

EGT2
ENGINEERING TRIPOS PART IIA

Monday 5 May 2025 9.30 to 11.10

Module 3C6

VIBRATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Attachment: 3C5 Dynamics and 3C6 Vibration data sheet (7 pages).

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) A thin horizontal string of length L has tension P and mass per unit length m . The distance along the string is x , and the small amplitude transverse displacement of the string is denoted $y(x, t)$. The string is fixed at both ends, and only transverse displacement should be considered in this part of the question.

(i) Find an expression for the mode shapes $u_n(x)$ and natural frequencies ω_n of the string. A full derivation is not needed. [10%]

(ii) At $t = 0$, a step force F_0 is applied to the string at $x = a$ as shown in Fig. 1(a). Find the transverse displacement of the string $y(x, t)$ using a modal summation expression, including a modal damping ratio ζ_n . [20%]

(iii) The string is instead pulled into a triangular shape using a force F_0 at $x = a$. The string is released from rest at $t = 0$. Using your answer from Part (a)(ii), or otherwise, find the transverse displacement of the string $y(x, t)$. [10%]

(b) The effect of finite string thickness is now considered. Consider the string from Part (a), now with circular cross-section of radius r , polar moment of area $J = \frac{1}{2}\pi r^4$, Young's modulus E , shear modulus G and density ρ (where $m = \rho\pi r^2$). The string can undergo small amplitude transverse, torsional and axial motion. The string is fixed at both ends for all degrees of freedom and has no bending stiffness.

(i) Find expressions for all of the mode shapes and natural frequencies of the string, including transverse, torsional and axial motion. A full derivation is not needed. [10%]

(ii) Obtain a modal summation expression for the driving point transfer function $H(a, a, \omega)$, from a vertical input force F applied at Point A on the surface of the string to the vertical output displacement also at Point A on the surface of the string, as illustrated in Fig. 1(b). Point A is horizontally level with the centre of the string. Take into account all relevant degrees of freedom and include modal damping. [20%]

(iii) For the case when $a = L/2$ and $\sqrt{G/\rho} = 1.2\sqrt{P/m}$, sketch the magnitude of $H(a, a, \omega)$ including the first four resonant peaks. Use a decibel scale for the amplitude and carefully label resonant frequencies, noting which mode they correspond to. [30%]

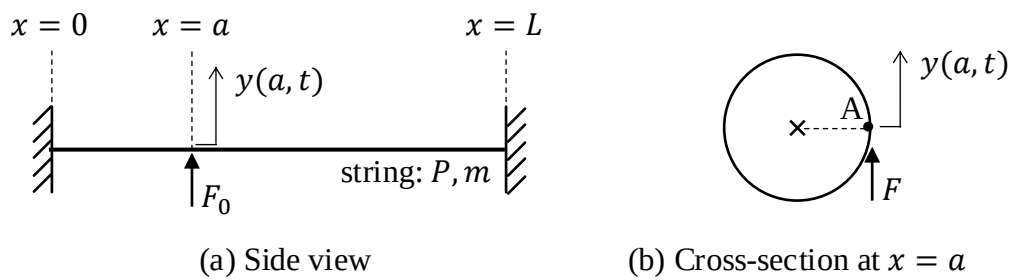


Fig. 1

2 A beam of length L , uniform cross-section of area A and second moment of area I is made from a material with density ρ and Young's modulus E . The distance from one end of the beam is x , and the small amplitude transverse deflection of the beam is y . The beam is pinned at both ends.

