

EGT2
ENGINEERING TRIPOS PART IIA

Thursday 30 April 2026 14.00 to 15.40

Module 3D2

GEOTECHNICAL ENGINEERING II

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Attachment: 3D1 & 3D2 Geotechnical Engineering Data Book (21 pages)

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A sample of London Clay is isotropically consolidated to an effective stress of 500 kPa and then allowed to swell to an effective stress of 100 kPa in a triaxial apparatus. Use the parameters for London Clay given in the 3D1 and 3D2 Geotechnical Engineering Data Book and the Cam-Clay model to answer the following questions.

- (a) Calculate the specific volume of the sample prior to shearing and determine the size of the current yield surface, i.e., the value of p'_c . Sketch the initial state and yield surface in both $p'-q$ and $v-\ln p'$ space. [25%]
- (b) The sample is sheared in drained triaxial compression by increasing the axial stress while keeping the cell pressure constant. Calculate the deviatoric stress and mean effective stress at yield. Determine the deviatoric stress, mean effective stress, and volumetric strain at critical state failure. [40%]
- (c) A second sample, identically consolidated and swelled, is sheared in undrained triaxial compression. Calculate the deviatoric stress and excess pore water pressure at both yield and critical state failure. [25%]
- (d) Discuss how the Cam-Clay model's predictions for heavily overconsolidated clay on the dry side of critical state compare to real soil behaviour. [10%]

2 A long strip foundation of width $B = 4$ m is to be constructed on a deposit of normally consolidated clay. The clay has unit weight $\gamma = 20 \text{ kN m}^{-3}$, angle of shearing resistance $\phi' = 25^\circ$, and undrained shear strength $s_u = 30 \text{ kPa}$. The water table is at ground level. The coefficient of earth pressure at rest, K_0 , may be estimated using Jaky's formula: $K_0 = 1 - \sin \phi'$. Element A is located directly beneath the edge of one side of the foundation at a depth of 5 m. When the foundation applies a bearing pressure q , the vertical and horizontal stress increments at Element A are $\Delta\sigma_v = 0.50q$ and $\Delta\sigma_h = 0.15q$.

(a) Calculate the in-situ total and effective stresses (σ_v , σ_h , σ'_v , σ'_h) at 5 m depth before the foundation is constructed. Express the initial stress state using the stress invariants t , s' , and s , and sketch this point on a graph of t against s' . [20%]

(b) For Element A, sketch the total stress path (TSP) and the effective stress path (ESP) on a graph of t against s' (and s) assuming undrained loading. Calculate the shear stress t and excess pore water pressure at yield (when the ESP first reaches the yield surface). [40%]

(c) For Element A, calculate the excess pore water pressure at ultimate failure, when the shear stress reaches the undrained shear strength s_u . Sketch the complete effective stress path from initial state through yield to ultimate failure. [25%]

(d) Without performing detailed calculations, describe how you would calculate the shear strain rate $\dot{\gamma}$ for Element A during each of the three phases of undrained loading: (i) the pre-yield elastic phase, (ii) the post-yield phase as the effective stress path follows the critical state line, and (iii) at ultimate failure when the undrained shear stress cap is reached. [15%]

3 An unsupported trench is constructed in saturated clay as shown in Fig. 1. The saturated unit weight and undrained shear strength of the clay are denoted by γ_s and s_u , respectively, and the trench depth is denoted by H .

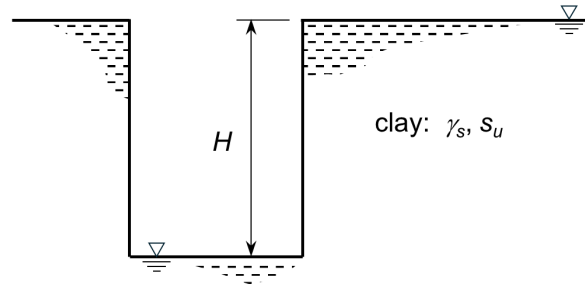


Fig. 1

- (a) Is there sufficient information to assess long-term stability of the trench? Why or why not? [10%]
- (b) Suppose the trench fills with water. In the short term, could this cause the trench to become unstable? Explain your reasoning. [10%]
- (c) Assume a circular failure surface and derive a formula for the safety factor SF in terms of s_u , H , and γ_s . Assume the circle has radius $r = H$ and is centred as shown in Fig. 2. [25%]

