

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 29 April 14.00 to 15.40

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

3C7 formulae sheet (2 pages).

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 An elastic fan blade has a uniform crescent shaped cross-section comprising two circular arcs as shown in Fig. 1. The blade is twisted by a torque T about the longitudinal axis that is normal to the cross-section shown in Fig. 1.

(a) Show that

$$\psi = C \left(1 - \frac{b^2}{r^2} \right) (r^2 - 2ar \cos \theta),$$

is a Prandtl stress function for this blade.

[30%]

(b) The torque twists the blade by an angle α per unit length along the longitudinal direction. Calculate the shear stress σ_{zy} .

[30%]

(c) Hence calculate the shear stresses $\sigma_{zy}(b, 0)$ and $\sigma_{zy}(2a, 0)$ at the surface in the thickest part of the section.

[20%]

(d) Does the maximum shear stress occur at the inner or outer surface at the thickest part of the section?

[20%]

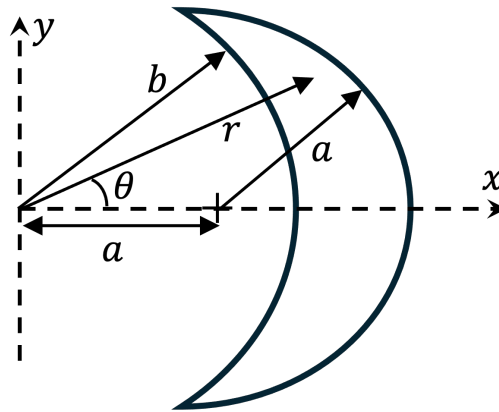


Fig. 1

- 2 (a) A thin annular disk has an outer radius b , inner radius a and thickness h . The disc is elastic with a Young's modulus E and Poisson's ratio ν . The inner radius is subjected to a pressure p and all other surfaces are free of traction. Calculate the radial displacement at the inner radius. [30%]
- (b) The annular disk is press-fitted onto a solid elastic cylinder of thickness $2h$ and initial radius $a + \delta$. The only loads are due to the interference δ that was present before assembly. If $b/a = 2$ and $\nu = 1/3$, estimate the fitting pressure in terms of E , δ and a . You may assume no variations along the longitudinal directions of the solid cylinder and annulus. [40%]
- (c) Plasticity in the annular disk obeys the Tresca yield criterion with a tensile yield strength Y . Calculate δ/a at which plasticity initiates in the annular disc in terms of Y/E . [10%]
- (d) Discuss whether the value of δ/a is an over or underestimate of the value at which plasticity would initiate. [20%]

3 (a) For the function $\phi = Br^n\theta$, where B is a constant, find all values of n such that ϕ is a valid Airy stress function. [20%]

(b) Using the stress function $\phi_2 = B_2r^2\theta$, determine the stress field (in polar co-ordinates) in the elastic half-space $y > 0$ shown in Fig. 2a. Hence sketch the boundary value problem for which ϕ_2 gives the elastic solution. [15%]

(c) Write the stress field calculated above in Cartesian co-ordinates. [25%]

(d) We wish to calculate the stress field within the half-space loaded by a uniform pressure p applied over a patch $-a \leq x \leq a$ on the surface $y = 0$ (Fig. 2b). Using superposition involving ϕ_2 , or otherwise, determine the stress distribution σ_{xy} on the plane $y = a$. [40%]

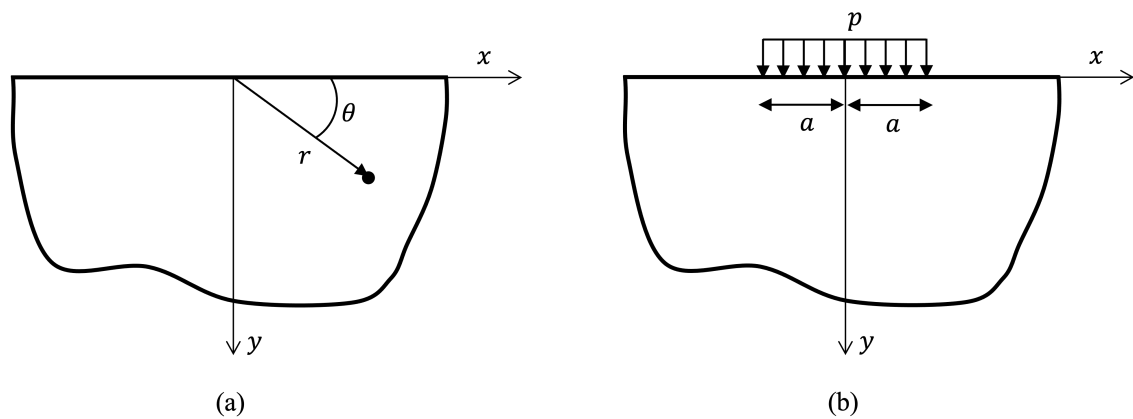


Fig. 2

4 (a) Explain briefly the differences between Lower and Upper Bound plasticity solutions and why Upper Bound is useful in modelling metal indentation and forming processes. [30%]

