

EGT2

ENGINEERING TRIPOS PART IIA

Wednesday 7 May 2025 9.30 to 11.10

Module 3D1

GEOTECHNICAL ENGINEERING I

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

Graph paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Attachment: 3D1 & 3D2 Geotechnical Engineering Databook (21 pages)

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Explain how you would construct a particle size distribution curve for a soil sample. [10%]

(b) Can a soil be classified based on the particle size distribution alone? What are the risks of doing this? [15%]

(c) Explain the differences in behaviour that you would expect between a soil with a high plasticity index and one with a low plasticity index. [15%]

(d) A standard Proctor compaction test was conducted on a sandy clay sample and the results are shown in Table 1. The specific gravity of the soil particles is 2.75.

Table 1

Water content (%)	2.40	4.55	7.09	11.02	14.68	16.21
Bulk density (kg m^{-3})	1948.7	2080.6	2200.7	2050.8	1900.0	1848.0

Draw the compaction curve and estimate the maximum dry density and optimum moisture content for this soil. [20%]

(e) Plot the variation of voids ratio with the water content and the zero air voids line. Explain what the zero air voids line signifies and why it is preferred by geotechnical practitioners. [20%]

(f) Explain from a micromechanical perspective why the density of the soil reduces on either side of the optimum moisture content. [20%]

2 (a) Explain briefly how an oedometer test is conducted on a soil sample. What soil parameters can be obtained from this test? [20%]

(b) An oedometer cell has an internal diameter of 50 mm and a sample height of 20 mm. A clay sample is placed in the cell and is subjected to vertical stress. The height of the soil sample is measured after 24 hours of each load increment. Measurements were obtained during the experiment as shown in Table 2.

Table 2

Effective stress (kPa)	Sample height (mm)
1	19.900
5	19.820
10	19.700
100	19.009
200	18.801
300	18.680
400	18.593
500	18.526
600	18.472
800	18.385
750	18.389
700	18.392
600	18.400
500	18.409

Calculate the Γ , λ and κ values of the clay sample. [25%]

(c) A 4 m high road embankment is to be constructed using a sandy clay soil compacted to a bulk unit weight of 23.2 kN m^{-3} , on a clay layer 8 m deep overlying bedrock. The sample tested in the oedometer in part (b) was taken as an undisturbed sample from the clay layer at a depth of 4 m. The water table is at the surface of the clay layer, the specific gravity of the clay particles is 2.71 and the initial void ratio of the clay is 0.71. Calculate the expected settlement of the clay surface beneath the embankment by assuming that the soil sample tested is representative of the whole clay layer. [35%]

(d) If the water table subsequently rises to 2 m above the original clay surface, recalculate the settlement. Justify any assumptions made. [20%]

3 A flexible temporary building is to be constructed utilising a square shallow foundation with a side of length 10 m which will exert a uniform bearing pressure of 400 kPa on the soil surface. The soil beneath the building comprises a 20 m thick layer of clay whose properties can be taken to be those given in the 3D1 & 3D2 Geotechnical Engineering Databook for Kaolin, underlain by rigid bedrock. Site investigation reveals that the clay has a saturated unit weight of approximately 16 kN m^{-3} and a specific volume of 2.75 at all depths, and has previously experienced an effective stress 100 kPa greater than that due to the clay's self-weight at all depths. The water table can be assumed to be at the soil surface.

- (a) Utilising the Fadum chart in Fig. 1, calculate the depth below the centre of the foundation at which the increase in vertical stress due to the building will equal 100 kPa. [20%]
- (b) Estimate the depth below the corner of the foundation at which the increase in vertical stress due to the building will equal 100 kPa. [10%]
- (c) Estimate the settlements of the corners and centre of the foundation due to the building load. [50%]
- (d) Describe the impacts that the settlements calculated in part (c) might have on the serviceability of the building. [20%]

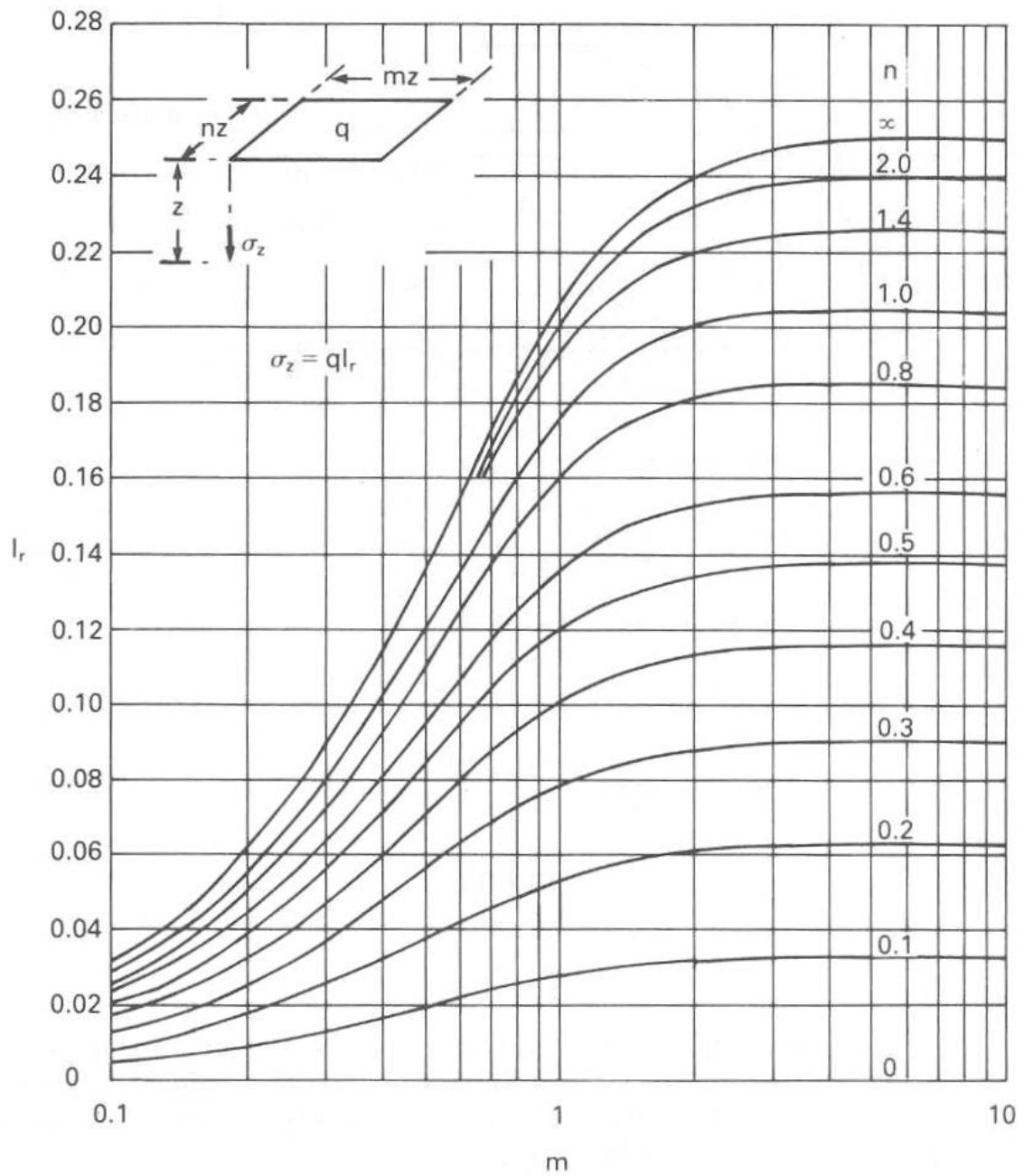


Fig. 1

4 A 5 m high flood embankment is to be constructed on a 10 m thick normally consolidated clay layer underlain by a sand stratum. Oedometer testing reveals the clay to have a constrained modulus E_0 of 2 MPa, a coefficient of consolidation for virgin compression c_v of $1 \text{ m}^2 \text{ year}^{-1}$, a unit weight of 16 kN m^{-3} , $\lambda = 0.25$ and $\kappa = 0.04$. The embankment exerts a bearing pressure of 80 kPa on the clay surface and the water table can be assumed to be 2 m below the clay surface.