(a) Find the mode shapes $u_n(x)$ and natural frequencies ω_n of the beam. A full derivation is not needed. [10%]

(b) Find a summation expression for the transfer function $G(x_1, x_2, \omega)$ from an input force at $x = x_1$ to an output *velocity* at $x = x_2$, including a modal damping ratio ζ_n . [20%]

(c) By considering small changes $V = V_0 + \Delta V$ and $\tilde{T} = \tilde{T}_0 + \Delta\tilde{T}$, use Rayleigh's quotient to show that the approximate new natural frequencies ω'_n of a modified structure can be written in the form:

$$\omega'_n \approx \omega_n (1 + \alpha_n \Delta V + \beta_n \Delta\tilde{T})$$

and find expressions for α_n and β_n in terms of V_0 and $\tilde{T}_0$. [20%]

(d) A mass M is connected to the centre of the beam, and a spring of stiffness K is added to each end of the beam, as shown in Fig. 2. Consider the trial mode shape:

$$u_n(x) = C_n + 2D_n \left(x - \frac{L}{2} \right) + \sin \left(\frac{n\pi x}{L} \right).$$

(i) Without calculation and by considering symmetry, justify why either C_n or D_n is zero for a given mode. [10%]

(ii) For the symmetric modes, find expressions for V and $\tilde{T}$. Write your answer in the form $V = V_0 + \Delta V$ and $\tilde{T} = \tilde{T}_0 + \Delta\tilde{T}$, where V_0 and $\tilde{T}_0$ are the quantities for an unmodified pinned-pinned beam. The integrals for V_0 and $\tilde{T}_0$ need not be evaluated. [30%]

(iii) For the symmetric modes, find an expression for the approximate natural frequencies of the beam with the added mass and springs, in terms of the coefficients C_n , D_n , the quantities V_0 and $\tilde{T}_0$, and the other system parameters. The integrals for V_0 and $\tilde{T}_0$ need not be evaluated. [10%]

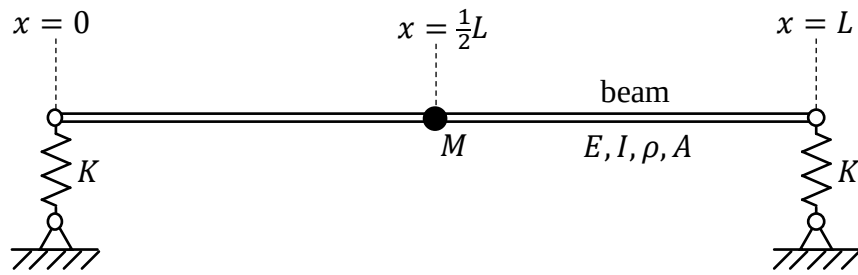


Fig. 2

3 The system shown in Fig. 3 consists of three masses, each with mass m , connected by two elastic springs each with stiffness k . The masses move horizontally without friction and gravity may be neglected. Damping is negligible. The absolute displacements of the masses are x_1, x_2, x_3 .

- (a) Determine, by inspection or otherwise, the natural frequencies and natural mode shapes of vibration of the system. [20%]
- (b) Sketch a log amplitude plot for the transfer function describing the displacement x_2 due to a sinusoidal force F applied to the mass with coordinate x_1 . Show salient values. [20%]
- (c) The system is initially at rest and in equilibrium. The force F is a unit impulse at time $t = 0$. Determine the displacement x_2 as a function of time t . [40%]
- (d) Estimate the natural frequency of the highest-frequency mode if the mass with coordinate x_3 is increased to $m(1 + \delta)$. [20%]

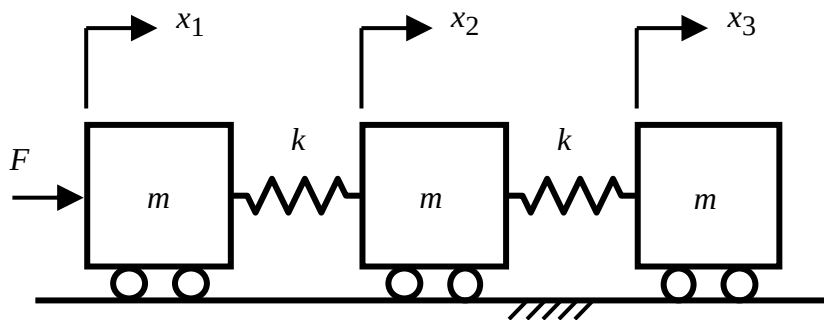


Fig. 3

4 The carriages of a commuter train are modelled as three beams each with length L and uniformly distributed mass m , as shown in Fig. 4. The carriages are coupled together with frictionless pivots. The vertical displacements at the ends of the carriages are y_1 , y_2 , y_3 , y_4 . The carriages are suspended on elastic springs with stiffness k , $2k$, $2k$, k . Damping is negligible.

