3C7/2025

Q1

(a)

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \frac{A}{3} y^3 + 2Bxy + Cy$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = Ax^2 y + Dy + Ey^3$$

$$\sigma_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y} = -Axy^2 - Bx^2 - Cx$$

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}$$

$$= 4Ay + 6Ey = 0$$

$$\Rightarrow E = -\frac{2}{3} A$$

The boundary conditions are

$$\sigma_{yy}(y = t) = q_1 \quad \& \quad \sigma_{yy}(y = -t) = q_2$$

$$\sigma_{xy}(y=\pm t)=0$$

$$\Rightarrow \quad Axt^2 + Bx^2 + Cx = 0$$

$$\Rightarrow \quad (At^2+C)x + Bx^2 = 0$$

$$\therefore \quad C = -At^2 \quad \& \quad B=0$$

$$\sigma_{yy}(y=t) = \frac{A}{3}t^3 + \cancel{Bxt}^{\,0} + Ct = q_1$$

$$\sigma_{yy}(y=-t) = -\frac{A}{3}t^3 - \cancel{Bxt}^{\,0} - Ct = q_2$$

$$\Rightarrow \quad q_1 = -q_2 \quad \&$$

$$-\frac{A}{3}t^3 + At^3 = q_2$$

$$\boxed{A = \frac{3}{2}\frac{q_2}{t^3}, \quad B=0, \quad C=-\frac{3}{2}\frac{q_2}{t}}$$

$$\boxed{E = -\frac{q_2}{t^3}}$$

(b) $$M = \int_{-t}^{t} \sigma_{xx}\big|_{x=0}\, y\, dy = 0$$

$$\sigma_{xx} = Ax^2 y + Dy + Ey^3$$

$$= \frac{3}{2}\frac{q_2}{t^3}\left(x^2 y - \frac{2}{3}y^3\right) + Dy$$

$$\Rightarrow \int_{-t}^{t} \left[\frac{3}{2}\frac{q_2}{t^3}\left(-\frac{2}{3}y^3\right) + Dy\right] y\, dy = 0$$

$$-\frac{2}{5}\frac{q_2}{t^3}t^5 + \frac{2}{3}Dt^3 = 0$$

$$\Rightarrow D = \frac{3}{5}\frac{q_2}{t}$$

(c) $$\sigma_{xx} = Ax^2 y + Dy + Ey^3$$

$$= \frac{3}{2}\frac{q_2}{t^3}\left(x^2 y - \frac{2}{3}y^3\right) + \frac{3}{5}\frac{q_2}{t}y$$

$$\sigma_{xy} = Axy^2 + Cx$$

$$= \frac{3}{2}\frac{q_2}{t^3}\left(xy^2 - t^2x\right)$$

$$\sigma_{yy} = \frac{A}{3}y^3 + C_y$$

$$= \frac{3}{2}\frac{q_2}{t^3}\left(\frac{y^3}{3} - t^2y\right)$$

$$S|_{x=0} = \int_{-t}^{t} \sigma_{xy}|_{x=0}\, dy$$

$$\sigma_{xy}|_{x=0} = 0$$

$$\Rightarrow \quad S|_{x=0} = 0.$$

Q2

(a) For plane stress

$$\sigma_{rr} = A - \frac{B}{r^2} - \frac{E\alpha}{r^2}\int_0^r r T_0\left(1-\frac{r^2}{b^2}\right)dr$$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} + \frac{E\alpha}{r^2}\int_0^r r T_0\left(1-\frac{r^2}{b^2}\right)dr - E\alpha T_0\left(1-\frac{r^2}{b^2}\right)$$

$$\int_0^r r T_0\left(1-\frac{r^2}{b^2}\right)dr = \frac{1}{2}T_0 r^2\left(1-\frac{1}{2}\frac{r^2}{b^2}\right)$$

$$\Rightarrow \sigma_{rr} = A - \frac{B}{r^2} - \frac{E\alpha T_0}{2}\left(1-\frac{1}{2}\frac{r^2}{b^2}\right)$$