- (a) Calculate the ultimate settlement of the clay beneath the embankment and the time to achieve 90% of this settlement. [20%]
- (b) In order to reduce the time taken for settlements to occur, the embankment height is initially increased to 10 m, with the additional fill being removed when the settlement reaches the ultimate settlement calculated in part (a). Assuming the stiffness of the soil to remain constant, calculate the time after which the additional fill should be removed. [25%]
- (c) Sketch the isochrones of excess pore pressure in the clay layer immediately before and after removal of the additional fill and during the further consolidation of the clay, marking salient values of excess pore pressure. [30%]
- (d) From the geometry of the isochrones, estimate how much additional settlement might occur after removal of the additional fill. [25%]

END OF PAPER

Engineering Tripos Part IIA

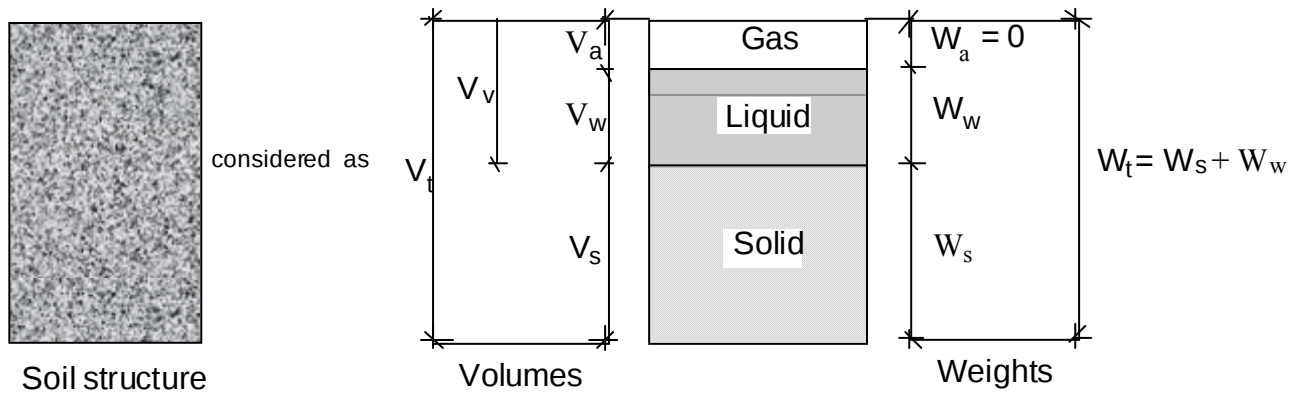
3D1 & 3D2

Geotechnical Engineering

Data Book 2024-2025

Contents	Page
General definitions	2
Soil classification	3
Seepage	4
One-dimensional compression	5
One-dimensional consolidation	6
Stress and strain components	7, 8
Elastic stiffness relations	9
Cam Clay	10, 11
Friction and dilation	12, 13, 14
Plasticity; cohesive material	15
Plasticity; frictional material	16
Empirical earth pressure coefficients	17
Cylindrical cavity expansion	17
Infinite slope analysis	17
Shallow foundation capacity	18- 19
Settlement of Shallow Foundations	20-21

General definitions



Specific gravity of solid

$$G_s$$

Voids ratio

$$e = V_v / V_s$$

Specific volume

$$v = V_t / V_s = 1 + e$$

Porosity

$$n = V_v / V_t = e / (1 + e)$$

Water content

$$w = (W_w / W_s)$$

Degree of saturation

$$S_r = V_w / V_v = (w G_s / e)$$

Unit weight of water

$$\gamma_w = 9.81 \text{ kN/m}^3$$

Unit weight of soil

$$\gamma = W_t / V_t = \left(\frac{G_s + S_r e}{1 + e} \right) \gamma_w$$

Buoyant saturated unit weight

$$\gamma' = \gamma - \gamma_w = \left(\frac{G_s - 1}{1 + e} \right) \gamma_w$$

Unit weight of dry solids

$$\gamma_d = W_s / V_t = \left(\frac{G_s}{1 + e} \right) \gamma_w$$

Air volume ratio

$$A = V_a / V_t = \left(\frac{e(1 - S_r)}{1 + e} \right)$$

Soil classification (BS1377)

Liquid limit w_L

Plastic Limit w_P

Plasticity Index $I_P = w_L - w_P$

Liquidity Index $I_L = \frac{w - w_P}{w_L - w_P}$

Activity = $\frac{\text{Plasticity Index}}{\text{Percentage of particles finer than } 2 \mu\text{m}}$

Sensitivity = $\frac{\text{Unconfined compressive strength of an undisturbed specimen}}{\text{Unconfined compressive strength of a remoulded specimen}}$ (at the same water content)

Classification of particle sizes:–

Boulders	larger than	200 mm	
Cobbles	between	200 mm	and 60 mm
Gravel	between	60 mm	and 2 mm
Sand	between	2 mm	and 0.06 mm
Silt	between	0.06 mm	and 0.002 mm
Clay	smaller than	0.002 mm (two microns)	

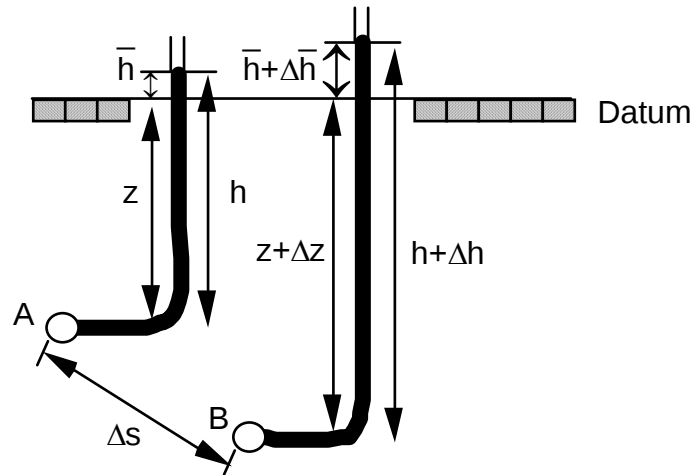
D equivalent diameter of soil particle

D_{10} , D_{60} etc. particle size such that 10% (or 60%) etc.) by weight of a soil sample is composed of finer grains.

C_U uniformity coefficient D_{60} / D_{10}

Seepage

Flow potential:
(piezometric level)



Total gauge pore water pressure at A: $u = \gamma_w h = \gamma_w (\bar{h} + z)$

$$\text{B: } u + \Delta u = \gamma_w (h + \Delta h) = \gamma_w (\bar{h} + z + \Delta \bar{h} + \Delta z)$$

Excess pore water pressure at A: $\bar{u} = \gamma_w \bar{h}$

$$\text{B: } \bar{u} + \Delta \bar{u} = \gamma_w (\bar{h} + \Delta \bar{h})$$

Hydraulic gradient A $\rightarrow$ B $i = - \frac{\Delta \bar{h}}{\Delta s}$

Hydraulic gradient (3D) $i = - \nabla \bar{h}$

Darcy's law $V = ki$

V = superficial seepage velocity

k = coefficient of permeability

Typical permeabilities:

$D_{10} > 10 \text{ mm}$	:	non-laminar flow
$10 \text{ mm} > D_{10} > 1 \mu\text{m}$	:	$k \cong 0.01 (D_{10} \text{ in mm})^2 \text{ m/s}$
clays	:	$k \cong 10^{-9} \text{ to } 10^{-11} \text{ m/s}$