(a) Show that the kinetic energy of the left-hand carriage is $(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_1\dot{y}_2)m/6$. Write an expression for the total kinetic energy of the system. Write an expression for the total potential energy of the system, neglecting the effect of gravity. [30%]

(b) Sketch the natural mode shapes of the system in order of increasing natural frequency. Indicate symmetric and antisymmetric modes. It should be not necessary to perform any calculation in this part of the question. [20%]

(c) The antisymmetric mode shapes may be parameterised as $\begin{bmatrix} 1 & \alpha & -\alpha & -1 \end{bmatrix}^T$. By minimising Rayleigh's quotient find the value of α for each of the antisymmetric modes. [35%]

(d) Explain why gravity can be neglected in the potential energy expression when analysing the vibration of this system. Give an example of a different system in which it would be important to include gravitational potential energy. [15%]

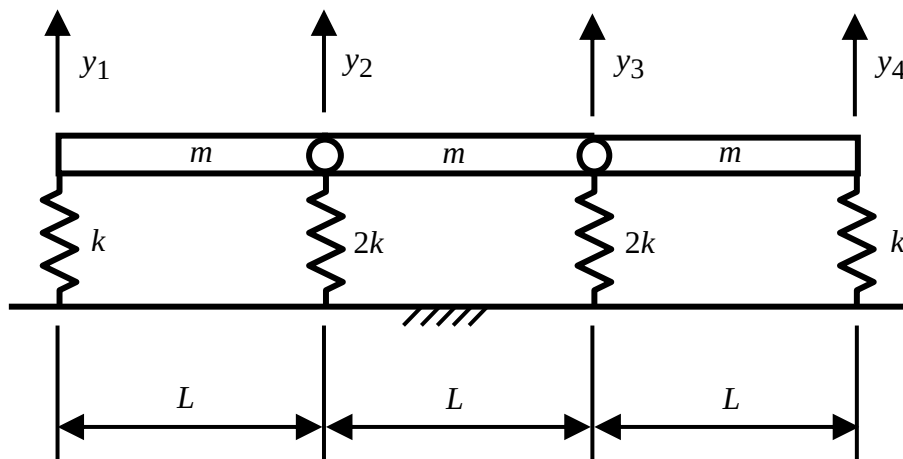


Fig. 4

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Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i :

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}^{(e)}$$

where $\mathbf{p} = M\mathbf{v}_G$, M is the total mass, $\mathbf{v}_G$ is the velocity of the centre of mass and $\mathbf{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\begin{aligned}\mathbf{Q}^{(e)} &= (\mathbf{r}_G - \mathbf{r}_P) \times \dot{\mathbf{p}} + \dot{\mathbf{h}}_G \\ &= \dot{\mathbf{h}}_P + \dot{\mathbf{r}}_P \times \mathbf{p}\end{aligned}$$

where $\mathbf{Q}^{(e)}$ is the total moment of external forces about P. Here $\mathbf{h}_P$ and $\mathbf{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\begin{aligned}\mathbf{h}_P &= \sum_i (\mathbf{r}_i - \mathbf{r}_P) \times m_i \dot{\mathbf{r}}_i \\ &= \mathbf{h}_G + (\mathbf{r}_G - \mathbf{r}_P) \times \mathbf{p}\end{aligned}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\mathbf{h}_P = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dm = \mathbf{I}\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\mathbf{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned}\text{and } A &= \int (y^2 + z^2) dm & B &= \int (z^2 + x^2) dm & C &= \int (x^2 + y^2) dm \\ D &= \int yz \, dm & E &= \int zx \, dm & F &= \int xy \, dm\end{aligned}$$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity Ω

