{1}{2} EI \int (u'')^2 \, dx + \frac{1}{2} K u(0)^2 + \frac{1}{2} K u(L)^2$$

$$\tilde{T} = \frac{1}{2} \rho A \int u^2 \, dx + \frac{1}{2} M u^2(L/2)$$

$$V_0 = \frac{1}{2} EI \int \sin^2 \frac{n\pi x}{L} \, dx$$

$$\tilde{T}_0 = \frac{1}{2} \rho A \int \left[C_n + 2D_n (x - L/2) + \sin \frac{n\pi x}{L} \right]^2 dx + \cancel{\frac{1}{2} M \left[\quad \right]^2}.$$

$$= \frac{1}{2} \rho A \int C_n^2 + 4 D_n^2 (x - L/2)^2 + \sin^2 \frac{n\pi x}{L}$$
$$+ 2 C_n \sin \frac{n\pi x}{L} + 4 D_n (x - L/2) \sin \frac{n\pi x}{L}$$
$$+ 4 C_n D_n (x - L/2) \, dx.$$

Q3 a)

mode 1 $\quad \underline{u}_1 = [1 \quad 1 \quad 1]^T \quad \omega_1 = 0$

mode 2 $\quad \underline{u}_2 = [+1 \quad 0 \quad -1]^T \quad \omega_2 = \sqrt{\frac{k}{m}}$

mode 3 $\quad \underline{u}_3 = [1 \quad -2 \quad 1]^T \quad \omega_3 = \sqrt{\frac{3k}{m}}$

check orthogonality $\quad [1 \quad 1 \quad 1] \cdot [1 \quad 0 \quad -1] = 0$ ✓

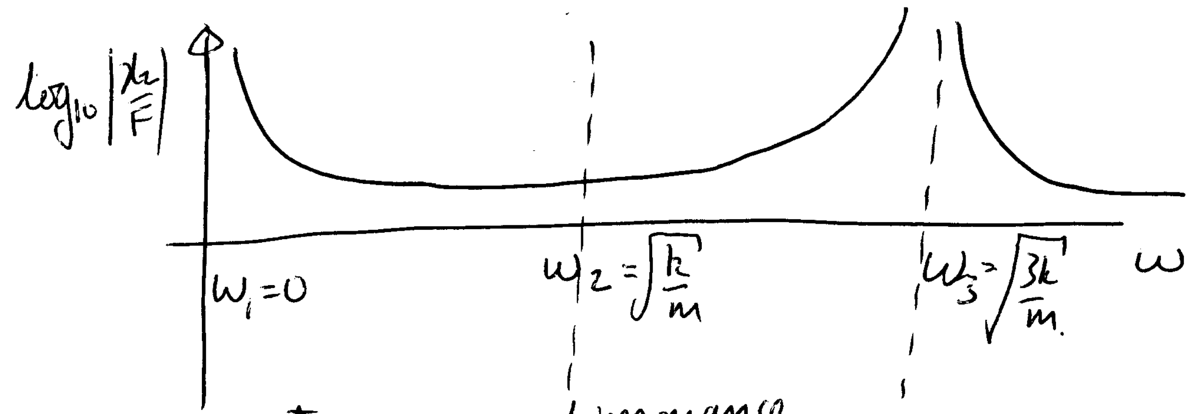
$[1 \quad 1 \quad 1] \cdot [1 \quad -2 \quad 1] = 0$ ✓

$[1 \quad 0 \quad -1] \cdot [1 \quad -2 \quad 1] = 0$ ✓

b)

$$H_{21} = \frac{x_2}{F} = \sum_{n=1}^{3} \frac{u_1^{(n)} u_2^{(n)}}{\omega_n^2 - \omega^2}$$

$$= \frac{\overbrace{u_1^{(1)} u_2^{(1)}}^{+ve}}{\omega_1^2 - \omega^2} + \frac{\overbrace{u_1^{(2)} u_2^{(2)}}^{0}}{\omega_2^2 - \omega^2} + \frac{\overbrace{u_1^{(3)} u_2^{(3)}}^{-ve}}{\omega_3^2 - \omega^2}$$



note
- no antiresonance.
- mode 2 does not contribute

c) normalise the modes $\underline{u}^{(j)T}[M]\underline{u}^{(k)} = 1 \quad j=k$ (data sheet)

mode 1 $\quad [\alpha \;\; \alpha \;\; \alpha]\begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} \alpha \\ \alpha \\ \alpha \end{Bmatrix} = 1$

$$3\alpha^2 m = 1$$
$$\alpha = \frac{1}{\sqrt{3m}} \qquad \therefore \underline{u}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}\frac{1}{\sqrt{3m}}$$

mode 3 $\quad [\beta \;\; -2\beta \;\; \beta]\begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} \beta \\ -2\beta \\ \beta \end{Bmatrix} = 1$

