Q1 (a)(i) fixed-free shaft,

1D wave equation so sinusoidal mode shapes

$$\Rightarrow \quad u_n(x) = A\sin kx + \cancel{B\cos kx}$$

0 as $u(0) = 0$.

$$\& \quad u_n'(L) = Ak\cos kL = 0$$

$$\Rightarrow kL = \pi/2,\ 3\pi/2,\ \dots$$

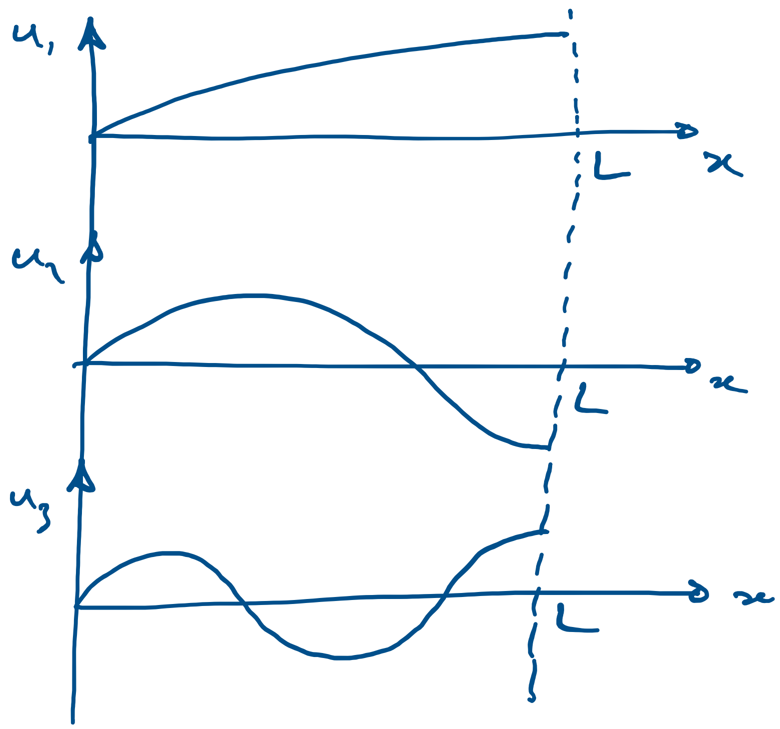
$$= (n - 1/2)\pi$$

$$u_n = A_n \sin\frac{(n-1/2)\pi x}{L} = A_n \sin\frac{(2n-1)\pi x}{2L}$$

$$\& \quad \omega_n = ck_n = \frac{(n-1/2)\pi}{L}\cdot\sqrt{G/\rho} = \frac{2n-1}{2L}\pi\sqrt{G/\rho}$$

[can quote results/by inspection as only asked to 'find'].

(ii)



no rigid body modes

(iii) $$H_A = \sum_n \frac{u_n(x_1)\,u_n(x_2)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

mass normalise : $u_n = A_n \sin kx$.

$$\int_0^L A_n^2 \sin^2 kx \; \rho J_1 \, dx = 1$$

$$\rho J_1 A_n^2 \int_0^L \tfrac{1}{2}\left(1 - \cos 2kx\right) dx = 1$$

$$\int_0^L \cos 2kx = 0$$

so $\rho J_1 A_n^2 \cdot L/2 = 1 \implies A_n^2 = \dfrac{2}{\rho J_1 L_1}$

$$\implies H_A = \frac{2}{\rho J_1 L_1} \sum_{n=1}^{\infty} \frac{\sin k_n x_1 \, \sin k_n x_2}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

where $\omega_n = \dfrac{2n-1}{2L}\pi\sqrt{\dfrac{G}{\rho}}$

& $k_n = \dfrac{2n-1}{2L}\cdot\pi$

(iv) torque $= -GJ_1\theta'$

$$H_A = \frac{\theta(x_2)}{T(x_1)}$$

$$H_B = \frac{-GJ_1\theta'(x_2)}{T(x_1)} = -GJ_1 \left.\frac{\partial H_A}{\partial x_2}\right|_{x_1 = L,\; x_2 = 0}.$$

$$\frac{\partial H_A}{\partial x_2} = \frac{2}{\rho J_1 L_1} \sum \frac{\sin k_n x_1 \, . \, k_n \cos k_n x_2}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}\Bigg|_{\substack{x_1 = L_1 \\ x_2 = 0.}}$$