Time derivatives of vectors must be replaced by the “rotating frame” form, so that for example

$$\dot{\mathbf{p}} + \Omega \times \mathbf{p} = \mathbf{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\mathbf{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler’s dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = Q_1$$

$$B\dot{\omega}_2 - (C - A)\omega_3\omega_1 = Q_2$$

$$C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = Q_3$$

where A , B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\mathbf{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the “Gyroscope equations”):

$$A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 = Q_1$$

$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2$$

$$C\dot{\omega}_3 = Q_3$$

where A , A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\mathbf{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange’s equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_i} \right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalised momenta p_i and the Hamiltonian $H(\mathbf{p}, \mathbf{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \quad H(\mathbf{p}, \mathbf{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\mathbf{p}, \mathbf{q})$ must be expressed as a function of the generalised momenta and the generalised displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

Special case when the kinetic energy is expressible using a mass matrix $\mathbf{M}$:

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} = \frac{1}{2} \mathbf{p}^T \mathbf{M}^{-1} \mathbf{p} \quad \text{and} \quad H = T + V$$

(b) Extension topics

The total time derivative of some function $f(\mathbf{p}, \mathbf{q}, t)$ can be expressed in terms of the Poisson bracket $\{f, H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}, \quad \{f, H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\mathbf{q}, \mathbf{Q}, t)$	$\mathbf{p} = \frac{\partial G_1}{\partial \mathbf{q}}$	$\mathbf{P} = -\frac{\partial G_1}{\partial \mathbf{Q}}$	$K = H + \frac{\partial G_1}{\partial t}$
2	$G_2(\mathbf{q}, \mathbf{P}, t)$	$\mathbf{p} = \frac{\partial G_2}{\partial \mathbf{q}}$	$\mathbf{Q} = \frac{\partial G_2}{\partial \mathbf{P}}$	$K = H + \frac{\partial G_2}{\partial t}$
3	$G_3(\mathbf{p}, \mathbf{Q}, t)$	$\mathbf{q} = -\frac{\partial G_3}{\partial \mathbf{p}}$	$\mathbf{P} = -\frac{\partial G_3}{\partial \mathbf{Q}}$	$K = H + \frac{\partial G_3}{\partial t}$
4	$G_4(\mathbf{p}, \mathbf{P}, t)$	$\mathbf{q} = -\frac{\partial G_4}{\partial \mathbf{p}}$	$\mathbf{Q} = \frac{\partial G_4}{\partial \mathbf{P}}$	$K = H + \frac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems

1. Equation of motion

The forced vibration of an N -degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

2. Kinetic Energy

$$T = \frac{1}{2} \dot{\mathbf{y}}^T \mathbf{M} \dot{\mathbf{y}}$$

3. Potential Energy

$$V = \frac{1}{2} \mathbf{y}^T \mathbf{K} \mathbf{y}$$

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2 \mathbf{M}\mathbf{u}^{(n)}$$

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)T} \mathbf{M} \mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)T} \mathbf{K} \mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^N q_j(t) \mathbf{u}^{(j)} = \mathbf{U} \mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the ‘quantity’ of the j th mode.

Continuous Systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

$$T = \frac{1}{2} \int \dot{y}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See Section 3 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x) u_k(x) dm = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses:

$$y(x, t) = \sum_j q_j(t) u_j(x)$$

where $y(x, t)$ is the displacement and q_j can be thought of as the ‘quantity’ of the j th mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + [\text{diag}(\omega_j^2)] \mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T \mathbf{f}$.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k , the transfer function is

$$H(j, k, \omega) = \frac{y_k}{f_j} = \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j, k, \omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j, k, t) = y_k(t) = \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j, k, t) \approx \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x, t) u_j(x) dm$ and $f(x, t)$ is the external applied force distribution.