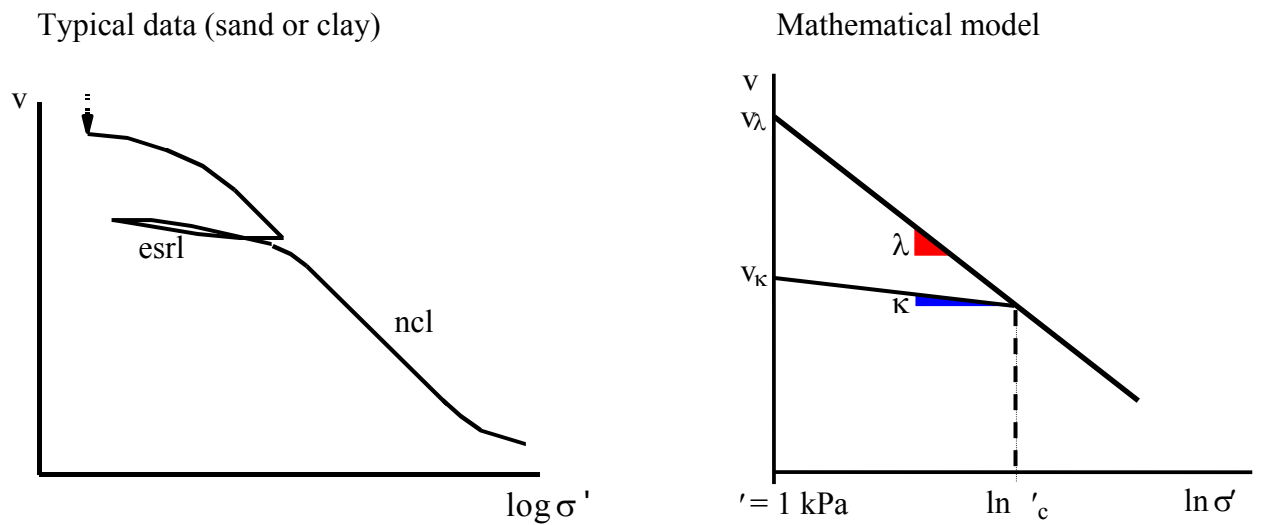
Saturated capillary zone

$$h_c = \frac{4T}{\gamma_w d} \quad : \quad \text{capillary rise in tube diameter } d, \text{ for surface tension } T$$

$$h_c \approx \frac{3 \times 10^{-5}}{D_{10}} \text{ m} \quad : \quad \text{for water at } 10^\circ\text{C}; \text{ note air entry suction is } u_c = - \gamma_w h_c$$

One-Dimensional Compression

• Fitting data



Plastic compression stress σ'_c is taken as the larger of the initial aggregate crushing stress and the historic maximum effective vertical stress. Clay muds are taken to begin with $\sigma'_c \approx 1$ kPa.

Plastic compression (normal compression line, ncl): $v = v_\lambda - \lambda \ln \sigma'$ for $\sigma' = \sigma'_c$

Elastic swelling and recompression line (esrl): $v = v_c + \kappa (\ln \sigma'_c - \ln \sigma'_v)$
 $= v_\kappa - \kappa \ln \sigma'_v$ for $\sigma' < \sigma'_c$

Equivalent parameters for $\log_{10}$ stress scale:

Terzaghi's compression index $C_c = \lambda \log_{10} e$

Terzaghi's swelling index $C_s = \kappa \log_{10} e$

• Deriving confined soil stiffnesses

Secant 1D compression modulus $E_o = (\Delta \sigma' / \Delta \epsilon)_o$

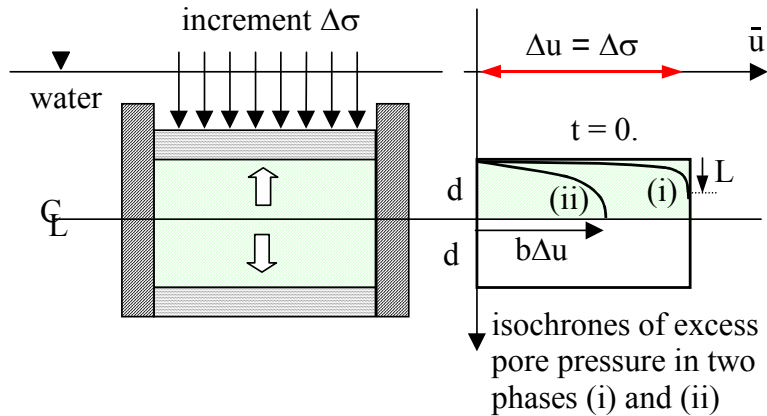
Tangent 1D plastic compression modulus $E_o = v \sigma' / \lambda$

Tangent 1D elastic compression modulus $E_o = v \sigma' / \kappa$

One-Dimensional Consolidation

Settlement	ρ	$= \int m_v (\Delta u - \bar{u}) dz$	$= \int (\Delta u - \bar{u}) / E_o dz$
Coefficient of consolidation	c_v	$= \frac{k}{m_v \gamma_w}$	$= \frac{kE_o}{\gamma_w}$
Dimensionless time factor	T_v	$= \frac{c_v t}{d^2}$	
Relative settlement	R_v	$= \frac{\rho}{\rho_{ult}}$	

• Solutions for initially rectangular distribution of excess pore pressure



Approximate solution by parabolic isochrones:

Phase (i) $L^2 = 12 c_v t$

$$R_v = \sqrt{\frac{4T_v}{3}} \quad \text{for } T_v < 1/12$$

Phase (ii) $b = \exp(1/4 - 3T_v)$

$$R_v = [1 - 2/3 \exp(1/4 - 3T_v)] \quad \text{for } T_v > 1/12$$

Solution by Fourier Series:

T_v	0	0.01	0.02	0.04	0.08	0.15	0.20	0.30	0.40	0.50	0.60	0.80	1.00
R_v	0	0.12	0.17	0.23	0.32	0.45	0.51	0.62	0.70	0.77	0.82	0.89	0.94

Stress and strain components

• Principle of effective stress (saturated soil)

$$\text{total stress } \sigma = \text{effective stress } \sigma' + \text{pore water pressure } u$$

• Principal components of stress and strain

sign convention	compression positive
total stress	$\sigma_1, \sigma_2, \sigma_3$
effective stress	$\sigma'_1, \sigma'_2, \sigma'_3$
strain	$\varepsilon_1, \varepsilon_2, \varepsilon_3$

• Simple Shear Apparatus (SSA) ($\varepsilon_2 = 0$; other principal directions unknown)

The only stresses that are readily available are the shear stress τ and normal stress σ applied to the top platen. The pore pressure u can be controlled and measured, so the normal effective stress σ' can be found. Drainage can be permitted or prevented. The shear strain γ and normal strain ε are measured with respect to the top platen, which is a plane of zero extension. Zero extension planes are often identified with slip surfaces.

$$\text{work increment per unit volume} \quad \delta W = \tau \delta \gamma + \sigma' \delta \varepsilon$$

• Biaxial Apparatus - Plane Strain (BA-PS) ($\varepsilon_2 = 0$; rectangular edges along principal axes)

Intermediate principal effective stress σ'_2 , in zero strain direction, is frequently unknown so that all conditions are related to components in the 1-3 plane.

mean total stress	$s = (\sigma_1 + \sigma_3)/2$
mean effective stress	$s' = (\sigma'_1 + \sigma'_3)/2 = s - u$
shear stress	$t = (\sigma'_1 - \sigma'_3)/2 = (\sigma_1 - \sigma_3)/2$

volumetric strain	$\varepsilon_v = \varepsilon_1 + \varepsilon_3$
shear strain	$\varepsilon_\gamma = \varepsilon_1 - \varepsilon_3$

$$\text{work increment per unit volume} \quad \delta W = \sigma'_1 \delta \varepsilon_1 + \sigma'_3 \delta \varepsilon_3$$

$$\delta W = s' \delta \varepsilon_v + t \delta \varepsilon_\gamma$$

providing that principal axes of strain increment and of stress coincide.

