Version VSD/2

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 30 April 2025 14.00 to 15.40

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
3C7 formulae sheet (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A cantilever beam of length L and thickness $2t$ is loaded by normal tractions q_1 and q_2 as sketched in Fig. 1. An Airy stress function

$$\phi = \frac{A}{6}x^2y^3 + \frac{B}{3}x^3y + \frac{C}{2}x^2y + \frac{D}{6}y^3 + \frac{E}{20}y^5$$

is a candidate to evaluate the stress fields in this cantilever.

(a) Show that for ϕ to represent the stress field, we require $q_1 = -q_2$ and thereby express A, B, C and E in terms of q_2. [40%]

(b) Obtain the value of D such that the net moment vanishes at $x = 0$. [35%]

(c) Give expressions for the stress components σ_{xx}, σ_{yy} and σ_{xy} in terms of q_2. What is the magnitude of the shear force at $x = 0$? [25%]

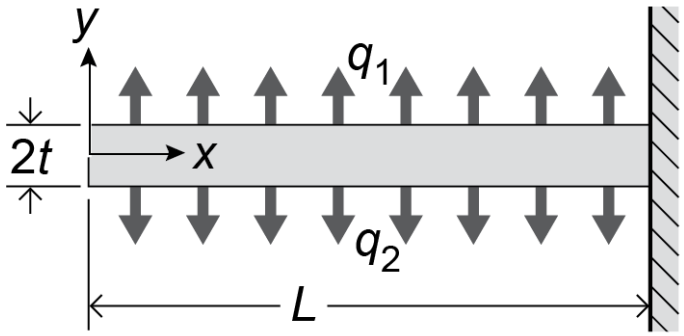

Fig. 1

2 A long cylindrical solid Ni/Al alloy wire of radius b is heated by an electric current. The wire loses heat to the environment and the steady-state temperature T of the wire is a function of only the radial co-ordinate r. This temperature is given in terms of the temperature T_0 at $r = 0$ and T_∞ at $r = b$ by

$$T = (T_0 - T_\infty)(1 - r^2/b^2) + T_\infty,$$

where $T_\infty \leq T_0$. The Young's modulus of the alloy is E, Poisson's ratio ν, coefficient of thermal expansion α and the tensile yield strength Y. Without loss of generality we may assume $T_\infty = 0$.

(a) Assuming plane strain conditions prevail (i.e. strain $\varepsilon_{zz} = 0$ along the length of wire), and neglecting the possibility of yield, determine the complete stress field in the wire. [35%]

(b) Sketch the stress components as a function of r/b assuming $\nu = 1/3$. [15%]

(c) Calculate the radial displacement at $r = b$. [25%]

(d) Assuming the alloy obeys the Tresca yield criterion, calculate the maximum value of T_0 for the wire to remain fully elastic. [25%]

3 (a) State two conditions for $\psi(x, y)$ to be a valid Prandtl stress function and briefly explain the reasons for these conditions. [20%]

(b) A thin-walled hollow box-section has an isosceles trapezoidal cross-section as sketched in Fig. 2a. The dimensions of the trapezoid are marked in terms of the parameter a and the wall thickness t. Using the Structures databook as appropriate, calculate the torsional stiffness of the box-section in terms of a, t and the shear modulus G of the material of the box-section. [25%]

(c) The box-section is now stiffened by adding two internal webs of length a as shown in Fig. 2b. These webs are of the same material as the original trapezoidal section and also have a thickness t. Calculate the torsional stiffness of this stiffened section and hence determine the factor by which the stiffness has increased compared to the answer in (b). (Hint: Recall the torque T for a multi-cell cross-section is given by $T = 2\sum_i A_i q_i$ where A_i and q_i are the enclosed area and shear flows, respectively in cell i.) [55%]

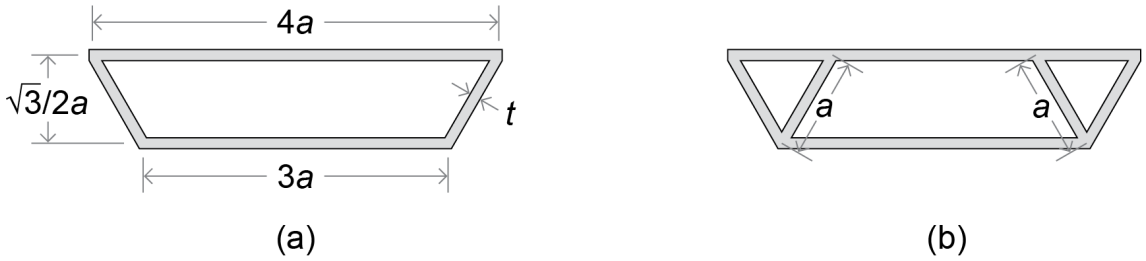

Fig. 2

4 (a) Sketch the von Mises and Tresca yield surfaces in the π-plane. [10%]