σ_{rr} is finite at $r=0 \Rightarrow B=0$

$\sigma_{rr}=0$ at $r=b \Rightarrow A = \frac{E\alpha T_0}{4}$

$$\therefore \sigma_{rr} = -\frac{E\alpha T_0}{4}\left(1-\frac{r^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = \frac{E\alpha T_0}{4} + \frac{E\alpha T_0}{2}\left(1 - \frac{1}{2}\frac{r^2}{b^2}\right) - E\alpha T_0\left(1 - \frac{r^2}{b^2}\right)$$

$$= -\frac{E\alpha T_0}{4}\left(1 - \frac{3r^2}{b^2}\right)$$

Now convert to plane strain $E \to \frac{E}{1-\nu^2}$

$$\alpha \to \alpha(1+\nu)$$

$$\Rightarrow \sigma_{rr} = -\frac{E\alpha T_0}{4(1-\nu)}\left(1 - \frac{r^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = -\frac{E\alpha T_0}{4(1-\nu)}\left(1 - \frac{3r^2}{b^2}\right)$$

$$\varepsilon_{zz} = 0 \Rightarrow \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

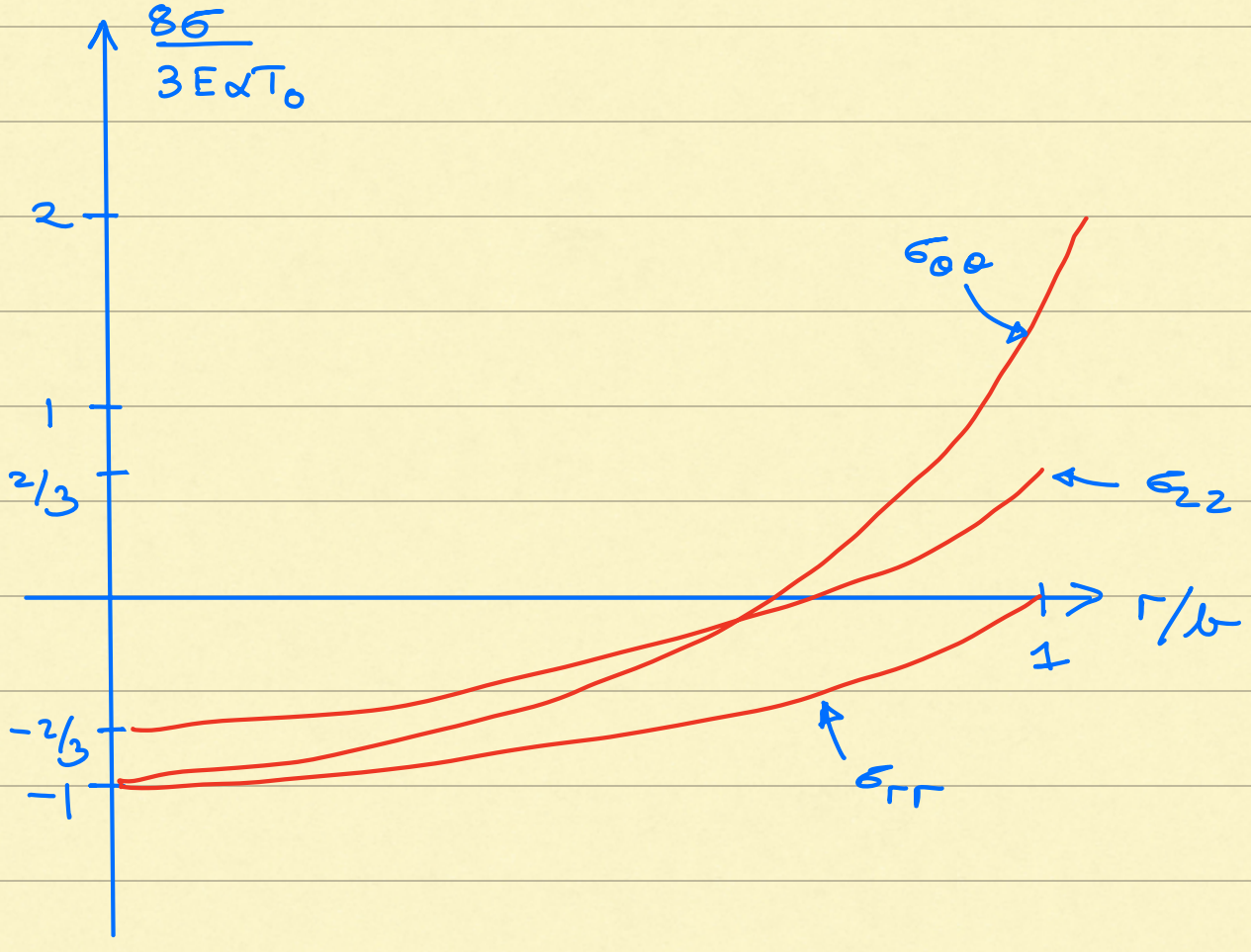
$$= -\frac{\nu E\alpha T_0}{2(1-\nu)}\left(1 - \frac{2r^2}{b^2}\right)$$