• **Triaxial Apparatus – Axial Symmetry (TA-AS)** (cylindrical element with radial symmetry)

total axial stress	$\sigma_a = \sigma'_a + u$
total radial stress	$\sigma_r = \sigma'_r + u$
total mean normal stress	$p = (\sigma_a + 2\sigma_r)/3$
effective mean normal stress	$p' = (\sigma'_a + 2\sigma'_r)/3 = p - u$
deviatoric stress	$q = \sigma'_a - \sigma'_r = \sigma_a - \sigma_r$
stress ratio	$\eta = q/p'$
axial strain	ϵ_a
radial strain	ϵ_r
volumetric strain	$\epsilon_v = \epsilon_a + 2\epsilon_r$
triaxial shear strain	$\epsilon_s = \frac{2}{3}(\epsilon_a - \epsilon_r)$
work increment per unit volume	$\delta W = \sigma'_a \delta \epsilon_a + 2\sigma'_r \delta \epsilon_r$
	$\delta W = p' \delta \epsilon_v + q \delta \epsilon_s$

Types of triaxial test include:

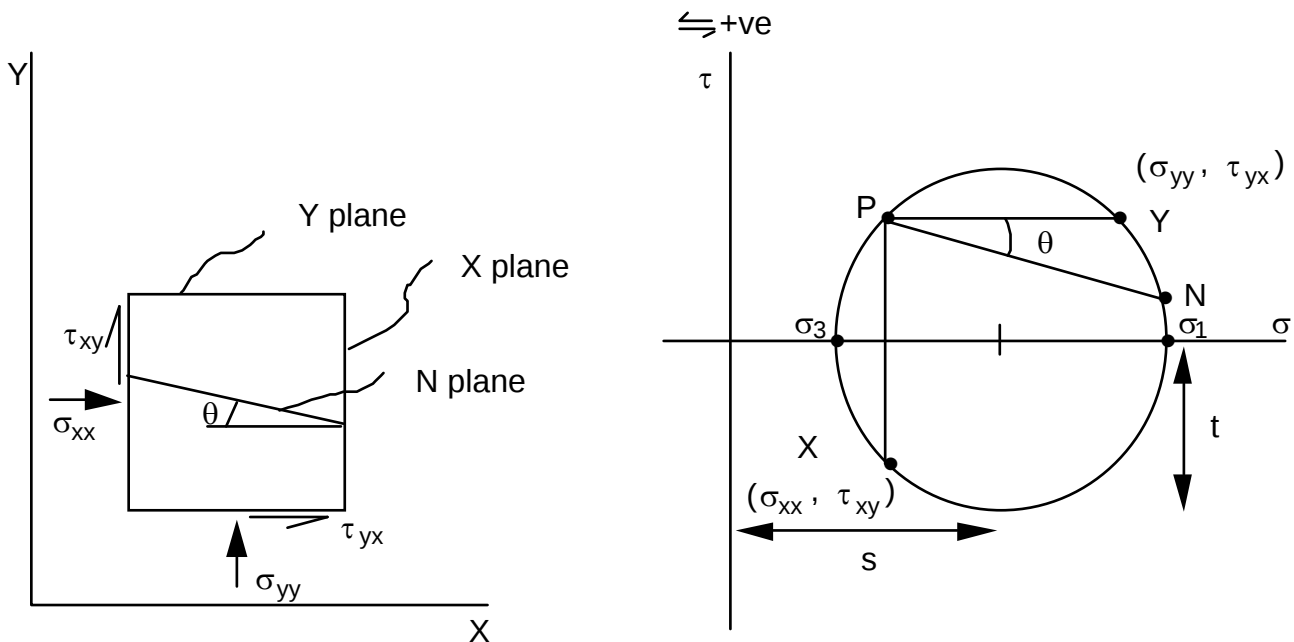
isotropic compression in which p' increases at zero q

triaxial compression in which q increases *either* by increasing σ_a *or* by reducing σ_r

triaxial extension in which q reduces *either* by reducing σ_a *or* by increasing σ_r

• **Mohr's circle of stress (1–3 plane)**

Sign of convention: compression, and counter-clockwise shear, positive



Poles of planes P : the components of stress on the N plane are given by the intersection N of the Mohr circle with the line PN through P parallel to the plane.

Elastic stiffness relations

These relations apply to tangent stiffnesses of over-consolidated soil, with a state point on some swelling and recompression line (κ -line), and remote from gross plastic yielding.

One-dimensional compression (axial stress and strain increments $d\sigma'$, $d\varepsilon$)

$$\text{compressibility} \quad m_v = d\varepsilon / d\sigma'$$

$$\text{constrained modulus} \quad E_o = 1 / m_v$$

Physically fundamental parameters

$$\text{shear modulus} \quad G' = dt / d\varepsilon_\gamma$$

$$\text{bulk modulus} \quad K' = dp' / d\varepsilon_v$$

Parameters which can be used for constant-volume deformations

$$\text{undrained shear modulus} \quad G_u = G'$$

$$\text{undrained bulk modulus} \quad K_u = \infty \quad (\text{neglecting compressibility of water})$$

Alternative convenient parameters

$$\text{Young's moduli} \quad E' \text{ (effective), } E_u \text{ (undrained)}$$

$$\text{Poisson's ratios} \quad \nu' \text{ (effective), } \nu_u = 0.5 \text{ (undrained)}$$

Typical value of Poisson's ratio for small changes of stress: $\nu' = 0.2$

$$\text{Relationships:} \quad G = \frac{E}{2(1 + \nu)}$$

$$K = \frac{E}{3(1 - 2\nu)}$$

$$E_o = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}$$

Cam Clay

• Interchangeable parameters for stress combinations at yield, and plastic strain increments

System	Effective normal stress	Plastic normal strain	Effective shear stress	Plastic shear strain	Critical stress ratio	Plastic normal stress	Critical normal stress
General	σ^*	ε^*	τ^*	γ^*	μ^*_{crit}	σ^*_c	σ^*_{crit}
SSA	σ'	ε	τ	γ	$\tan \phi_{\text{crit}}$	σ'_c	σ'_{crit}
BA-PS	s'	ε_v	t	ε_γ	$\sin \phi_{\text{crit}}$	s'_c	s'_{crit}
TA-AS	p'	ε_v	q	ε_s	M	p'_c	p'_{crit}

• General equations of plastic work

Plastic work and dissipation

$$\sigma^* \delta \varepsilon^* + \tau^* \delta \gamma^* = \mu^*_{\text{crit}} \sigma^* \delta \gamma^*$$

Plastic flow rule – normality

$$\frac{d\tau^*}{d\sigma^*} \cdot \frac{d\gamma^*}{d\varepsilon^*} = -1$$

• General yield surface

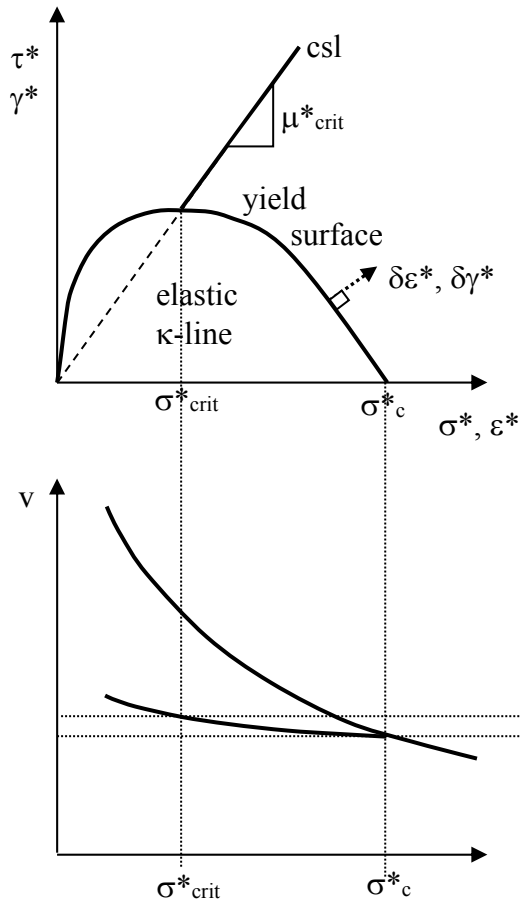
$$\frac{\tau^*}{\sigma^*} = \mu^* = \mu^*_{\text{crit}} \cdot \ln \left[\frac{\sigma^*_c}{\sigma^*} \right]$$

• Parameter values which fit soil data

	London Clay	Weald Clay	Kaolin	Dog's Bay Sand	Ham River Sand
λ^*	0.161	0.093	0.26	0.334	0.163
κ^*	0.062	0.035	0.05	0.009	0.015
Γ^* at 1 kPa	2.759	2.060	3.767	4.360	3.026
$\sigma^*_{c, \text{ virgin}}$ kPa	1	1	1	Loose 500 Dense 1500	Loose 2500 Dense 15000
ϕ_{crit}	23°	24°	26°	39°	32°
M_{comp}	0.89	0.95	1.02	1.60	1.29
M_{extn}	0.69	0.72	0.76	1.04	0.90
w_L	0.78	0.43	0.74	-----	-----
w_P	0.26	0.18	0.42	-----	-----
G_s	2.75	2.75	2.61	2.75	2.65

Note: 1) parameters λ^* , κ^* , Γ^* , $\sigma^*_{c, \text{ virgin}}$ should depend to a small extent on the deformation mode, e.g. SSA, BA-PS, TA-AS, etc. This may be neglected unless further information is given.
2) Sand which is loose, or loaded cyclically, compacts more than Cam Clay allows.