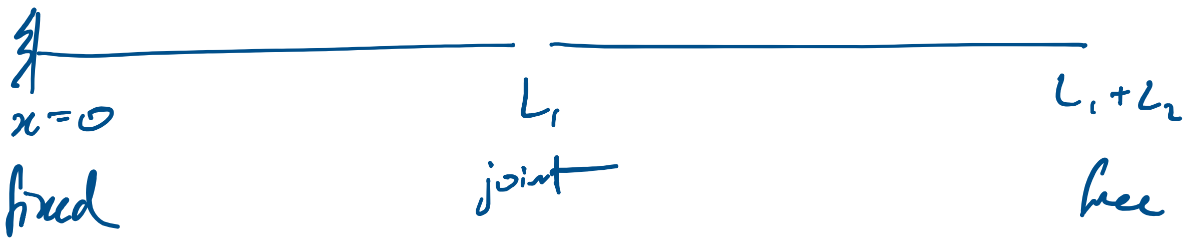
$$(-1)^{n-1}$$

$$= \frac{2}{\rho J_1 L_1} \sum \frac{\sin (2n-1)\pi \, . \, k_n}{\sim\sim\sim}$$

$$H_B = -GJ\frac{\partial H_A}{\partial x_2} = + \frac{2G\cancel{J}}{\rho \cancel{J} L} \, . \, \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)\pi/2L}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

$$H_B = \frac{\pi G}{\rho L^2} \, . \, \sum_{n=1}^{\infty} \frac{(-1)^n . (2n-1)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

(b) (i)



$A_1 = 0$ to satisfy $u_1(0) = 0$.

$B_2 = 0$ to satisfy $u_2'(L_1+L_2) = 0$

(ii) continuity $\Rightarrow$ $u_1(L_1) = u_2(L_1)$ —— (1)

& force balance $\Rightarrow$ $GJ_1 u_1'(L_1) = GJ_2 u_2'(L_1)$ —— (2)

(iii) $u_1 = B_1 \sin kx$

$u_2 = A_2 \cos(k(x - L_1 - L_2))$

$(1) \Rightarrow \quad B_1 \sin kL_1 = A_2 \cos kL_2$

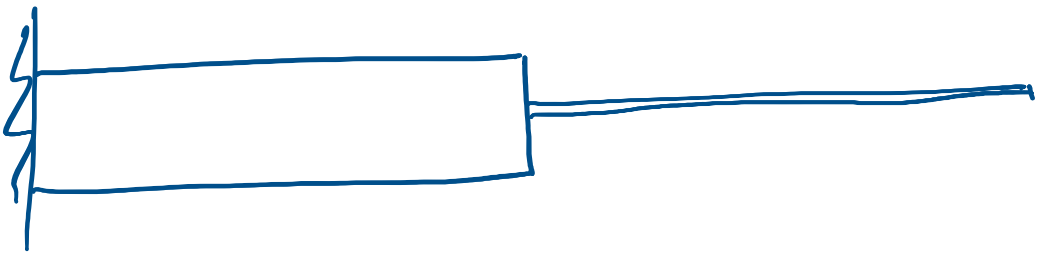
$(2) \Rightarrow \quad \cancel{G} J_1 B_1 \cancel{k} \cos kL_1 = -\cancel{G} J_2 A_2 \cancel{k} \sin(-kL_2)$

$$B_1 J_1 \cos kL_1 = A_2 J_2 \sin kL_2$$

$\frac{(2)}{(1)} \Rightarrow \quad J_2 \tan kL_2 = J_1 \cot kL_1$ //

or $\quad J_2 \sin kL_1 \sin kL_2 = J_1 \cos kL_1 \cos kL_2$ //

(iv) If $J_2 \ll J_1$ then system will be like:



So second shaft has negligible effect on first, & first looks like fixed end to second...

$\Rightarrow \quad \omega_{n1} = \frac{2n-1}{2L_1} \pi \sqrt{G/\rho}$ (fixed free modes of shaft 1)

& $\omega_{n2} = \frac{2n-1}{2L_2} \pi \sqrt{G/\rho}$ (fixed free modes of shaft 2) //.

Ie. weakly coupled.