(b) $\nu = \frac{1}{3}$

$$\sigma_{rr} = -\frac{3E\alpha T_0}{8}\left(1 - \frac{r^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = -\frac{3E\alpha T_0}{8}\left(1 - \frac{3r^2}{b^2}\right)$$

$$\sigma_{zz} = -\frac{E\alpha T_0}{4}\left(1 - \frac{2r^2}{b^2}\right)$$



(c)

$$\frac{u}{r} = \varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr} - \nu\sigma_{zz}\right)$$

$$u = \frac{r}{E}\left[-\frac{E\alpha T_0}{4(1-\nu)}\right]\left[1 - \frac{3r^2}{b^2} - \nu\left(1 - \frac{r^2}{b^2}\right) - 2\nu^2\left(1 - \frac{2r^2}{b^2}\right)\right]$$

at $r = b$

$$u = -\frac{\alpha b T_0}{4(1-\nu)}\left(-2 + 2\nu^2\right) = \frac{\alpha b T_0 (1+\nu)}{2}$$

(d) $$\gamma_{max} = \frac{max\left(|\sigma_{rr} - \sigma_{\theta\theta}|, |\sigma_{\theta\theta} - \sigma_{zz}|, |\sigma_{zz} - \sigma_{rr}|\right)}{2}$$

From (b) we see γ_{max} at $r = b$ with

$$\gamma_{max} = \left.\frac{|\sigma_{\theta\theta} - \sigma_{rr}|}{2}\right|_{r=b}$$

$$= \frac{E\alpha T_0}{4(1-\nu)} \frac{r^2}{b^2}\Big|_{r=b} = \frac{E\alpha T_0}{4(1-\nu)}$$

At yield $\gamma_{max} = \frac{Y}{2}$

$$\Rightarrow T_0 = \frac{2(1-\nu)Y}{E\alpha}$$

Q3

(a) (i) $\nabla^2 \psi = \text{constant}$

where constant $= -2G\beta$

This condition ensures compatibility & Hooke's law are satisfied

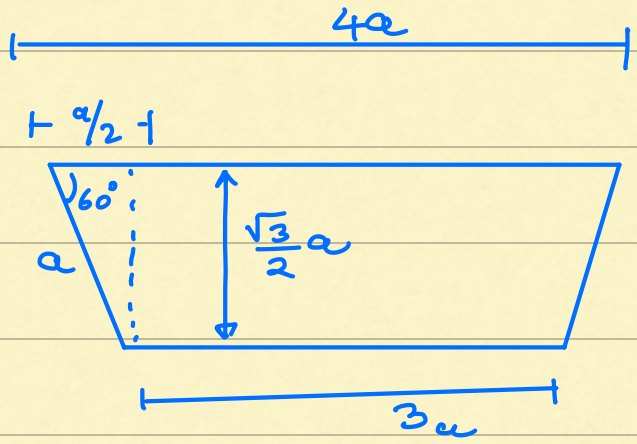
(ii) $\psi = 0$ on boundary

This condition implies torque $T = 2\iint \psi \, dA$

(b) Torsional stiffness of thin-walled cross-section

$$\frac{T}{G\beta} = \frac{4A_e^2}{\oint \frac{ds}{t}}$$

$$A_e = \frac{1}{2}(3a + 7a)\, a\frac{\sqrt{3}}{2} = \frac{7\sqrt{3}}{4} a^2$$

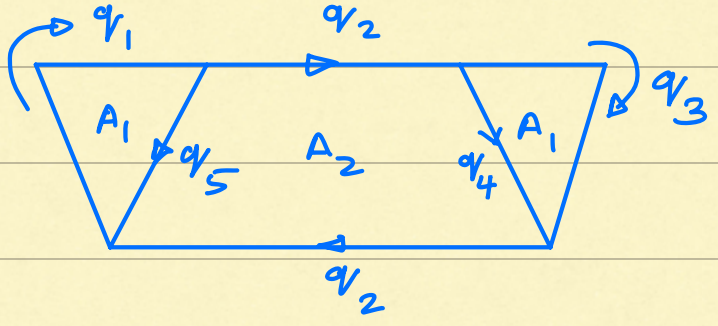


$$\oint \frac{ds}{t} = \frac{1}{t}(2a + 7a) = \frac{9a}{t}$$

$$\Rightarrow \frac{T}{q\beta} = \frac{49 \times 3 \times a^4 t}{9 \times 4a}$$

$$= \frac{49a^3 t}{12}$$

(c)



