EGT2
ENGINEERING TRIPOS PART IIA

Friday 8 May 2026 9.30 to 11.10

Module 3C6

VIBRATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed.

Attachment: 3C5 Dynamics and 3C6 Vibration data sheet 2023 (7 pages).

Engineering Data Book.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A marine propulsion system consists of two shafts joined together. The shafts have length L_1, L_2, and polar second moments of area J_1, J_2, where the subscripts denote shafts 1 and 2. Both shafts are made from the same material with shear modulus G and density ρ. Shaft 1 is fixed at $x = 0$ (the engine mounting), shaft 2 is free at $x = L_1 + L_2$ (the propeller end, with the propeller mass neglected), and they are joined at $x = L_1$.

(a) For shaft 1 alone (fixed at $x = 0$, free at $x = L_1$):

(i) Find an expression for the mode shapes and natural frequencies. [10%]

(ii) Sketch the first three mode shapes, including rigid-body modes if they exist.

[10%]

(iii) Find a modal summation expression including damping for the transfer function $H_A(\omega)$ from input torque at $x = x_1$ to angular displacement at $x = x_2$, where x_1 and x_2 are arbitrary positions along the shaft. [10%]

(iv) By considering the internal torque in a shaft, or otherwise, find a modal summation expression for the transfer function $H_B(\omega)$ from input torque at $x = L_1$ to the reaction torque at $x = 0$. [20%]

(b) The two shafts are now joined at $x = L_1$. The general solution for free torsional vibration in each shaft section may be written as $\theta(x, t) = u(x)e^{i\omega t}$, where:

$$u_1(x) = A_1 \cos(kx) + B_1 \sin(kx) \qquad \text{for } 0 \le x \le L_1$$
$$u_2(x) = A_2 \cos(k\,(x - L_1 - L_2)) + B_2 \sin(k\,(x - L_1 - L_2)) \qquad \text{for } L_1 \le x \le L_1 + L_2$$

where A_1, B_1, A_2, B_2 are constant coefficients and $k = \omega/c$ and $c^2 = G/\rho$.

(i) Which coefficients must be zero to satisfy the boundary conditions? Justify your answer. [5%]

(ii) Write down the equations that must be satisfied at the junction $x = L_1$. [10%]

(iii) Derive an equation whose solutions give the natural frequencies of the coupled system. [25%]

(iv) Without detailed calculation, what do you expect the natural frequencies to be for the case $J_2 \ll J_1$ (and L_2 of similar magnitude to L_1)? [10%]

2 (a) A beam of length L, uniform cross-section of area A and second moment of area I is made from a material with density ρ and Young's modulus E. The distance from one end of the beam is x, and the small amplitude transverse deflection of the beam is $y(x, t)$. The beam is free at both ends.

(i) Starting from the equation for transverse vibration of a beam, derive an equation whose solutions give the natural wavenumbers k_n of a free-free beam. [20%]

(ii) Using a graphical construction, or otherwise, find approximate solutions for the first three non-zero values of k_1L, k_2L, k_3L. [10%]

(iii) Find the rati of the natural frequencies ω_n/ω_1 for $n = 2$ and $n = 3$. [10%]

(iv) Using Rayleigh's quotient, or otherwise, how do the natural frequencies ω_n scale with the beam thickness h? Write your answer in the form $\omega_n \propto h^p$ and find p. You do not need to find the constant of proportionality. [10%]

(b) The beams used for a Marimba (a tuned percussion instrument) often have reduced thickness near the centre of each beam so that their first two bending natural frequencies are tuned to a musical ratio. A simplified model consists of a beam with a short notch of width b at the centre which locally reduces the second moment of area to $I = I_0 - \Delta I$. The change in mass due to the notch can be neglected, and you may use the approximation:

$$\int_{L/2-b/2}^{L/2+b/2} f(x)dx \approx bf(L/2).$$

for any smooth function $f(x)$.

(i) Assuming that the mode shapes have not been significantly changed, show that the new natural frequency ω'_n can be written in the form: $\omega'^2_n \approx \omega_n^2(1 - A\phi_n)$, where $\phi_n = \frac{Lf(L/2)}{\int_0^L f(x)dx}$

and find the non-dimensional constant A and function $f(x)$ in terms of the beam and notch properties, and the mode shapes $u_n(x)$ and its derivatives. [30%]

(ii) Show that the ratio of the second to first natural frequencies can be written:

$$\left(\frac{\omega'_2}{\omega'_1}\right)^2 = B\frac{1 - C\phi_2}{1 - C\phi_1}$$

and find expressions for B and C. [10%]

(iii) Given the factors ϕ_n for the first two modes are $\phi_1 = 3.2$ and $\phi_2 = 0.08$, determine the value of C required to tune the beam so that $\omega_2/\omega_1 = 4$. Comment on whether this is physically achievable if $b/L = 0.1$. [10%]

3 Part of a satellite is an unconstrained rod of length L, mass M, cross sectional area A, and Young's modulus E as shown in Fig. 1(a). A four-element discrete model of the rod, used to calculate axial vibration transmission, is shown in Fig. 1(b). Each element has mass m and is joined to adjacent elements with springs of stiffness k.