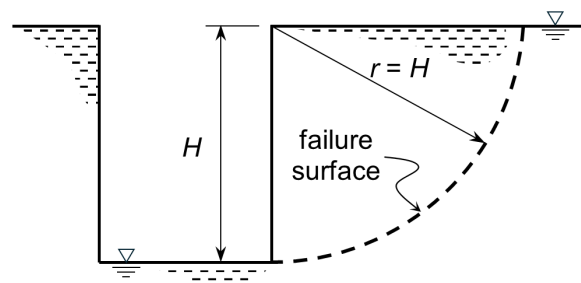


Fig. 2

- (d) Assume a straight failure surface inclined at angle θ from the horizontal, as shown in Fig. 3. Derive a formula for the safety factor SF in terms of s_u , H , and γ_s . The final expression should not depend on θ , i.e., θ should be determined as part of the solution. [45%]

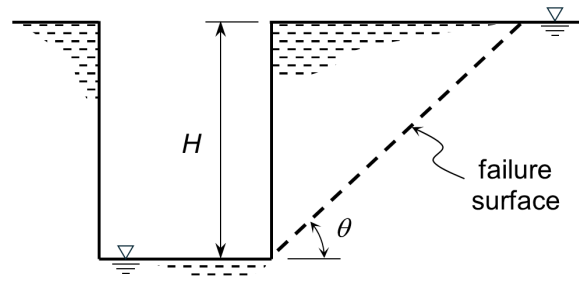


Fig. 3

- (e) Which of the two failure mechanisms considered in parts (c) and (d) is more likely to occur? Explain your reasoning. [10%]

4 A retaining wall for an excavation is supported by two props as shown in Fig. 4. The wall height equals the excavation depth H . One prop is installed at the top of the wall, and the other is at depth d . The upper layer of soil is sand with layer thickness t , saturated unit weight $\gamma_s = 21 \text{ kN m}^{-3}$, and friction angle $\phi' = 35^\circ$. The lower layer is clay with saturated unit weight $\gamma_s = 19 \text{ kN m}^{-3}$ and undrained shear strength $s_u = 30 \text{ kPa}$. The groundwater level coincides with the surface. Assume the wall is smooth.

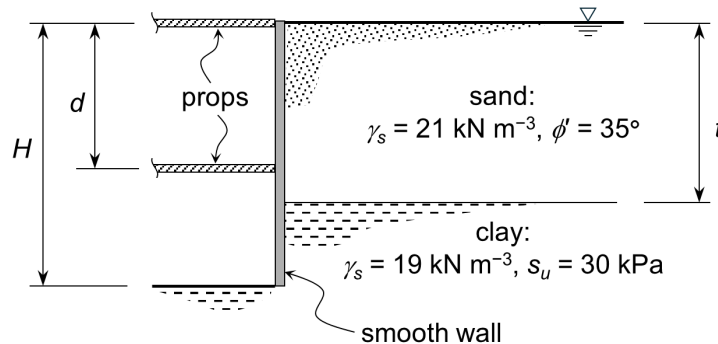


Fig. 4

- (a) Suppose $H = 9 \text{ m}$ and $t = 5 \text{ m}$. Evaluate the distribution of vertical effective stress within the soil. Sketch the distribution of vertical effective stress and indicate key values. [10%]
- (b) What is the minimum force exerted by the soil on the retaining wall in the short term? Again suppose $H = 9 \text{ m}$ and $t = 5 \text{ m}$, as in part (a). [30%]
- (c) If $d = 7 \text{ m}$, what are the forces in the props corresponding to the force evaluated in part (b)? Would it be reasonable to use these values to design the props? Justify your answer. [35%]
- (d) Consider the general case where d is variable, and suppose that the sand layer extends to the base of the wall ($t = H$). Consider the possibility that the upper prop is accidentally removed. What are the possible states of limiting equilibrium? Without performing detailed calculations, sketch the corresponding distribution of horizontal stress in each case. Which case corresponds to the minimum thrust on the wall? Justify your answer. [25%]