$$A_1 = \frac{\sqrt{3}}{4}a^2$$

$$A_2 = \frac{5\sqrt{3}}{4}a^2$$

By symmetry $q_1 = q_3$ & $q_5 = -q_4$

Equilibrium $\Rightarrow \quad q_1 = q_2 + q_5$

$\Rightarrow q_5 = q_1 - q_2$

Compatibility

$$\frac{q_1(2a) + (q_1 - q_2)a}{2A_1 G t} =$$

$$\frac{q_2(3a + 2a) + 2(q_2 - q_1)a}{2A_2 G t}$$

$$\frac{3q_1 a - q_2 a}{\frac{\sqrt{3}}{4}a^2} = \frac{7q_2 a - 2q_1 a}{\frac{5\sqrt{3}}{4}a^2}$$

$$15q_1 - 5q_2 = 7q_2 - 2q_1$$

$$17q_1 = 12q_2 \Rightarrow q_1 = \frac{12}{17}q_2$$

$$T = 2A_1 q_1 + 2A_2 q_2 + 2A_1 q_1$$

$$T = 4\frac{\sqrt{3}}{4}a^2 q_1 + 2\times 5\frac{\sqrt{3}}{4}a^2 \times \frac{17}{12} q_1$$

$$T = \sqrt{3}a^2 q_1 \left(1 + \frac{85}{24}\right)$$

$$= \sqrt{3}a^2 q_1 \left(\frac{109}{24}\right)$$

$$q_1 = \frac{24T}{\sqrt{3}a^2 \times 109}$$

Torsional stiffness $= \frac{T}{G\beta}$

$$\beta = \frac{q_1(2a) + (q_1 - q_2)a}{2A_1 G t}$$

$$\beta = \frac{q_1\left[3a - \frac{17}{12}a\right]}{2\times\frac{\sqrt{3}}{4}a^2 G t} = 2q_1 \frac{19a}{12}\frac{1}{\sqrt{3}a^2 G t}$$

$$= q_1 \frac{19a}{6\sqrt{3}a^2 G t}$$

$$\frac{T}{G\beta} = \frac{T}{G} \quad \frac{1}{19\,a\,\psi_1} \times 6\sqrt{3}\,a^2 G t$$

$$= \frac{6\sqrt{3}\,a\,t}{19 \times 24} \times 109\sqrt{3}\,a^2$$

$$\boxed{\frac{T}{G\beta} = \frac{18 \times 109}{19 \times 24}\, a^3 t = 4.303\, a^3 t}$$

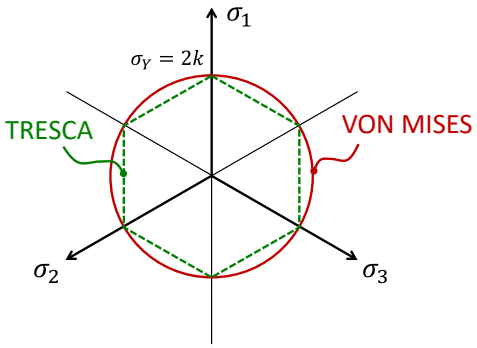
Original stiffness $= \frac{49}{12} a^3 t = 4.083\, a^3 t$

$\Rightarrow$ stiffness increase $= \left(\frac{4.303}{4.083} - 1\right) \times 100$

$$\boxed{= 5.4\%}$$

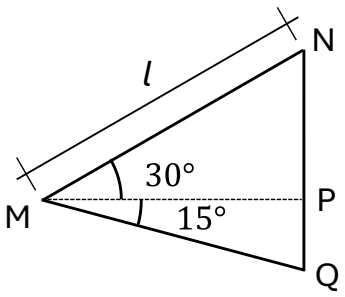
Q4

(a)



(b) the Upper Bound Theorem is useful in modelling metal forming processes because it provides a way to estimate the upper limit of the force or energy required for deformation. By assuming a kinematically admissible velocity field that satisfies boundary conditions and material flow constraints, the theorem ensures that the calculated forces or energy are always higher than the actual values, providing a conservative and safe estimate for process design. The theorem helps to model complex forming processes like forging, extrusion, and rolling, without solving the full field equations and can be used to optimise forming parameters, such as die design.