Q2/(a) free-free beam: $u'' = u''' = 0$ @ $x = 0, L$

$$u = D_1 \cos kx + D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx$$

$$u' = \left(-D_1 \sin + D_2 \cos + D_3 \sinh + D_4 \cosh\right) k$$

$$u'' = \left(-D_1 \cos - D_2 \sin + D_3 \cosh + D_4 \sinh\right) k^2$$

$$u''' = \left(D_1 \sin - D_2 \cos + D_3 \sinh + D_4 \cosh\right) k^3$$

$x = 0 \Rightarrow$

$$-D_1 + D_3 = 0 \quad \Rightarrow \quad D_1 = D_3$$

$$-D_2 + D_4 = 0 \quad \Rightarrow \quad D_2 = D_4.$$

$x = L \Rightarrow$

$$\begin{bmatrix} \cosh - \cos & \sinh - \sin \\ \sinh + \sin & \cosh - \cos \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\det = 0 \Rightarrow$

$$(\cosh - \cos)^2 - (\sinh - \sin)(\sinh + \sin) = 0.$$

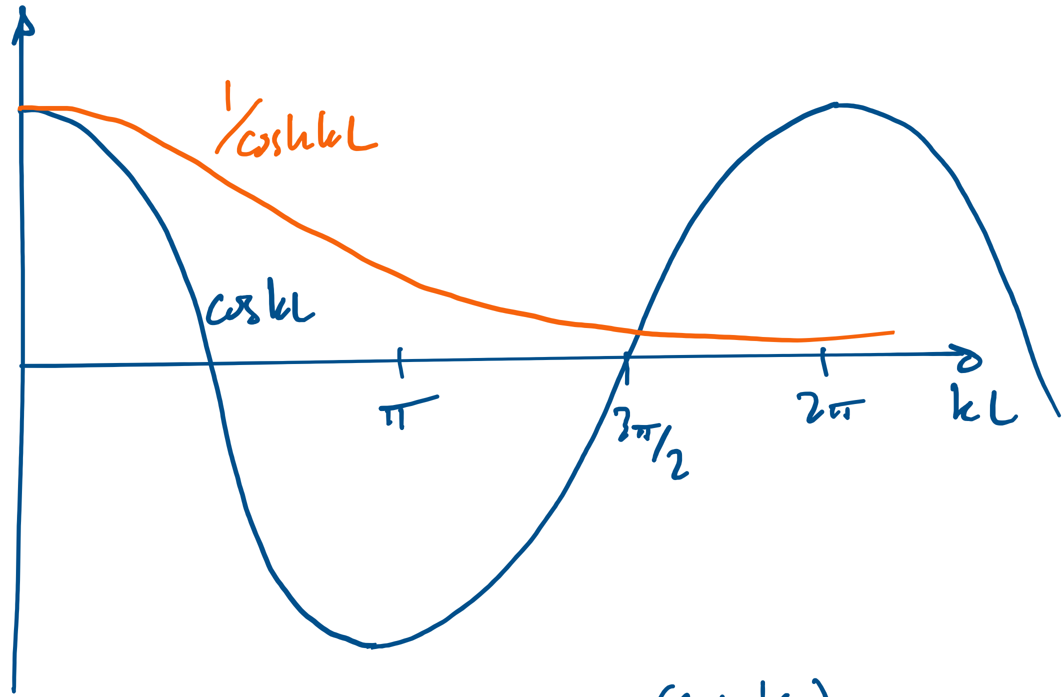
$$\overbrace{\cosh^2 - 2\cos\cosh + \cos^2 - \sinh^2}^{1} + \sin^2 = 0.$$

$$\cosh^2 - 2\cos\cosh + \underbrace{\cos^2 - \sinh^2 + \sin^2}_{1} = 0.$$

$$2 - 2\cos\cosh = 0$$

$$\Rightarrow \quad \cos kL \cosh kL = 1$$

(ii)



Intersections @ $kL \simeq (n+\tfrac{1}{2})\pi$

$\simeq 4.712,\ 7.854,\ 10.996$

(iii) $\omega \propto k^2$

$n = 1,\ 2,\ 3$

$k^2 = 22.203,\ \cancel{61.685},\ 120.912$

$\frac{\omega_n}{\omega_1} = 1,\ 2.78,\ 5.45$ //

(iv) $\omega_n^2 = \frac{V}{\tilde{T}} = \frac{\frac{1}{2}EI\int u''^2\,dx}{\frac{1}{2}\rho A\int u^2\,dx}$ for true mode shape u.