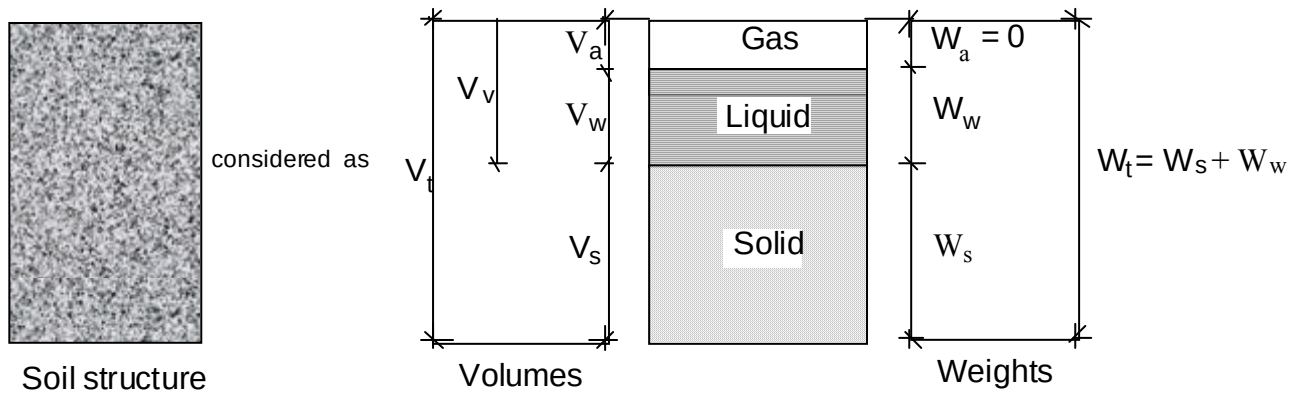
END OF PAPER

Engineering Tripos Part IIA

3D1 & 3D2 Geotechnical Engineering Data Book

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General definitions



Specific gravity of solid

$$G_s$$

Voids ratio

$$e = V_v / V_s$$

Specific volume

$$v = V_t / V_s = 1 + e$$

Porosity

$$n = V_v / V_t = e / (1 + e)$$

Water content

$$w = (W_w / W_s)$$

Degree of saturation

$$S_r = V_w / V_v = (w G_s / e)$$

Unit weight of water

$$\gamma_w = 9.81 \text{ kN/m}^3$$

Unit weight of soil

$$\gamma = W_t / V_t = \left(\frac{G_s + S_r e}{1 + e} \right) \gamma_w$$

Buoyant saturated unit weight

$$\gamma' = \gamma - \gamma_w = \left(\frac{G_s - 1}{1 + e} \right) \gamma_w$$

Unit weight of dry solids

$$\gamma_d = W_s / V_t = \left(\frac{G_s}{1 + e} \right) \gamma_w$$

Air volume ratio

$$A = V_a / V_t = \left(\frac{e(1 - S_r)}{1 + e} \right)$$

Soil classification (BS1377)

Liquid limit w_L

Plastic Limit w_P

Plasticity Index $I_P = w_L - w_P$

Liquidity Index $I_L = \frac{w - w_P}{w_L - w_P}$

Activity = $\frac{\text{Plasticity Index}}{\text{Percentage of particles finer than } 2 \mu\text{m}}$

Sensitivity = $\frac{\text{Unconfined compressive strength of an undisturbed specimen}}{\text{Unconfined compressive strength of a remoulded specimen}}$ (at the same water content)

Classification of particle sizes:–

Boulders	larger than	200 mm	
Cobbles	between	200 mm	and 60 mm
Gravel	between	60 mm	and 2 mm
Sand	between	2 mm	and 0.06 mm
Silt	between	0.06 mm	and 0.002 mm
Clay	smaller than	0.002 mm (two microns)	

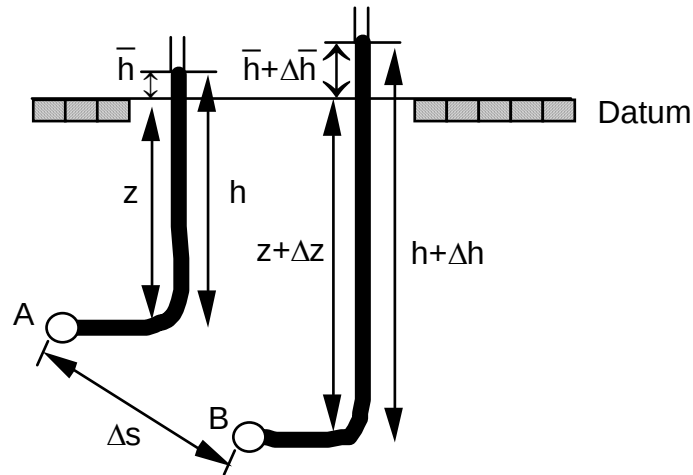
D equivalent diameter of soil particle

D_{10} , D_{60} etc. particle size such that 10% (or 60%) etc.) by weight of a soil sample is composed of finer grains.