• The yield surface in (σ^*, τ^*, v) space



ncl: normal compression line

$$v = N - \lambda \ln \sigma^*$$

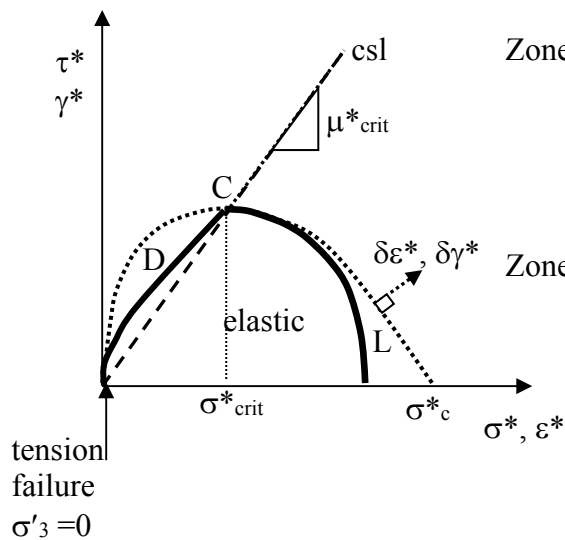
csl: critical state line

$$v = \Gamma - \lambda \ln \sigma^*$$

where $N = \Gamma + \lambda - \kappa$

• Regions of limiting soil behaviour

Variation of Cam Clay yield surface

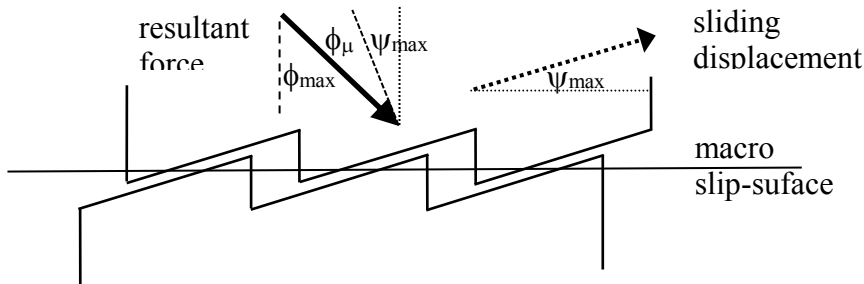


Zone D: denser than critical, "dry",
dilation or negative excess pore pressures,
Hvorslev strength envelope,
friction-dilatancy theory,
unstable shear rupture, progressive failure

Zone L: looser than critical, "wet",
compaction or positive excess pore pressures,
Modified Cam Clay yield surface,
stable strain-hardening continuum

Strength of soil: friction and dilation

• Friction and dilatancy: the saw-blade model of direct shear

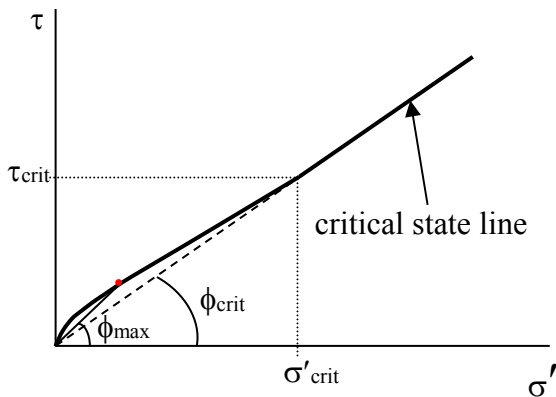


Intergranular angle of friction at sliding contacts ϕ_{μ}

Angle of dilation $\psi_{\max}$

Angle of internal friction $\phi_{\max} = \phi_{\mu} + \psi_{\max}$

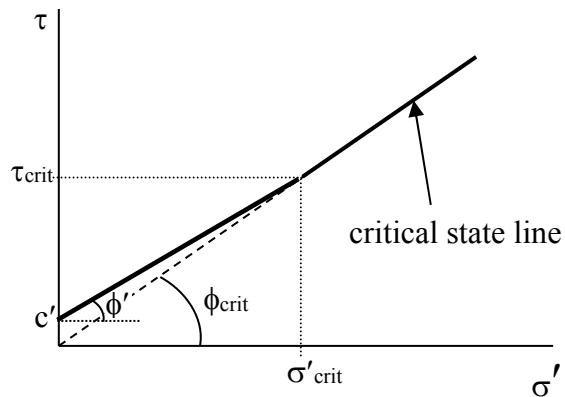
• Friction and dilatancy: secant and tangent strength parameters



Secant angle of internal friction

$$\begin{aligned}\tau &= \sigma' \tan \phi_{\max} \\ \phi_{\max} &= \phi_{\text{crit}} + \Delta\phi \\ \Delta\phi &= f(\sigma'_{\text{crit}}/\sigma')\end{aligned}$$

typical envelope fitting data:
power curve
 $(\tau/\tau_{\text{crit}}) = (\sigma'/\sigma'_{\text{crit}})^{\alpha}$
with $\alpha \approx 0.85$



Tangent angle of shearing envelope

$$\begin{aligned}\tau &= c' + \sigma' \tan \phi' \\ c' &= f(\sigma'_{\text{crit}})\end{aligned}$$

typical envelope:
straight line
 $\tan \phi' = 0.85 \tan \phi_{\text{crit}}$
 $c' = 0.15 \tau_{\text{crit}}$

• Friction and dilation: data of sands

The inter-granular friction angle of quartz grains, $\phi_\mu \approx 26^\circ$. Turbulent shearing at a critical state causes ϕ_{crit} to exceed this. The critical state angle of internal friction ϕ_{crit} is a function of the uniformity of particle sizes, their shape, and mineralogy, and is developed at large shear strains irrespective of initial conditions. Typical values of ϕ_{crit} ($\pm 2^\circ$) are:

well-graded, angular quartz or feldspar sands	40°
uniform sub-angular quartz sand	36°
uniform rounded quartz sand	32°

Relative density $I_D = \frac{(e_{\text{max}} - e)}{(e_{\text{max}} - e_{\text{min}})}$ where:

e_{max} is the maximum void ratio achievable in quick-tilt test

e_{min} is the minimum void ratio achievable by vibratory compaction

Relative crushability $I_C = \ln(\sigma_c / p')$ where:

σ_c is the aggregate crushing stress, taken to be a material constant, typical values being: 80 000 kPa for quartz silt, 20 000 kPa for quartz sand, 5 000 kPa for carbonate sand.

p' is the mean effective stress at failure which may be taken as approximately equal to the effective stress σ' normal to a shear plane.