$$6\beta^2 m = 1,$$
$$\beta = \frac{1}{\sqrt{6m}} \qquad \therefore \underline{u}^{(3)} = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}\frac{1}{\sqrt{6m}}$$

unit impulse response

$$g(j,k,t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t. \quad \text{(data sheet)}$$

$$g(j=1,k=2,t) = \sum_{n=1}^{3} \frac{u_1^{(n)} u_2^{(n)}}{\omega_n} \sin \omega_n t$$

$$= \underbrace{\left(\frac{1}{\sqrt{3m}}\right)^2 \cancel{\frac{\sin \omega_1 t}{\omega_1}}^{=t \text{ for } \omega_1 = 0}}_{n=1,\ \omega_1=0} + \underbrace{0}_{n=2} + \underbrace{\left(\frac{1}{\sqrt{6m}}\right)\left(\frac{-2}{\sqrt{6m}}\right)\frac{\sin \omega_3 t}{\omega_3}}_{n=3,\ \omega_3 = \sqrt{\frac{3k}{m}}}$$

$$= \frac{t}{3m} - \frac{2}{6m}\sqrt{\frac{m}{3k}} \sin\sqrt{\frac{3k}{m}}\, t$$

$$= \frac{t}{3m} - \frac{1}{3\sqrt{3mk}} \sin\sqrt{\frac{3k}{m}}\, t$$

d) Use Rayleigh's quotient

$$V = \tfrac{1}{2}k(x_1-x_2)^2 + \tfrac{1}{2}k(x_2-x_3)^2$$

$$\hat{T} = \tfrac{1}{2}m x_1^2 + \tfrac{1}{2}m x_2^2 + \tfrac{1}{2}m(1+\delta)x_3^2$$

$$R = \frac{V}{\tilde{T}} = \frac{k}{m}\,\frac{(x_1-x_2)^2+(x_2-x_3)^2}{(x_1^2+x_2^2+x_3^2(1+\delta))}$$

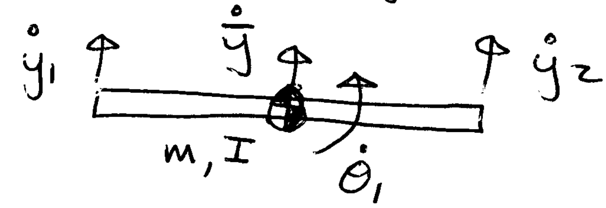
mode 3 $[1, -2 \; 1]^T$

$$\therefore R = \frac{k}{m}\,\frac{3^2+3^2}{1^2+2^2+1^2(1+\delta)} = \frac{k}{m}\,\frac{18}{6+\delta}.$$

$$\therefore \underline{\underline{\omega_3 \simeq \sqrt{\frac{k}{m}}\sqrt{\frac{18}{6+\delta}}}}$$

compared to $\omega_3 = \sqrt{\frac{k}{m}}\sqrt{3}$ from part (a).

Q4. a) left hand carriage beam.



$$T_1 = \tfrac{1}{2} m \dot{\bar{y}}^2 + \tfrac{1}{2} I \dot{\theta}_1^2$$

$$\boxed{\begin{aligned} I &= \tfrac{1}{12} m L^2 \\ \dot{\bar{y}} &= \frac{\dot{y}_1 + \dot{y}_2}{2} \\ \dot{\theta}_1 &= \frac{\dot{y}_2 - \dot{y}_1}{L} \end{aligned}}$$

$$\therefore T_1 = \tfrac{1}{2} m \left(\frac{\dot{y}_1 + \dot{y}_2}{2}\right)^2 + \tfrac{1}{2} \frac{m \cancel{L^2}}{12} \left(\frac{\dot{y}_2 - \dot{y}_1}{\cancel{L}}\right)^2$$

$$= m\left(\tfrac{1}{8}\left(\dot{y}_1^2 + 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2\right) + \tfrac{1}{24}\left(\dot{y}_2^2 - 2\dot{y}_1\dot{y}_2 + \dot{y}_1^2\right)\right)$$

$$= \frac{m}{24}\left(4\dot{y}_1^2 + 4\dot{y}_1\dot{y}_2 + 4\dot{y}_2^2\right)$$

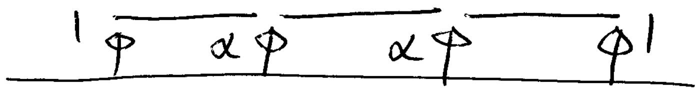
$$\underline{\underline{T_1 = \frac{m}{6}\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_1\dot{y}_2\right)}}$$

$$\text{total } T = \frac{m}{6}\left(\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_1\dot{y}_2\right) + \left(\dot{y}_2^2 + \dot{y}_3^2 + \dot{y}_2\dot{y}_3\right) + \left(\dot{y}_3^2 + \dot{y}_4^2 + \dot{y}_3\dot{y}_4\right)\right)$$

$$\underline{\underline{T = \frac{m}{6}\left(\dot{y}_1^2 + 2\dot{y}_2^2 + 2\dot{y}_3^2 + \dot{y}_4^2 + \dot{y}_1\dot{y}_2 + \dot{y}_2\dot{y}_3 + \dot{y}_3\dot{y}_4\right)}}$$