C_U uniformity coefficient D_{60} / D_{10}

Seepage

Flow potential:
(piezometric level)



Total gauge pore water pressure at A: $u = \gamma_w h = \gamma_w (\bar{h} + z)$

$$\text{B: } u + \Delta u = \gamma_w (h + \Delta h) = \gamma_w (\bar{h} + z + \Delta \bar{h} + \Delta z)$$

Excess pore water pressure at A: $\bar{u} = \gamma_w \bar{h}$

$$\text{B: } \bar{u} + \Delta \bar{u} = \gamma_w (\bar{h} + \Delta \bar{h})$$

Hydraulic gradient A $\rightarrow$ B $i = - \frac{\Delta \bar{h}}{\Delta s}$

Hydraulic gradient (3D) $i = - \nabla \bar{h}$

Darcy's law $V = ki$

V = superficial seepage velocity

k = coefficient of permeability

Typical permeabilities:

$D_{10} > 10 \text{ mm}$	:	non-laminar flow
$10 \text{ mm} > D_{10} > 1 \mu\text{m}$	:	$k \cong 0.01 (D_{10} \text{ in mm})^2 \text{ m/s}$
clays	:	$k \cong 10^{-9} \text{ to } 10^{-11} \text{ m/s}$

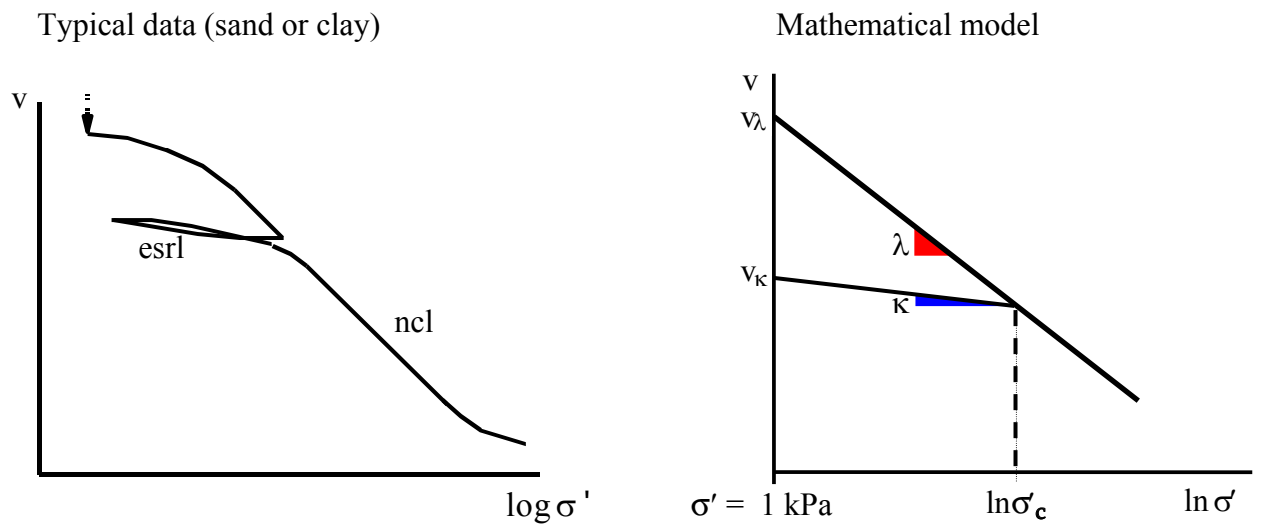
Saturated capillary zone

$$h_c = \frac{4T}{\gamma_w d} \quad : \quad \text{capillary rise in tube diameter } d, \text{ for surface tension } T$$

$$h_c \approx \frac{3 \times 10^{-5}}{D_{10}} \text{ m} \quad : \quad \text{for water at } 10^\circ\text{C}; \text{ note air entry suction is } u_c = - \gamma_w h_c$$

One-Dimensional Compression

• Fitting data



Plastic compression stress σ'_c is taken as the larger of the initial aggregate crushing stress and the historic maximum effective vertical stress. Clay muds are taken to begin with $\sigma'_c \approx 1$ kPa.