Now $I \propto h^3$ & $A \propto h$

so $\omega_n^2 \propto h^2$ & $\omega_n \propto h^1$, ie $p=1$ //

(b) (i) $\omega_n'^2 = \frac{V}{T} = \frac{V_0 + V'}{\tilde{T}}$

$$= \omega_n^2 \left(1 + V'/V_0\right)$$

now $V = \frac{1}{2} EI \int_0^L u''^2 \, dx$

$$= V_0 - \frac{1}{2} E \Delta I \int_{L/2 - b/2}^{L/2 + b/2} u''^2 \, dx$$

$$= V_0 \underbrace{- \frac{1}{2} E \Delta I \, b \, u''^2(L/2)}_{V'}$$ using given approximation.

$$V_0 = \frac{1}{2} EI \int_0^L u''^2 \, dx.$$

so $A\phi_n = \dfrac{\frac{1}{2} E \Delta I b u''^2(L/2)}{\frac{1}{2} EI \int_0^L u''^2 \, dx.}$

$$= \underbrace{\frac{\Delta I}{I} \cdot \frac{b}{L}}_{} \cdot \underbrace{\frac{L u_n''^2(L/2)}{\int_0^L u_n''^2 \, dx}}_{}$$

$$= A \quad . \quad \phi_n \quad //.$$

& $f(x) = [u_n''(x)]^2$ //.

(ii) $$\left(\frac{\omega_2'}{\omega_1'}\right)^2 = \frac{\omega_2^2(1-A\phi_2)}{\omega_1^2(1-A\phi_1)}$$

$$= \left(\frac{\omega_2}{\omega_1}\right)^2 \frac{1-A\phi_2}{1-A\phi_1}$$

$$= 2.78^2 \cdot \frac{1-A\ \phi_2}{1-A\ \phi_1}$$

ie $B = 2.78^2 = 7.73$

& $C = A = \frac{\Delta I}{I} \cdot \frac{b}{L}$

(iii) $$7.73 \frac{1-0.08C}{1-3.2C} = 16.$$

$$7.73(1-0.08C) = 16(1-3.2C)$$

$$50.6C = 8.27$$

$$C = 0.16$$

ie $\frac{\Delta I}{I} \cdot \frac{b}{L} = 0.16$

If $b/L = 0.1$, need $\frac{\Delta I}{I} = 1.6$, but can't have $\Delta I > I$

so not possible

Q3

(a) $k = \frac{3EA}{L}$, $m = M/4$

Mass matrix by inspection $[M] = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$

Stiffness equation by energy:

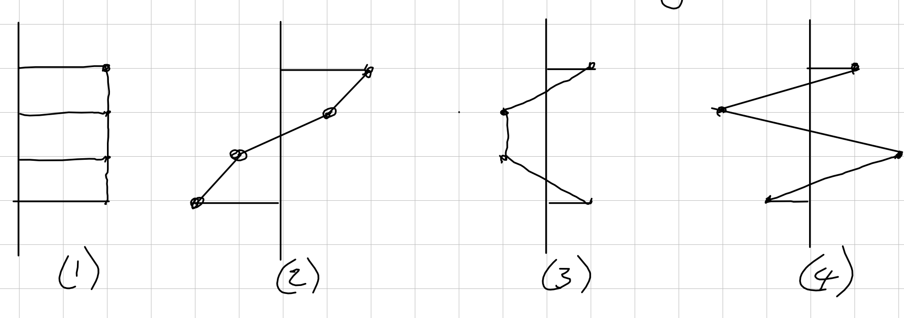
$$V = \tfrac{1}{2}k\left[(y_2-y_1)^2 + (y_3-y_2)^2 + (y_4-y_3)\right]$$

$$= \tfrac{1}{2}k\left[y_1^2 + 2y_2^2 + 2y_3^2 + y_4^2 - 2y_1y_2 - 2y_2y_3 - 2y_3y_4\right]$$

$$= \tfrac{1}{2}\underline{q}^T[K]\underline{q}$$

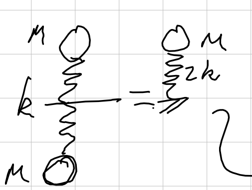
where $[K] = k\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

(b) Modes are symmetric or antisymmetric



(c) Mode 1: Rigid body translation $\omega_1^2 = 0$ Exact

Mode 3: The spring between masses 2 and 3 is unstretched, so masses 1 & 2 vibrate independently of masses 3 & 4. Therefore the frequency is as if each pair vibrates with a node at mid spring



$\omega_3^2 = 2k/m$

Exact

3 cont.