Dilatancy contribution to the peak angle of internal friction is $\Delta\phi = (\phi_{\text{max}} - \phi_{\text{crit}}) = f(I_R)$

Relative dilatancy index $I_R = I_D I_C - 1$ where:

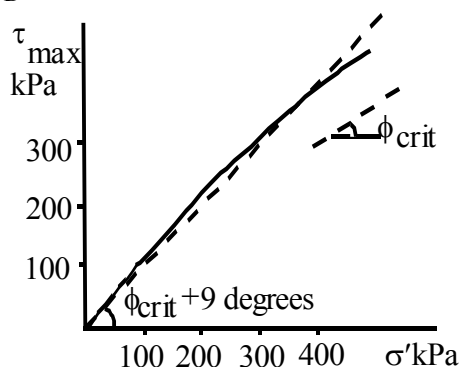
$I_R < 0$ indicates compaction, so that I_D increases and $I_R \rightarrow 0$ ultimately at a critical state

$I_R > 4$ to be limited to $I_R = 4$ unless corroborative dilatant strength data is available

The following empirical correlations are then available

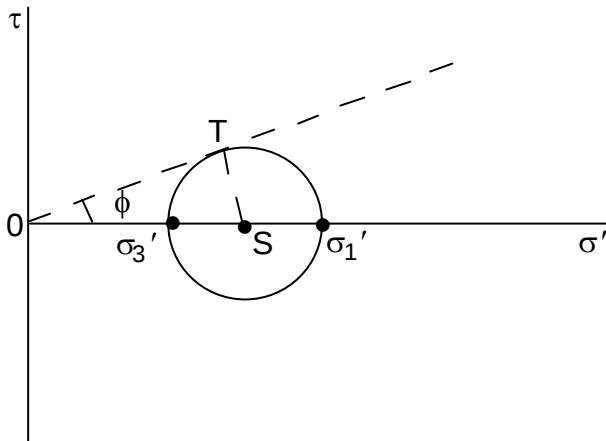
plane strain conditions	$(\phi_{\text{max}} - \phi_{\text{crit}})$	$= 0.8 \psi_{\text{max}}$	$= 5 I_R$ degrees
triaxial strain conditions	$(\phi_{\text{max}} - \phi_{\text{crit}})$	$= 3 I_R$ degrees	
all conditions	$(-\delta\varepsilon_v / \delta\varepsilon_1)_{\text{max}}$	$= 0.3 I_R$	

The resulting peak strength envelope for triaxial tests on a quartz sand at an initial relative density $I_D = 1$ is shown below for the limited stress range 10 - 400 kPa:



$$\phi_{\text{max}} > \phi_{\text{crit}} + 9^\circ \quad \text{for } I_D = 1, \sigma' = 400 \text{ kPa}$$

• Mobilised (secant) angle of shearing ϕ in the 1 – 3 plane



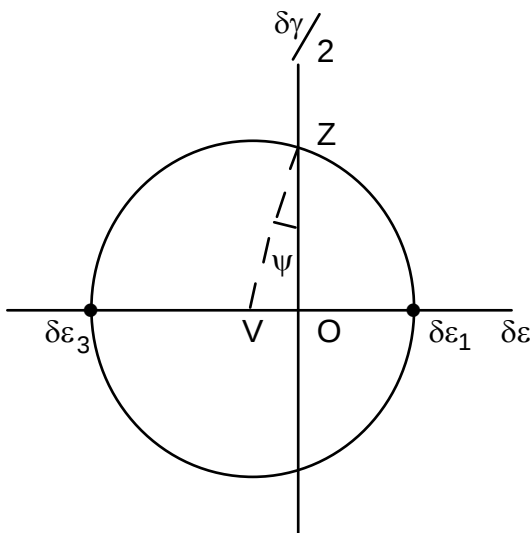
$$\begin{aligned}\sin \phi &= TS/OS \\ &= \frac{(\sigma_1' - \sigma_3')/2}{(\sigma_1' + \sigma_3')/2} \\ \left[\frac{\sigma_1'}{\sigma_3'} \right] &= \frac{(1 + \sin \phi)}{(1 - \sin \phi)}\end{aligned}$$

Angle of shearing resistance:

at peak strength $\phi_{\max}$ at $\left[\frac{\sigma_1'}{\sigma_3'} \right]_{\max}$

at critical state ϕ_{crit} after large shear strains

• Mobilised angle of dilation in plane strain ψ in the 1 – 3 plane



$$\begin{aligned}\sin \psi &= VO/VZ \\ &= -\frac{(\delta \epsilon_1 + \delta \epsilon_3)/2}{(\delta \epsilon_1 - \delta \epsilon_3)/2} \\ &= -\frac{\delta \epsilon_v}{\delta \epsilon_\gamma}\end{aligned}$$

$$\left[\frac{\delta \epsilon_1}{\delta \epsilon_3} \right] = -\frac{(1 - \sin \psi)}{(1 + \sin \psi)}$$

at peak strength $\psi = \psi_{\max}$ at $\left[\frac{\sigma_1'}{\sigma_3'} \right]_{\max}$

at critical state $\psi = 0$ since volume is constant

Plasticity: Cohesive material $\tau_{\max} = c_u$ (or s_u)

• Limiting stresses

Tresca $|\sigma_1 - \sigma_3| = q_u = 2c_u$

von Mises $(\sigma_1 - p)^2 + (\sigma_2 - p)^2 + (\sigma_3 - p)^2 = \frac{2}{3} q_u^2 = 2c_u^2$

where q_u is the undrained triaxial compression strength, and c_u is the undrained plane shear strength.

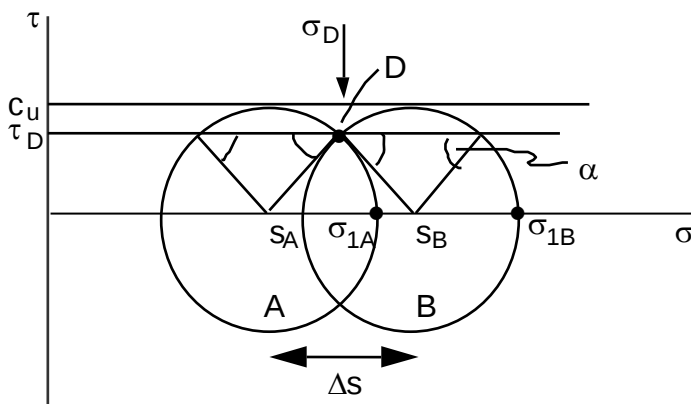
Dissipation per unit volume in plane strain deformation following either Tresca or von Mises,

$$\delta D = c_u \delta \varepsilon_\gamma$$

For a relative displacement x across a slip surface of area A mobilising shear strength c_u , this becomes

$$D = A c_u x$$

• Stress conditions across a discontinuity



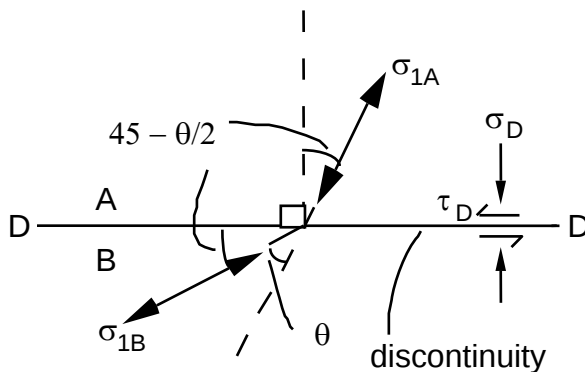
Rotation of major principal stress θ

$$s_B - s_A = \Delta s = 2c_u \sin \theta$$

$$\sigma_{1B} - \sigma_{1A} = 2c_u \sin \theta$$

In limit with $\theta \rightarrow 0$

$$ds = 2c_u d\theta$$



Useful example:

$$\theta = 30^\circ$$

$$\sigma_{1B} - \sigma_{1A} = c_u$$

$$\tau_D / c_u = 0.87$$

σ_{1A} = major principal stress in zone A

σ_{1B} = major principal stress in zone B

Plasticity: Frictional material $(\tau/\sigma')_{\max} = \tan \phi$

• Limiting stresses

$$\sin \phi = (\sigma'_{1f} - \sigma'_{3f}) / (\sigma'_{1f} + \sigma'_{3f}) = (\sigma_{1f} - \sigma_{3f}) / (\sigma_{1f} + \sigma_{3f} - 2u_s)$$

where σ'_{1f} and σ'_{3f} are the major and minor principal effective stresses at failure, σ_{1f} and σ_{3f} are the major and minor principle total stresses at failure, and u_s is the steady state pore pressure.