(a) Determine suitable values of m and k in terms of M, E, A and L and write down mass and stiffness matrices for the discrete model in terms of m and k. [20%]

(b) Without calculating eigenvalues or eigenvectors, sketch plausible mode shapes for each of the four natural frequencies of the discrete model. [20%]

(c) Using Rayleigh's method or otherwise, estimate the four natural frequencies of the discrete model in terms of m and k. Which of the calculated frequencies is exact for the discrete model? [30%]

(d) The base of the rod is excited by an axial sinusoidal force $F\sin\omega t$. Sketch a graph of the magnitude of the displacement of coordinate y_3 (on a dB scale) as a function of angular frequency ω. [20%]

(e) The base of the rod is now forced to vibrate with a vertical *displacement* of $\pm\Delta$ at a frequency corresponding to the lowest non-zero natural frequency of the free-free rod. What is the amplitude and phase of the displacement at the top of the rod? [10%]

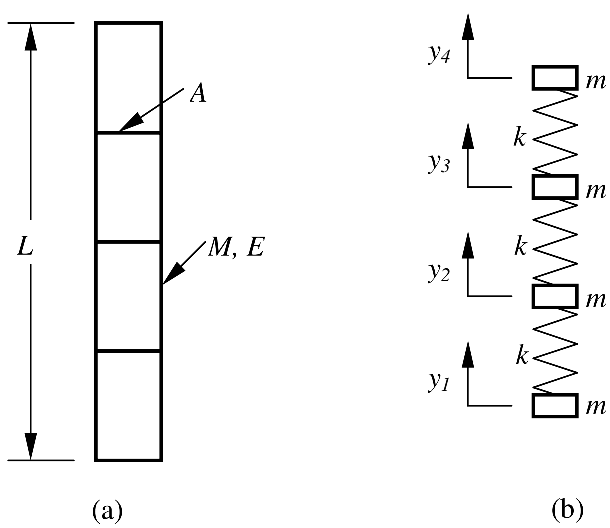

Fig. 1

4 Vibration of a three-wheeled 'economy' vehicle can be modelled as shown in Fig. 2. A light, rigid, triangular frame of length $4a$ and width $4b$ is supported at its corners by three identical suspension springs of stiffness k, resting on a flat, rigid road surface. The mass distribution can be modelled by three identical lumped masses m, located at the positions shown. Small vibration of the system is described by the three generalised coordinates z, θ, ϕ which are the vertical displacement, 'roll' and 'pitch' rotations about the mid-lines of the frame.

(a) Derive expressions for the potential and kinetic energies, about the position of static equilibrium and hence show that the mass and stiffness matrices are

$$[M] = m\begin{bmatrix} 3 & 0 & -a \\ 0 & 2b^2 & 0 \\ -a & 0 & 3a^2 \end{bmatrix} \quad \text{and} \quad [K] = k\begin{bmatrix} 3 & 0 & -2a \\ 0 & 8b^2 & 0 \\ -2a & 0 & 12a^2 \end{bmatrix}. \qquad [30\%]$$

(b) One vibration mode can be deduced without calculation. Describe it and give its natural frequency. [10%]

(c) Find the remaining two vibration modes and their natural frequencies. Sketch these two mode shapes and identify which mode is the 'pitch' mode (the one with larger oscillation about ϕ). Calculate and show the positions of the nodal points. [40%]

(d) The frequency of the 'pitch' mode is to be increased by 10% by changing the stiffness of the spring at A. Use Rayleigh's principle to estimate the percentage change in stiffness required. [20%]