Use Rayleigh for modes 2 & 4. Either guess the mode shapes or assume $\underset{\sim}{q} = [1 \; \alpha \; -\alpha \; -1]^T$ and minimise to find α

$$R = \frac{V_{max}}{T^*} = \frac{\frac{1}{2}k\left[(\alpha-1)^2 + (-\alpha-\alpha)^2 + (-1+\alpha)^2\right]}{\frac{1}{2}m\left[1^2+\alpha^2+\alpha^2+1^2\right]}$$

$$= \frac{k}{m}\frac{(6\alpha^2 - 4\alpha + 2)}{(2 + 2\alpha^2)} \qquad \text{——— (1)}$$

$$\frac{dR}{d\alpha} = \frac{k}{m}\left[\frac{(2+2\alpha^2)(12\alpha - 4) - (6\alpha^2 - 4\alpha + 2)(4\alpha)}{(2+2\alpha^2)^2}\right] = 0$$

Numerator: $8\alpha^2 + 16\alpha - 8 = 0$

$\hookrightarrow \alpha^2 + 2\alpha - 1 = 0 \longrightarrow \alpha = -1 \pm \sqrt{2}$

ie $\alpha_2 = 0.414$

& $\alpha_4 = -2.414$

from which (1) $\rightarrow$

for $\alpha_2 = -1+\sqrt{2}$: $R = \omega_2^2 = \frac{k}{m}\left(\frac{6(0.414)^2 - 4(0.414) + 2}{2 + 2(0.414)^2}\right)$

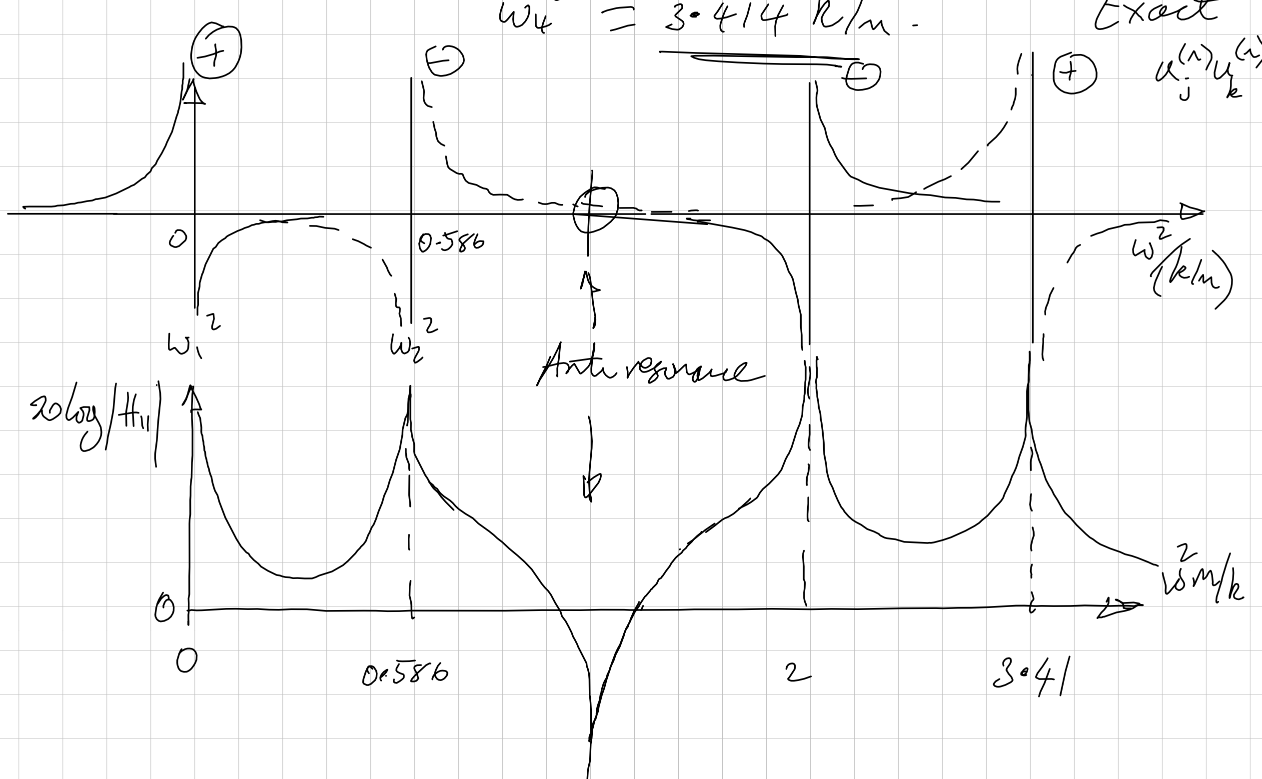
$$\underline{\underline{\omega_2^2 = 0.586 \; k/m}}$$