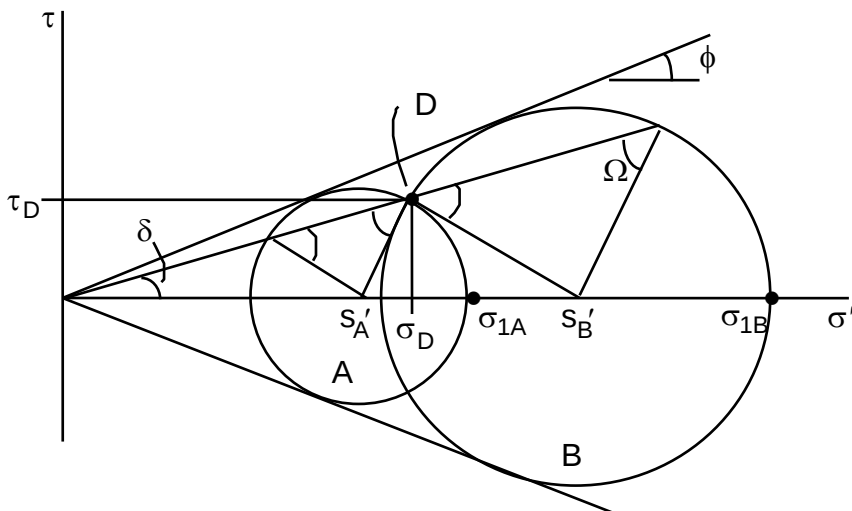
Active pressure:

$$\begin{aligned}\sigma'_v &> \sigma'_h \\ \sigma'_1 &= \sigma'_v \text{ (assuming principal stresses are horizontal and vertical)} \\ \sigma'_3 &= \sigma'_h \\ K_a &= (1 - \sin \phi) / (1 + \sin \phi)\end{aligned}$$

Passive pressure:

$$\begin{aligned}\sigma'_h &> \sigma'_v \\ \sigma'_1 &= \sigma'_h \text{ (assuming principal stresses are horizontal and vertical)} \\ \sigma'_3 &= \sigma'_v \\ K_p &= (1 + \sin \phi) / (1 - \sin \phi) = 1 / K_a\end{aligned}$$

• Stress conditions across a discontinuity



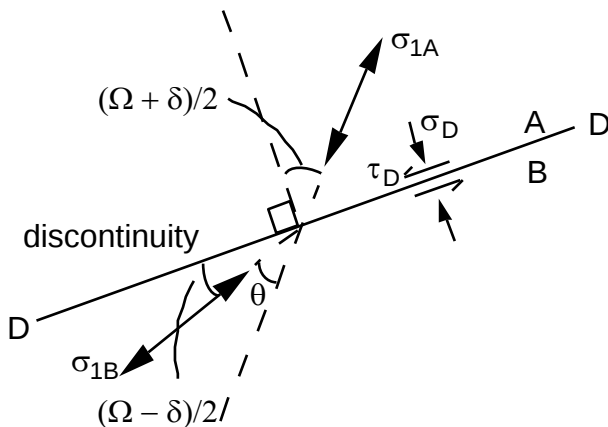
Rotation of major principal stress

$$\theta = \pi/2 - \Omega$$

σ_{1A} = major principal stress in zone A

σ_{1B} = major principal stress in zone B

$$\tan \delta = \tau_D / \sigma'_D$$



$$\sin \Omega = \sin \delta / \sin \phi$$

$$s'_B / s'_A = \sin(\Omega + \delta) / \sin(\Omega - \delta)$$

In limit, $d\theta \rightarrow 0$ and $\delta \rightarrow \phi$

$$ds' = 2s' \cdot d\theta \tan \phi$$

Integration gives $s'_B / s'_A = \exp(2\theta \tan \phi)$

Empirical earth pressure coefficients following one-dimensional strain

Coefficient of earth pressure in 1D plastic compression (normal compression)

$$K_{o,nc} = 1 - \sin \phi_{crit}$$

Coefficient of earth pressure during a 1D unloading-reloading cycle (overconsolidated soil)

$$K_o = K_{o,nc} \left[1 + \frac{(n-1)(n_{max}^\alpha - 1)}{(n_{max} - 1)} \right]$$

where n is current overconsolidation ratio (OCR) defined as $\sigma'_{v,max} / \sigma'_v$

n_{max} is maximum historic OCR defined as $\sigma'_{v,max} / \sigma'_{v,min}$

α is to be taken as $1.2 \sin \phi_{crit}$

Cylindrical cavity expansion

Expansion $\delta A = A - A_o$ caused by increase of pressure $\delta \sigma_c = \sigma_c - \sigma_o$

At radius r : small displacement $\rho = \frac{\delta A}{2\pi r}$

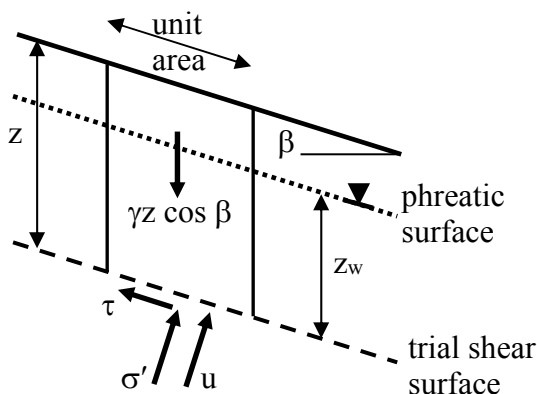
small shear strain $\gamma = \frac{2\rho}{r}$

Radial equilibrium: $r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0$

Elastic expansion (small strains) $\delta \sigma_c = G \frac{\delta A}{A}$

Undrained plastic-elastic expansion $\delta \sigma_c = c_u \left[1 + \ln \frac{G}{c_u} + \ln \frac{\delta A}{A} \right]$

Infinite slope analysis



$$\begin{aligned} u &= \gamma_w z_w \cos^2 \beta \\ \sigma &= \gamma z \cos^2 \beta \\ \sigma' &= (\gamma z - \gamma_w z_w) \cos^2 \beta \\ \tau &= \gamma z \cos \beta \sin \beta \end{aligned}$$

$$\tan \phi_{mob} = \frac{\tau}{\sigma'} = \frac{\tan \beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}$$

Shallow foundation design

Tresca soil, with undrained strength s_u

Vertical loading

The vertical bearing capacity, q_f , of a shallow foundation for undrained loading (Tresca soil) is:

$$\frac{V_{ult}}{A} = q_f = s_c d_c N_c s_u + \gamma h$$

V_{ult} and A are the ultimate vertical load and the foundation area, respectively. h is the embedment of the foundation base and γ (or γ') is the appropriate density of the overburden.

The exact bearing capacity factor N_c for a plane strain surface foundation (zero embedment) on uniform soil is:

$$N_c = 2 + \pi \quad (\text{Prandtl, 1921})$$

Shape correction factor:

For a rectangular footing of length L and breadth B (Eurocode 7):

$$s_c = 1 + 0.2 B / L$$

The exact solution for a rough circular foundation ($D = B = L$) is $q_f = 6.05 s_u$, hence $s_c = 1.18 \sim 1.2$.

Embedment correction factor:

A fit to Skempton's (1951) embedment correction factors, for an embedment of h , is:

$$d_c = 1 + 0.33 \tan^{-1} (h/B) \quad (\text{or } h/D \text{ for a circular foundation})$$

Combined V-H loading

A curve fit to Green's lower bound plasticity solution for V-H loading is:

$$\text{If } V/V_{ult} > 0.5: \quad \frac{V}{V_{ult}} = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{H}{H_{ult}}} \quad \text{or} \quad \frac{H}{H_{ult}} = 1 - \left(2 \frac{V}{V_{ult}} - 1 \right)^2$$

$$\text{If } V/V_{ult} < 0.5: \quad H = H_{ult} = B s_u$$

Combined V-H-M loading

With lift-off: combined Green-Meyerhof

$$\text{Without lift-off:} \quad \left(\frac{V}{V_{ult}} \right)^2 + \left[\frac{M}{M_{ult}} \left(1 - 0.3 \frac{H}{H_{ult}} \right) \right]^2 + \left| \left(\frac{H}{H_{ult}} \right)^3 \right| - 1 = 0 \quad (\text{Taiebet \& Carter 2000})$$

Frictional (Coulomb) soil, with friction angle ϕ

Vertical loading

The vertical bearing capacity, q_f , of a shallow foundation under drained loading (Coulomb soil) is:

$$\frac{V_{ult}}{A} = q_f = s_q N_q \sigma'_{v0} + s_\gamma N_\gamma \frac{\gamma' B}{2}$$

The bearing capacity factors N_q and N_γ account for the capacity arising from surcharge and self-weight of the foundation soil respectively. σ'_{v0} is the in situ effective stress acting at the level of the foundation base.