Plastic compression (normal compression line, ncl): $v = v_\lambda - \lambda \ln \sigma'$ for $\sigma' = \sigma'_c$

Elastic swelling and recompression line (esrl): $v = v_c + \kappa (\ln \sigma'_c - \ln \sigma'_v)$
 $= v_\kappa - \kappa \ln \sigma'_v$ for $\sigma' < \sigma'_c$

Equivalent parameters for $\log_{10}$ stress scale:

Terzaghi's compression index $C_c = \lambda \log_{10} e$

Terzaghi's swelling index $C_s = \kappa \log_{10} e$

• Deriving confined soil stiffnesses

Secant 1D compression modulus $E_o = (\Delta \sigma' / \Delta \epsilon)_o$

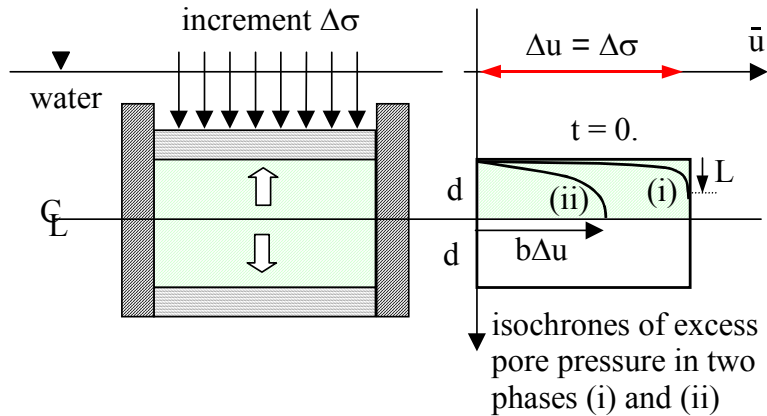
Tangent 1D plastic compression modulus $E_o = v \sigma' / \lambda$

Tangent 1D elastic compression modulus $E_o = v \sigma' / \kappa$

One-Dimensional Consolidation

Settlement	ρ	$= \int m_v (\Delta u - \bar{u}) dz$	$= \int (\Delta u - \bar{u}) / E_o dz$
Coefficient of consolidation	c_v	$= \frac{k}{m_v \gamma_w}$	$= \frac{kE_o}{\gamma_w}$
Dimensionless time factor	T_v	$= \frac{c_v t}{d^2}$	
Relative settlement	R_v	$= \frac{\rho}{\rho_{ult}}$	

• Solutions for initially rectangular distribution of excess pore pressure



Approximate solution by parabolic isochrones:

Phase (i) $L^2 = 12 c_v t$

$$R_v = \sqrt{\frac{4T_v}{3}} \quad \text{for } T_v < 1/12$$

Phase (ii) $b = \exp(1/4 - 3T_v)$

$$R_v = [1 - 2/3 \exp(1/4 - 3T_v)] \quad \text{for } T_v > 1/12$$

Solution by Fourier Series:

T_v	0	0.01	0.02	0.04	0.08	0.15	0.20	0.30	0.40	0.50	0.60	0.80	1.00
R_v	0	0.12	0.17	0.23	0.32	0.45	0.51	0.62	0.70	0.77	0.82	0.89	0.94

Stress and strain components

- **Principle of effective stress (saturated soil)**

$$\text{total stress } \sigma = \text{effective stress } \sigma' + \text{pore water pressure } u$$

- **Principal components of stress and strain**

sign convention	compression positive
total stress	$\sigma_1, \sigma_2, \sigma_3$
effective stress	$\sigma'_1, \sigma'_2, \sigma'_3$
strain	$\varepsilon_1, \varepsilon_2, \varepsilon_3$

- **Simple Shear Apparatus (SSA)** ($\varepsilon_2 = 0$; other principal directions unknown)

The only stresses that are readily available are the shear stress τ and normal stress σ applied to the top platen. The pore pressure u can be controlled and measured, so the normal effective stress σ' can be found. Drainage can be permitted or prevented. The shear strain γ and normal strain ε are measured with respect to the top platen, which is a plane of zero extension. Zero extension planes are often identified with slip surfaces.

$$\text{work increment per unit volume} \quad \delta W = \tau \delta \gamma + \sigma' \delta \varepsilon$$

- **Biaxial Apparatus - Plane Strain (BA-PS)** ($\varepsilon_2 = 0$; rectangular edges along principal axes)

Intermediate principal effective stress σ'_2 , in zero strain direction, is frequently unknown so that all conditions are related to components in the 1-3 plane.