Exact

& for $\alpha_4 = -1-\sqrt{2}$: $R = \omega_4^2 = k/m\left(\frac{6(-2.414)^2 - 4(-2.414) + 2}{2 + 2(-2.414)^2}\right)$

$$\underline{\underline{\omega_4^2 = 3.414 \; k/m.}}$$

Exact



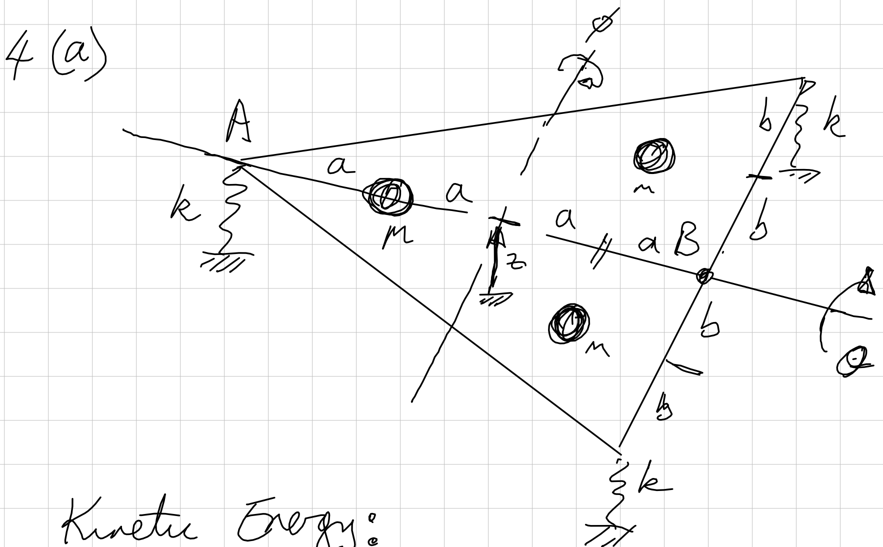
3 (cont)

(e) Mode 2 must be symmetric, so in mode 2,
$y_1 = -y_4$

So when column is excited in mode 2, the response at the top must be of same magnitude as at the base, but 180° out of phase. (NB No damping is needed to maintain a finite response for a displacement input)

Q4

4 (a)



Kinetic Energy:

$$T = \tfrac{1}{2}m(\dot{z}+a\dot{\phi})^2 + \tfrac{1}{2}m(\dot{z}-a\dot{\phi}-b\dot{\theta})^2 + \tfrac{1}{2}m(\dot{z}-a\dot{\phi}+b\dot{\theta})^2$$

$$= \tfrac{1}{2}m\left(3\dot{z}^2 + 3a^2\dot{\phi}^2 + 2b^2\dot{\theta}^2 - 2a\dot{z}\dot{\phi}\right)$$

$$= [\dot{z}\ \dot{\theta}\ \dot{\phi}]\, m \underbrace{\begin{bmatrix} 3 & 0 & -a \\ 0 & 2b^2 & 0 \\ -a & 0 & 3a^2 \end{bmatrix}}_{[M]} \begin{Bmatrix} \dot{z} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix}$$

Potential Energy:

$$V = \tfrac{1}{2}k(z+2a\phi)^2 + \tfrac{1}{2}k(z-2a\phi+2b\theta)^2 + \tfrac{1}{2}k(z-2a\phi-2b\theta)^2$$

$$= \tfrac{1}{2}k\left(3z^2 + 12a^2\phi^2 + 8b^2\theta^2 - 4az\phi\right)$$

$$= [z\ \theta\ \phi]\, k \underbrace{\begin{bmatrix} 3 & 0 & -2a \\ 0 & 8b^2 & 0 \\ -2a & 0 & 12a^2 \end{bmatrix}}_{[K]} \begin{Bmatrix} z \\ \theta \\ \phi \end{Bmatrix}$$