For a strip footing on weightless soil, the exact solution for N_q is:

$$N_q = \tan^2(\pi/4 + \phi/2) e^{(\pi \tan \phi)} \quad (\text{Prandtl 1921})$$

An empirical relationship to estimate N_γ from N_q is (Eurocode 7):

$$N_\gamma = 2 (N_q - 1) \tan \phi$$

Curve fits to exact solutions for $N_\gamma = f(\phi)$ are (Davis & Booker 1971):

$$\text{Rough base:} \quad N_\gamma = 0.1054 e^{9.6\phi}$$

$$\text{Smooth base:} \quad N_\gamma = 0.0663 e^{9.3\phi}$$

Shape correction factors:

For a rectangular footing of length L and breadth B (Eurocode 7):

$$s_q = 1 + (B \sin \phi) / L$$

$$s_\gamma = 1 - 0.3 B / L$$

For circular footings take $L = B$.

Combined V-H loading

The Green/Sokolovski lower bound solution gives a V-H failure surface.

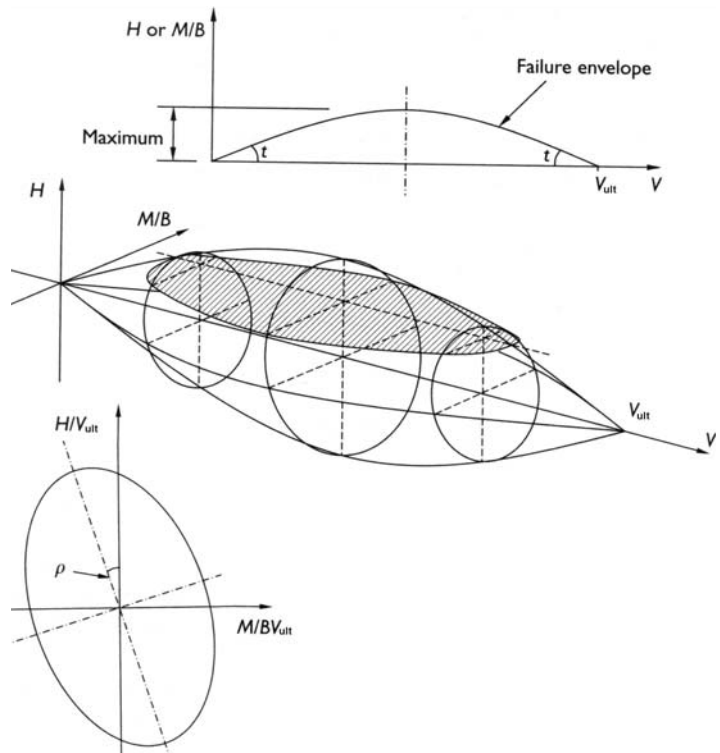
Combined V-H-M loading

With lift-off- drained conditions - use Butterfield & Gottardi (1994) failure surface shown above

$$\left[\frac{H/V_{ult}}{t_h} \right]^2 + \left[\frac{M/BV_{ult}}{t_m} \right]^2 + \left[\frac{2C(M/BV_{ult})(H/V_{ult})}{t_h t_m} \right] = \left[\frac{V}{V_{ult}} \left(1 - \frac{V}{V_{ult}} \right) \right]^2$$

where $C = \tan \left(\frac{2\rho(t_h - t_m)(t_h + t_m)}{2t_h t_m} \right)$ (Butterfield & Gottardi, 1994)

Typically, $t_h \sim 0.5$, $t_m \sim 0.4$ and $\rho \sim 15^\circ$. Note that t_h is the friction coefficient, $H/V = \tan \phi$, during sliding.



Settlement of Shallow Foundations

Elastic stress distributions below point, strip and circular loads

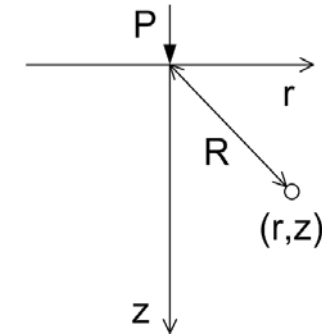
Point loading (Boussinesq solution)

Vertical stress $\sigma_z = \frac{3Pz^3}{2\pi R^5}$

Radial stress $\sigma_r = \frac{P}{2\pi R^2} \left[\frac{3r^2z}{R^3} - \frac{(1-2\nu)R}{R+z} \right]$

Tangential stress $\sigma_\theta = \frac{P(1-2\nu)}{2\pi R^2} \left[\frac{R}{R+z} - \frac{z}{R} \right]$

Shear stress $\tau_{rz} = \frac{3Prz^2}{2\pi R^5}$



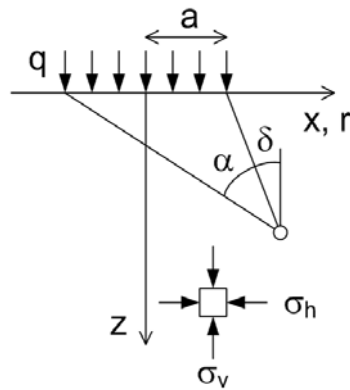
Uniformly-loaded strip

Vertical stress $\sigma_v = \frac{q}{\pi} [\alpha + \sin \alpha \cos(\alpha + 2\delta)]$

Horizontal stress $\sigma_h = \frac{q}{\pi} [\alpha - \sin \alpha \cos(\alpha + 2\delta)]$

Shear stress $\tau_{vh} = \frac{q}{\pi} \sin \alpha \sin(\alpha + 2\delta)$

Principal stresses $\sigma_1 = \frac{q}{\pi} (\alpha + \sin \alpha) \quad \sigma_3 = \frac{q}{\pi} (\alpha - \sin \alpha)$



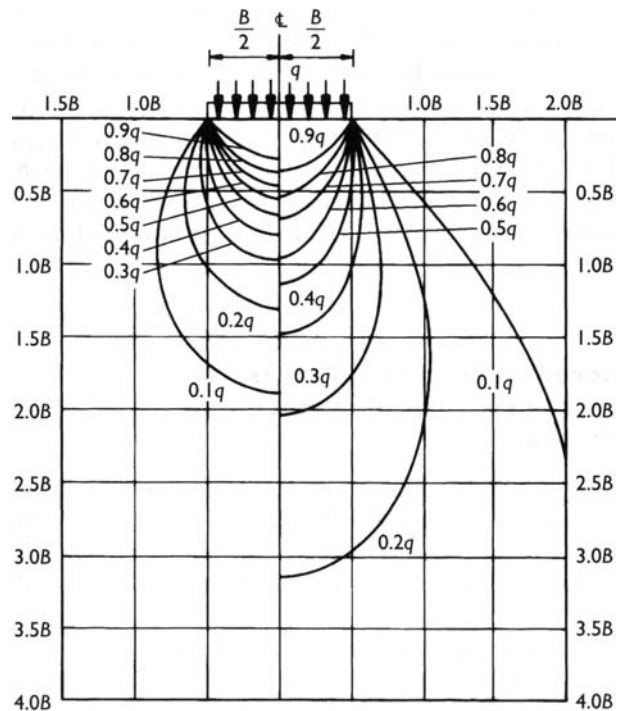
Uniformly-loaded circle radius a (on centerline, r=0)

Vertical stress

$$\sigma_v = q \left[1 - \left(\frac{1}{1 + (a/z)^2} \right)^{\frac{3}{2}} \right]$$

Horizontal stress

$$\sigma_h = \frac{q}{2} \left[(1 + 2\nu) - \frac{2(1 + \nu)z}{(a^2 + z^2)^{1/2}} + \frac{z^3}{(a^2 + z^2)^{3/2}} \right]$$



Contours of vertical stress below uniformly-loaded circular (left) and strip footings (right)