(b) Figure 3 shows a plane-strain extrusion process in which a billet is extruded sideways from a container as a punch moves from left to right under a pressure p . The billet material is assumed to be rigid-perfectly plastic, with a shear yield stress of magnitude k . The billet thickness is t , which is also equal to the width of the slot through which the material is extruded. Velocity discontinuities for an Upper Bound analysis are indicated by double lines. The material emerges at an angle α , which is controlled by the position of point P . As shown in Fig. 3, point P is located at a distance xt from the left edge of the slot, where x is an unknown parameter to be determined using the Upper Bound analysis.

(i) If the interface between the material and the interior of the container is well lubricated, explain why $\alpha = \tan^{-1}(2)$. [30%]

(ii) In practice, friction acts at the interface between the material and the container, producing a frictional shear stress equal to fk , where $0 < f < 1$. If $f = 0.2$, determine the expected value of x . [20%]

(iii) If $f = 0.4$, estimate the extrusion force $F = pt$ required per unit width of material, in terms of k and t . [20%]

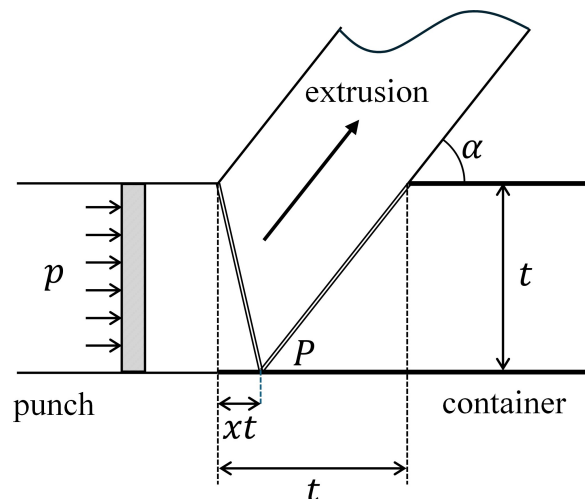


Fig. 3

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Module 3C7: Mechanics of Solids
ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	<u>Discs and tubes</u>	<u>Spheres</u>
Equilibrium	$\sigma_{\theta\theta} = \frac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \frac{1}{2r} \frac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \frac{B}{r^2} - \frac{3+\nu}{8} \rho\omega^2 r^2 - \frac{E\alpha}{r^2} \int_c^r rTdr$	$\sigma_{rr} = A - \frac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \frac{B}{r^2} - \frac{1+3\nu}{8} \rho\omega^2 r^2 + \frac{E\alpha}{r^2} \int_c^r rTdr - E\alpha T$	$\sigma_{\theta\theta} = A + \frac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\bar{E} = \frac{E}{1-\nu^2}$; $\bar{\nu} = \frac{\nu}{1-\nu}$; $\bar{\alpha} = \alpha(1+\nu)$

	<u>Cartesian coordinates</u>	<u>Polar coordinates</u>
Strains	$\varepsilon_{xx} = \frac{\partial u}{\partial x}$ $\varepsilon_{yy} = \frac{\partial v}{\partial y}$ $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$	$\varepsilon_{rr} = \frac{\partial u}{\partial r}$ $\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}$ $\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}$
Compatibility	$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\frac{\partial}{\partial r} \left\{ r \frac{\partial \gamma_{r\theta}}{\partial \theta} \right\} = \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial \varepsilon_{\theta\theta}}{\partial r} \right\} - r \frac{\partial \varepsilon_{rr}}{\partial r} + \frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity		
with no thermal strains	$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} (\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right\} (\sigma_{rr} + \sigma_{\theta\theta}) = 0$
or body forces)		
Equilibrium	$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$ $\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$	$\frac{\partial}{\partial r} (r\sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$ $\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r} (r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4 \phi = 0$ (in elasticity)	$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} \left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right\} = 0$	$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right\}$ $\times \left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right\} = 0$
Airy Stress Function	$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$ $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$ $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$ $\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$ $\sigma_{r\theta} = -\frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx} (= \tau_x) = \frac{\partial \psi}{\partial y}, \quad \sigma_{zy} (= \tau_y) = -\frac{\partial \psi}{\partial x}$

Equilibrium: $T = 2 \int_A \psi dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2} \int_V \underline{\varepsilon}^T [D] \underline{\varepsilon} dV$, $W = \underline{P}^T \underline{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_p , can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_p & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_p \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_p .

Expanding: $\sigma_p^3 - I_1 \sigma_p^2 + I_2 \sigma_p - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2} \{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3} \{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\}}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2} \frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$