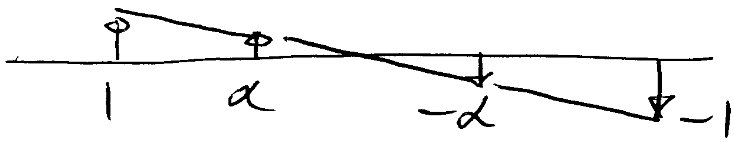
$$\underline{\underline{\text{total } V = \tfrac{1}{2} k\left(y_1^2 + 2y_2^2 + 2y_3^2 + y_4^2\right)}}$$

b) mode 1



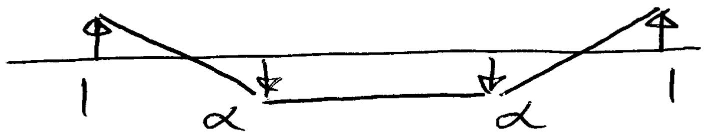
0 nodes symmetric

mode 2



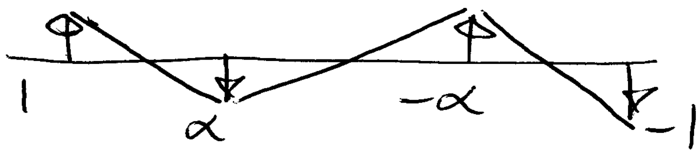
1 node anti-symmetric

mode 3



2 nodes symmetric

mode 4.



3 nodes antisymmetric

c) antisymmetric mode $[1 \quad \alpha \quad -\alpha \quad -1]^T$

$$R = \frac{V}{\tilde{T}} = \frac{\frac{k}{2}(1^2+2\alpha^2+2\alpha^2+1^2)}{\frac{m}{6}(1^2+2\alpha^2+2\alpha^2+1^2+\alpha-\alpha^2+\alpha)}$$

$$R = \frac{3k}{m}\,\frac{4\alpha^2+2}{3\alpha^2+2\alpha+2}$$

$$\frac{dR}{d\alpha} = \frac{3k}{m}\,\frac{(3\alpha^2+2\alpha+2)8\alpha-(4\alpha^2+2)(6\alpha+2)}{(3\alpha^2+2\alpha+2)^2}$$

equate to zero

$$24\alpha^3+16\alpha^2+16\alpha = 24\alpha^3+8\alpha^2+12\alpha+4$$

$$8\alpha^2+4\alpha-4=0$$

$$2\alpha^2+\alpha-1=0$$

$$\alpha = \frac{-1 \pm \sqrt{1^2-4.2.(-1)}}{2.2}$$

$$\alpha = -\frac{1}{4} \pm \frac{3}{4}$$

$$\alpha = -1 \quad \text{or} \quad +\frac{1}{2}$$

mode 4   mode 2

d) consider equation of motion for y_1

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) - \frac{\partial T}{\partial y_1} + \frac{\partial V}{\partial y_1} = 0. \quad \text{(data sheet)}$$

$$\frac{\partial T}{\partial \dot{y}_1} = \frac{m}{6}(2\dot{y}_1 + \dot{y}_2) \qquad \frac{\partial T}{\partial y_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) = \frac{m}{6}(2\ddot{y}_1 + \ddot{y}_2)$$

V with gravity terms: $V = \frac{1}{2}k(y_1^2 + 2y_2^2 + 2y_3^2 + y_4^2)$
$+ \frac{1}{2}mg(y_1 + 2y_2 + 2y_3 + y_4)$

$$\frac{\partial V}{\partial y_1} = \frac{1}{2}k(2y_1) + \frac{1}{2}mg$$

EoM:

$$\frac{m}{6}(2\ddot{y}_1 + \ddot{y}_2) + \frac{1}{2}k(2y_1) + \frac{1}{2}mg = 0$$

steady state $\qquad ky_1 + \frac{1}{2}mg = 0$

$$y_1 = -\frac{1}{2}\frac{mg}{k}$$

this is simply the static deflection due to weight.

Thus gravitational potential energy only affects equilibrium position, there is no influence on stiffness.

An example of a system that does require account of gravitational potential energy is a pendulum.

[above solution to part (d) is longer than required]