(c) (i)



$\overline{\mathrm{NP}} = l \sin 30°$

$\overline{\mathrm{MP}} = l \cos 30°$

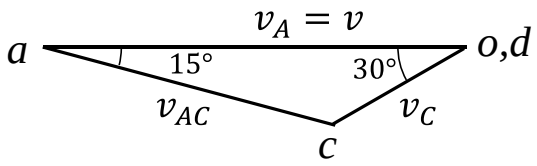
$$\overline{\mathrm{MQ}} = \frac{\overline{\mathrm{MP}}}{\cos 15°} = \frac{l \cos 30°}{\cos 15°}$$

$\overline{\mathrm{PQ}} = \overline{\mathrm{MQ}} \sin 15° = l \cos 30° \tan 15°$

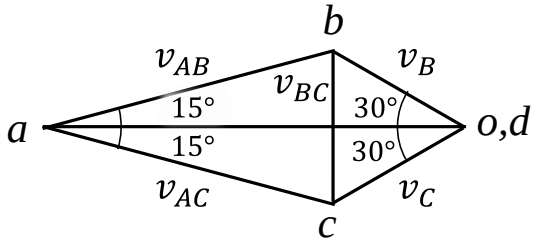
$\overline{\mathrm{NQ}} = \overline{\mathrm{NP}} + \overline{\mathrm{PQ}} = l\,[\sin 30° + \cos 30° \tan 15°] = 0.732l$

$$\overline{\mathrm{MQ}} = \frac{l \cos 30°}{\cos 15°} = 0.897l$$

(ii)



Absolute velocity of work-piece A is horizontal, $v_A = v$. Absolute velocity of block C, v_C, is parallel to MN, at $30°$on the horizontal. Velocity of block C relative to block A is parallel to MQ, at an angle $-15°$ on the horizontal. Hence find point c.



Velocity of block B relative to block C is vertical. Velocity of block B relative to block A is at an angle of $15°$ on the horizontal. Hence find point b.

$$v_B = v_C = \frac{v_{BC}}{2}\frac{1}{\sin 30°}$$

$$v_{AB} = v_{AC} = \frac{v_{BC}}{2}\frac{1}{\sin 15°}$$

$$v = v_{AB}\cos 15° + v_B \cos 30° = \frac{v_{BC}}{2}[\cot 15° + \cot 30°]$$

$$v_{BC} = \frac{2v}{\cot 30° + \cot 15°} = 0.366v$$

$$v_B = v_C = \frac{v}{[\cot 15° + \cot 30°]}\frac{1}{\sin 30°} = 0.366v$$

$$v_{AB} = \frac{v}{[\cot 15° + \cot 30°]}\frac{1}{\sin 15°} = 0.707v$$

(iii)

$\dot{W}_{ext} = Fv$

$\dot{W}_{int} = k(2 \times \overline{MQ} \times v_{AC} + \overline{\mathrm{NQ}} \times v_{BC}) = 1.536klv$

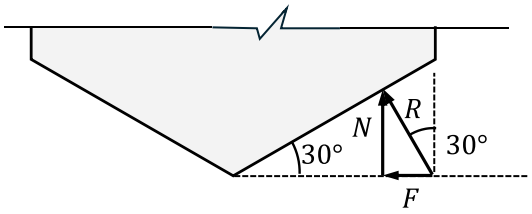
Note: no plastic work along MN, because the contact between the workpiece and the tool is assumed smooth.

By setting:

$\dot{W}_{ext} = \dot{W}_{int}$

$F = 1.536kl$

(iv)



Because the contact between tool D and block C is smooth, the force, R, between them must be orthogonal to MN. Hence:

$F/N = \tan 30° = 0.577$

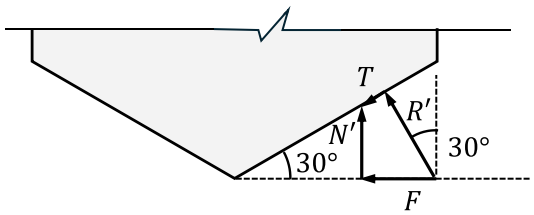
$$N = \frac{F}{0.577} = \frac{1.536}{0.577}kl = 2.660kl$$

(v)

$\dot{W}_{ext} = Fv$

$\dot{W}_{int} = k(2 \times \overline{MQ} \times v_{AC} + \overline{\mathrm{NQ}} \times v_{BC} + f \times \overline{\mathrm{MN}} \times v_C) = klv(1.536 + 0.366f)$

$F = kl(1.536 + 0.366f)$



This time the contact between is tool D and block C is frictional, so there is a tangential force at the interface, $T = fkl$. Horizontal equilibrium requires that:

$F = T\cos 30° + R\sin 30°$

While vertical equilibrium requires that:

$N = R\cos 30° - T\sin 30°$

Eliminating R, one obtains:

$$N = \frac{F}{\tan 30°} - T\left(\frac{\cos 30°}{\tan 30°} + \sin 30°\right) = 1.732F - 2T = kl(2.660 - 1.366f)$$