(b) The second row is an equation in θ only ie θ is decoupled from z & ϕ

$$\Rightarrow 2b^2 m\ddot{\theta} + 8b^2 k\theta = 0 \Rightarrow \omega^2 = \frac{8b^2k}{2b^2m} = 4k/m$$

(roll motion about longitudinal axis) $\quad = 2\sqrt{k/m}$

4 (cont) For the other two modes, find ω^2 from

$$\left| [K] - \omega^2 [M] \right| = 0$$

$$\begin{vmatrix} 3k - \omega^2 3m & -2ak + \omega^2 am \\ -2ak + \omega^2 am & 12a^2k - \omega^2 3a^2m \end{vmatrix} = 0$$

$$\Rightarrow (3k - 3m\omega^2)(12a^2k - 3a^2m\omega^2) - (-2ak + am\omega^2)^2 = 0$$

$$\Longrightarrow 8\cancel{a^2}m^2\omega^4 - 41\cancel{a^2}km\omega^2 + 32\cancel{a^2}k^2 = 0$$

$\div k^2 \quad 8\lambda^2 - 41\lambda + 32 = 0$ where $\lambda = \omega^2 m/k$

$$\lambda = \frac{41 \pm \sqrt{657}}{16} = \underline{\underline{0{\cdot}96,\ 4{\cdot}16}}$$

$$\Longrightarrow \omega^2 = 0{\cdot}96\, k/m \quad \& \quad \omega^2 = 4{\cdot}16\, k/m$$

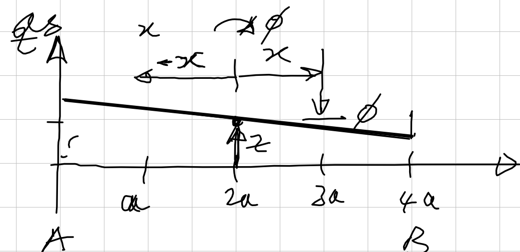
<u>Mode Shapes</u> $\quad [K]\underline{\phi} - \omega^2 [M]\underline{\phi} = 0$

Row 1 $\quad (3k - 3\omega^2 m)z + (-2ka + \omega^2 am)\phi = 0$

$$(3 - 3\lambda)z + (-2a + \lambda a)\phi = 0$$

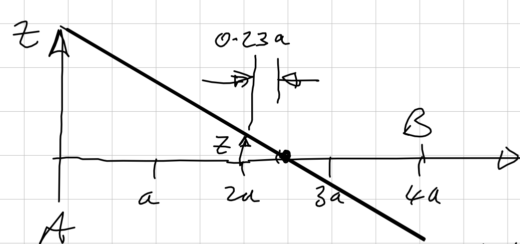
$$\Longrightarrow z/a\phi = \frac{2 - \lambda}{3 - 3\lambda} = 8{\cdot}7, \quad 0{\cdot}23$$

So the mode shapes are:



$\lambda = 0{\cdot}96$: $\omega^2 = 0{\cdot}96 \dfrac{k}{m}$

<u>Bounce</u>



$\lambda = 4{\cdot}16 \quad \omega^2 = 4{\cdot}16\, k/m$

<u>Pitch</u>

Node positions: $\quad z - \phi x = 0$

$$x = z/\phi = 8{\cdot}7a$$

$$z - \phi x = 0$$

$$x = z/\phi = 0{\cdot}23a$$

4(d)

For a 10% increase in frequency assume the same mode shape $\frac{z}{a\phi} \simeq 0.23$ and increase the stiffness of the spring at A to be $k(1+\varepsilon)$

Then Rayleigh gives:

$$\frac{\omega_{new}^2}{\omega_{old}^2} = (1.1)^2 = \frac{(V_{max}/T^*)_{new}}{(V_{max}/T^*)_{old}} = \frac{(V_{max})_{new}}{(V_{max})_{old}}$$

since the KE's don't change.

So $$(1.1)^2 = \frac{k(1+\varepsilon)(z+2a\phi)^2 + 2k(z-2a\phi)^2}{k(z+2a\phi)^2 + k(z-2a\phi)^2}$$

with $z/a\phi = 0.23$

So $$1.1^2 = \frac{(1+\varepsilon)(0.23+2)^2 + (0.23-2)^2}{(0.23+2)^2 + (0.23-2)^2}$$

ie $$\varepsilon = \frac{1.1^2(8.11) - 3.13}{4.97} - 1 = \underline{\underline{16.5\%}}$$