mean total stress	$s = (\sigma_1 + \sigma_3)/2$
mean effective stress	$s' = (\sigma'_1 + \sigma'_3)/2 = s - u$
shear stress	$t = (\sigma'_1 - \sigma'_3)/2 = (\sigma_1 - \sigma_3)/2$
volumetric strain	$\varepsilon_v = \varepsilon_1 + \varepsilon_3$
shear strain	$\varepsilon_\gamma = \varepsilon_1 - \varepsilon_3$
work increment per unit volume	$\delta W = \sigma'_1 \delta \varepsilon_1 + \sigma'_3 \delta \varepsilon_3$
	$\delta W = s' \delta \varepsilon_v + t \delta \varepsilon_\gamma$

providing that principal axes of strain increment and of stress coincide.

• **Triaxial Apparatus – Axial Symmetry (TA-AS)** (cylindrical element with radial symmetry)

total axial stress	$\sigma_a = \sigma'_a + u$
total radial stress	$\sigma_r = \sigma'_r + u$
total mean normal stress	$p = (\sigma_a + 2\sigma_r)/3$
effective mean normal stress	$p' = (\sigma'_a + 2\sigma'_r)/3 = p - u$
deviatoric stress	$q = \sigma'_a - \sigma'_r = \sigma_a - \sigma_r$
stress ratio	$\eta = q/p'$
axial strain	ε_a
radial strain	ε_r
volumetric strain	$\varepsilon_v = \varepsilon_a + 2\varepsilon_r$
triaxial shear strain	$\varepsilon_s = \frac{2}{3}(\varepsilon_a - \varepsilon_r)$
work increment per unit volume	$\delta W = \sigma'_a \delta \varepsilon_a + 2\sigma'_r \delta \varepsilon_r$
	$\delta W = p' \delta \varepsilon_v + q \delta \varepsilon_s$

Types of triaxial test include:

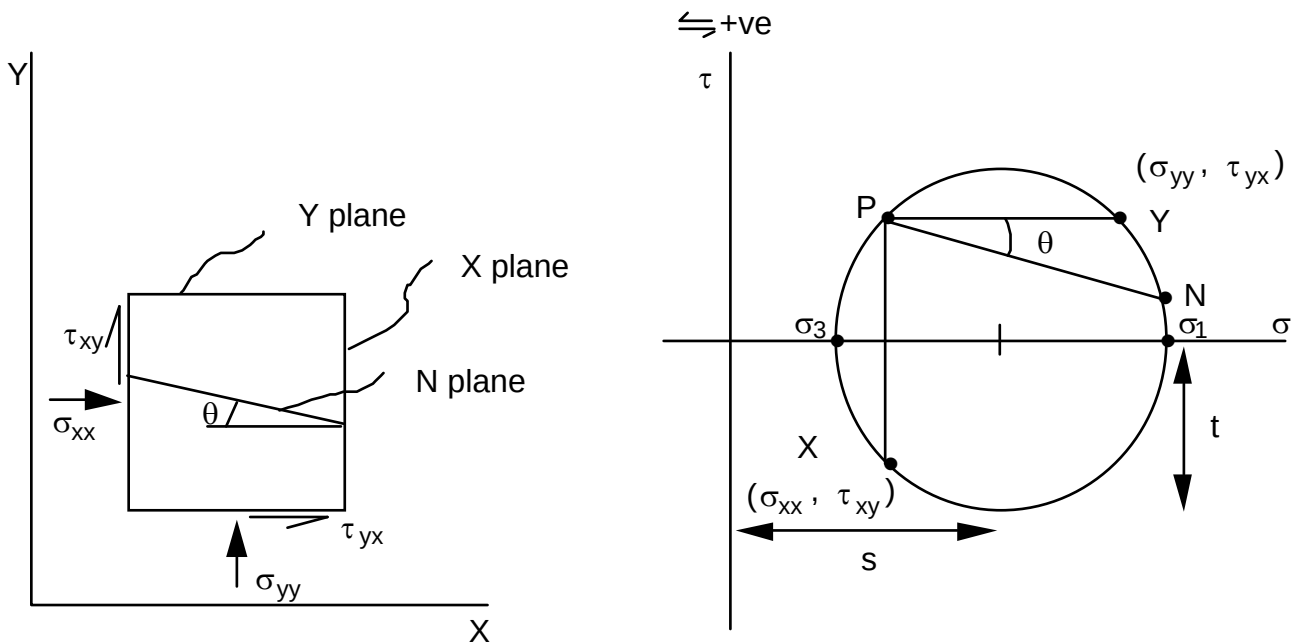
isotropic compression in which p' increases at zero q

triaxial compression in which q increases *either* by increasing σ_a *or* by reducing σ_r

triaxial extension in which q reduces *either* by reducing σ_a *or* by increasing σ_r

• **Mohr's circle of stress (1–3 plane)**

Sign of convention: compression, and counter-clockwise shear, positive



Poles of planes P : the components of stress on the N plane are given by the intersection N of the Mohr circle with the line PN through P parallel to the plane.