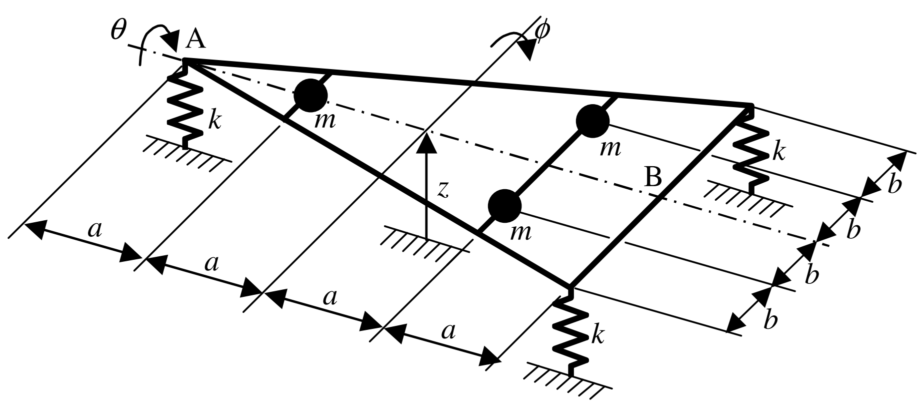

Fig. 2

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Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\boldsymbol{v}_{\mathrm{G}}$, M is the total mass, $\boldsymbol{v}_{\mathrm{G}}$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_{\mathrm{G}}$$
$$= \dot{\boldsymbol{h}}_{\mathrm{P}} + \dot{\boldsymbol{r}}_{\mathrm{P}} \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here $\boldsymbol{h}_{\mathrm{P}}$ and $\boldsymbol{h}_{\mathrm{G}}$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_{\mathrm{P}}) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_{\mathrm{G}} + (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm = \boldsymbol{I}\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $A = \int (y^2 + z^2) dm \qquad B = \int (z^2 + x^2) dm \qquad C = \int (x^2 + y^2) dm$

$D = \int yz \; dm \qquad E = \int zx \; dm \qquad F = \int xy \; dm$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= Q_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= Q_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= Q_3 \end{aligned}$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the"Gyroscope equations"):

$$\begin{aligned} A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 &= Q_1 \\ A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 &= Q_2 \\ C\dot{\omega}_3 &= Q_3 \end{aligned}$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalised momenta p_i and the Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \qquad H(\boldsymbol{p}, \boldsymbol{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\boldsymbol{p}, \boldsymbol{q})$ must be expressed as a function of the generalised momenta and the generalised displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

Special case when the kinetic energy is expressible using a mass matrix $\boldsymbol{M}$:

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1} \boldsymbol{p} \qquad \text{and} \qquad H = T + V$$

(b) Extension topics

The total time derivative of some function $f(\boldsymbol{p}, \boldsymbol{q}, t)$ can be expressed in terms of the Poisson bracket $\{f, H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}, \qquad \{f, H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\boldsymbol{q}, \boldsymbol{Q}, t)$	$\boldsymbol{p} = \frac{\partial G_1}{\partial \boldsymbol{q}}$	$\boldsymbol{P} = -\frac{\partial G_1}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_1}{\partial t}$
2	$G_2(\boldsymbol{q}, \boldsymbol{P}, t)$	$\boldsymbol{p} = \frac{\partial G_2}{\partial \boldsymbol{q}}$	$\boldsymbol{Q} = \frac{\partial G_2}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_2}{\partial t}$
3	$G_3(\boldsymbol{p}, \boldsymbol{Q}, t)$	$\boldsymbol{q} = -\frac{\partial G_3}{\partial \boldsymbol{p}}$	$\boldsymbol{P} = -\frac{\partial G_3}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_3}{\partial t}$
4	$G_4(\boldsymbol{p}, \boldsymbol{P}, t)$	$\boldsymbol{q} = -\frac{\partial G_4}{\partial \boldsymbol{p}}$	$\boldsymbol{Q} = \frac{\partial G_4}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems / **Continuous Systems**

1. Equation of motion

Discrete Systems: The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

Continuous Systems: The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

2. Kinetic Energy

Discrete Systems:

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

Continuous Systems:

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

Discrete Systems:

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

Continuous Systems: See Section 3 for examples.

4. Natural frequencies and mode shapes

Discrete Systems: The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2 \mathbf{M}\mathbf{u}^{(n)}$$

Continuous Systems: The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

Discrete Systems:

$$\mathbf{u}^{(j)^T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)^T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

Continuous Systems:

$$\int u_j(x) u_k(x) \mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

Discrete Systems: The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

Continuous Systems: The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_{j} q_j(t) u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n}\sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n}e^{-\omega_n\zeta_n t}\sin\omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1,x_2,t) = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n}\sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1,x_2,t) \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n}e^{-\omega_n\zeta_n t}\sin\omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\tilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

3.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned}
&\cos\alpha\cosh\alpha + 1 = 0, \quad && \alpha_1 = 1.8751\\
&\cos\alpha\cosh\alpha - 1 = 0, \quad && \alpha_1 = 4.7300\\
&\tan\alpha - \tanh\alpha = 0, \quad && \alpha_1 = 3.9266
\end{aligned}$$