(b) Explain briefly why the Upper Bound theorem is useful in modelling metal forming processes. [10%]

(c) Figure 3 shows a rigid-perfectly plastic metal workpiece A of unit depth into the plane of the paper. The workpiece material has a tensile yield strength σ_Y and follows the Tresca failure criterion. It drawn at speed v by a force F under a rigid tool D, which is loaded by a force N (Fig. 3). At steady-state, the tip of the tool is at the same level as the incoming free surface of the workpiece with a wave of plastically deforming material (regions B and C) at the leading edge of the tool. The angles between the various regions are as indicated in Fig. 3 and the length of contact between the tool (D) and block C is l.

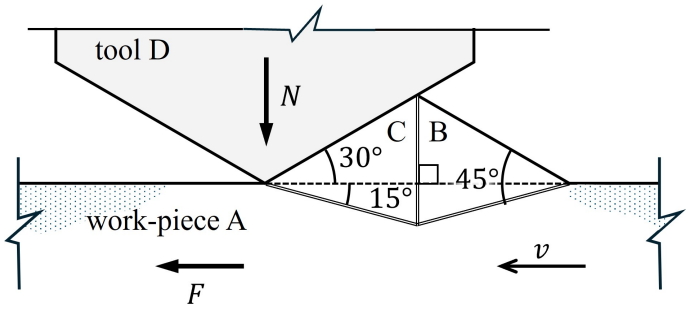

Fig. 3

Assuming plane strain conditions:

(i) Find the length of the interfaces between regions B and C and regions A and C in terms of l. [15%]

(ii) Draw, to scale, a velocity diagram for this mode of deformation. [20%]

(iii) Assuming that there is no friction between the tool and the work-piece, estimate the value of the force F in terms of the shear yield strength of the material $k = (\sigma_Y/2)$ and the length l. Is your estimate an upper bound or lower bound? [10%]

(iv) By considering the equilibrium of the tool, obtain the relationship between l and the applied load N. [10%]

(v) Repeat the calculations in (iii) and (iv), this time assuming that at the contact between tool D and block C there is a frictional shear stress $\tau = fk$, with $0 \leq f \leq 1$. Plot normalised forces $F/(kl)$ and $N/(kl)$ as a function of f. [25%]

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Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{\mathrm{d}(r\sigma_{rr})}{\mathrm{d}r} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{\mathrm{d}(r^2\sigma_{rr})}{\mathrm{d}r}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,\mathrm{d}r$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,\mathrm{d}r - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \bar{E} = \dfrac{E}{1-\nu^2}\ ;\ \bar{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \bar{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity with no thermal strains or body forces)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (=\tau_x) = \dfrac{\partial\psi}{\partial y}, \quad \sigma_{zy}\ (=\tau_y) = -\dfrac{\partial\psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi\, dA$

Governing equation for elastic torsion: $\nabla^2\psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\prod = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^{\mathrm{T}}\,[D]\,\underset{\sim}{\varepsilon}\,\mathrm{d}V$, $\mathrm{W} = \underset{\sim}{P}^{\mathrm{T}}\,\underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\ \{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\}^{1/2}$

Equivalent strain increment $\mathrm{d}\bar{\varepsilon} = \sqrt{\frac{2}{3}}\ \{\mathrm{d}\varepsilon_1^2 + \mathrm{d}\varepsilon_2^2 + \mathrm{d}\varepsilon_3^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $\mathrm{d}\varepsilon_1 : \mathrm{d}\varepsilon_2 : \mathrm{d}\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{\mathrm{d}\varepsilon_1}{\sigma'_1} = \frac{\mathrm{d}\varepsilon_2}{\sigma'_2} = \frac{\mathrm{d}\varepsilon_3}{\sigma'_3} = \frac{\mathrm{d}\varepsilon_1-\mathrm{d}\varepsilon_2}{\sigma_1-\sigma_2} = \frac{\mathrm{d}\varepsilon_2-\mathrm{d}\varepsilon_3}{\sigma_2-\sigma_3} = \frac{\mathrm{d}\varepsilon_3-\mathrm{d}\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{\mathrm{d}\bar{\varepsilon}}{\bar{\sigma}}\ .$$