Elastic stiffness relations

These relations apply to tangent stiffnesses of over-consolidated soil, with a state point on some swelling and recompression line (κ -line), and remote from gross plastic yielding.

One-dimensional compression (axial stress and strain increments $d\sigma'$, $d\varepsilon$)

$$\text{compressibility} \quad m_v = d\varepsilon / d\sigma'$$

$$\text{constrained modulus} \quad E_o = 1 / m_v$$

Physically fundamental parameters

$$\text{shear modulus} \quad G' = dt / d\varepsilon_\gamma$$

$$\text{bulk modulus} \quad K' = dp' / d\varepsilon_v$$

Parameters which can be used for constant-volume deformations

$$\text{undrained shear modulus} \quad G_u = G'$$

$$\text{undrained bulk modulus} \quad K_u = \infty \quad (\text{neglecting compressibility of water})$$

Alternative convenient parameters

$$\text{Young's moduli} \quad E' \text{ (effective), } E_u \text{ (undrained)}$$

$$\text{Poisson's ratios} \quad \nu' \text{ (effective), } \nu_u = 0.5 \text{ (undrained)}$$

Typical value of Poisson's ratio for small changes of stress: $\nu' = 0.2$

$$\text{Relationships:} \quad G = \frac{E}{2(1 + \nu)}$$

$$K = \frac{E}{3(1 - 2\nu)}$$

$$E_o = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}$$

Cam Clay

• Interchangeable parameters for stress combinations at yield, and plastic strain increments

System	Effective normal stress	Plastic normal strain	Effective shear stress	Plastic shear strain	Critical stress ratio	Plastic normal stress	Critical normal stress
General	σ^*	ε^*	τ^*	γ^*	μ^*_{crit}	σ^*_c	σ^*_{crit}
SSA	σ'	ε	τ	γ	$\tan \phi_{\text{crit}}$	σ'_c	σ'_{crit}
BA-PS	s'	ε_v	t	ε_γ	$\sin \phi_{\text{crit}}$	s'_c	s'_{crit}
TA-AS	p'	ε_v	q	ε_s	M	p'_c	p'_{crit}

• General equations of plastic work

Plastic work and dissipation

$$\sigma^* \delta \varepsilon^* + \tau^* \delta \gamma^* = \mu^*_{\text{crit}} \sigma^* \delta \gamma^*$$

Plastic flow rule – normality

$$\frac{d\tau^*}{d\sigma^*} \cdot \frac{d\gamma^*}{d\varepsilon^*} = -1$$

• General yield surface

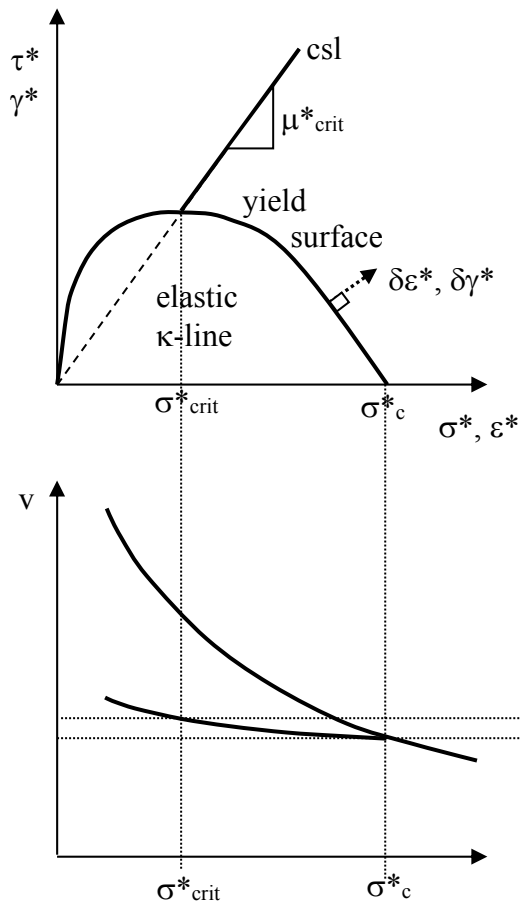
$$\frac{\tau^*}{\sigma^*} = \mu^* = \mu^*_{\text{crit}} \cdot \ln \left[\frac{\sigma^*_c}{\sigma^*} \right]$$