For force F at frequency ω applied at point x_1 , and displacement y measured at point x_2 , the transfer function is

$$H(x_1, x_2, \omega) = \frac{y}{F} = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1, x_2, \omega) = \frac{y}{F} \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1) u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

For a unit impulse applied at $t = 0$ at point x_1 , the response at point x_2 is

$$g(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j, k, t) = y_k(t) = \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} [1 - \cos \omega_n t]$$

for $t \geq 0$ (with no damping), or

$$h(j, k, t) \approx \sum_{n=1}^N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} [1 - e^{-\omega_n \zeta_n t} \cos \omega_n t]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1 , the response at point x_2 is

$$h(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} [1 - \cos \omega_n t]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} [1 - e^{-\omega_n \zeta_n t} \cos \omega_n t]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The “Rayleigh quotient” for a discrete system is

$$\frac{V}{\tilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be

- (1) $\geq$ the smallest squared natural frequency;
- (2) $\leq$ the largest squared natural frequency;
- (3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\frac{V}{\tilde{T}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P , mass per unit length m , transverse displacement $y(x, t)$, applied lateral force $f(x, t)$ per unit length.

Equation of motion

Potential energy

Kinetic energy

$$m \frac{\partial^2 y}{\partial t^2} - P \frac{\partial^2 y}{\partial x^2} = f(x, t)$$

$$V = \frac{1}{2} P \int \left(\frac{\partial y}{\partial x} \right)^2 dx$$

$$T = \frac{1}{2} m \int \left(\frac{\partial y}{\partial t} \right)^2 dx$$

3.2 Torsional vibration of a circular shaft

Shear modulus G , density ρ , external radius a , internal radius b if shaft is hollow, angular displacement $\theta(x, t)$, applied torque $\tau(x, t)$ per unit length. The polar moment of area is given by $J = (\pi/2)(a^4 - b^4)$.

Equation of motion

Potential energy

Kinetic energy

$$\rho J \frac{\partial^2 \theta}{\partial t^2} - G J \frac{\partial^2 \theta}{\partial x^2} = \tau(x, t)$$

$$V = \frac{1}{2} G J \int \left(\frac{\partial \theta}{\partial x} \right)^2 dx$$

$$T = \frac{1}{2} \rho J \int \left(\frac{\partial \theta}{\partial t} \right)^2 dx$$

3.3 Axial vibration of a rod or column

Young's modulus E , density ρ , cross-sectional area A , axial displacement $y(x, t)$, applied axial force $f(x, t)$ per unit length.

Equation of motion

Potential energy

Kinetic energy

$$\rho A \frac{\partial^2 y}{\partial t^2} - E A \frac{\partial^2 y}{\partial x^2} = f(x, t)$$

$$V = \frac{1}{2} E A \int \left(\frac{\partial y}{\partial x} \right)^2 dx$$

$$T = \frac{1}{2} \rho A \int \left(\frac{\partial y}{\partial t} \right)^2 dx$$

3.4 Bending vibration of an Euler beam

Young's modulus E , density ρ , cross-sectional area A , second moment of area of cross-section I , transverse displacement $y(x, t)$, applied transverse force $f(x, t)$ per unit length.

Equation of motion

Potential energy

Kinetic energy

$$\rho A \frac{\partial^2 y}{\partial t^2} + E I \frac{\partial^4 y}{\partial x^4} = f(x, t)$$

$$V = \frac{1}{2} E I \int \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx$$

$$T = \frac{1}{2} \rho A \int \left(\frac{\partial y}{\partial t} \right)^2 dx$$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned} \cos \alpha \cosh \alpha + 1 &= 0, & \alpha_1 &= 1.8751 \\ \cos \alpha \cosh \alpha - 1 &= 0, & \alpha_1 &= 4.7300 \\ \tan \alpha - \tanh \alpha &= 0, & \alpha_1 &= 3.9266 \end{aligned}$$