• Parameter values which fit soil data

	London Clay	Weald Clay	Kaolin	Dog's Bay Sand	Ham River Sand
λ^*	0.161	0.093	0.26	0.334	0.163
κ^*	0.062	0.035	0.05	0.009	0.015
Γ^* at 1 kPa	2.759	2.060	3.767	4.360	3.026
$\sigma^*_{c, \text{ virgin}}$ kPa	1	1	1	Loose 500 Dense 1500	Loose 2500 Dense 15000
ϕ_{crit}	23°	24°	26°	39°	32°
M_{comp}	0.89	0.95	1.02	1.60	1.29
M_{extn}	0.69	0.72	0.76	1.04	0.90
w_L	0.78	0.43	0.74	-----	-----
w_P	0.26	0.18	0.42	-----	-----
G_s	2.75	2.75	2.61	2.75	2.65

Note: 1) parameters λ^* , κ^* , Γ^* , $\sigma^*_{c, \text{ virgin}}$ should depend to a small extent on the deformation mode, e.g. SSA, BA-PS, TA-AS, etc. This may be neglected unless further information is given.
2) Sand which is loose, or loaded cyclically, compacts more than Cam Clay allows.

• The yield surface in (σ^*, τ^*, v) space



ncl: normal compression line

$$v = N - \lambda \ln \sigma^*$$

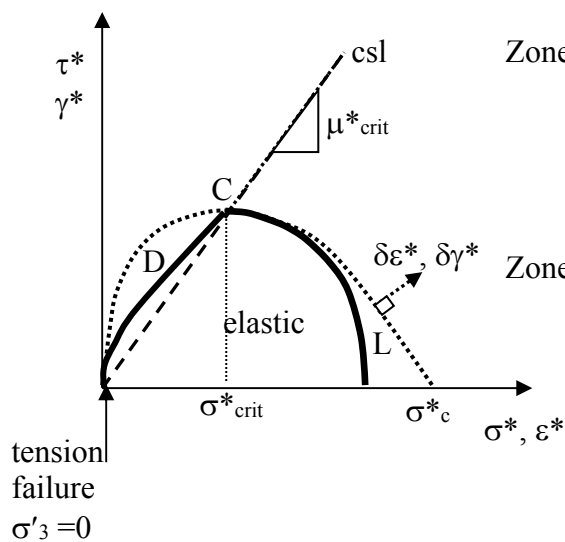
csl: critical state line

$$v = \Gamma - \lambda \ln \sigma^*$$

where $N = \Gamma + \lambda - \kappa$

• Regions of limiting soil behaviour

Variation of Cam Clay yield surface

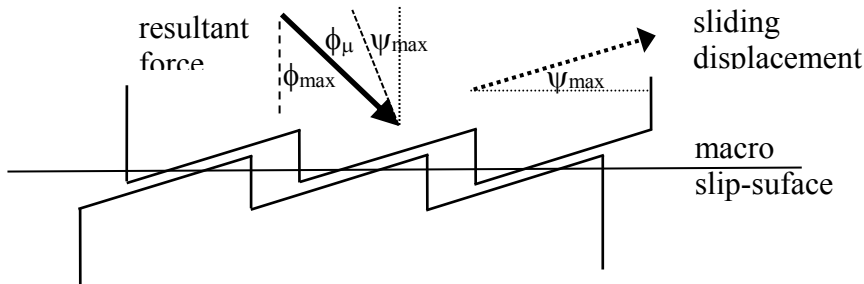


Zone D: denser than critical, "dry",
dilation or negative excess pore pressures,
Hvorslev strength envelope,
friction-dilatancy theory,
unstable shear rupture, progressive failure

Zone L: looser than critical, "wet",
compaction or positive excess pore pressures,
Modified Cam Clay yield surface,
stable strain-hardening continuum

Strength of soil: friction and dilation

• Friction and dilatancy: the saw-blade model of direct shear

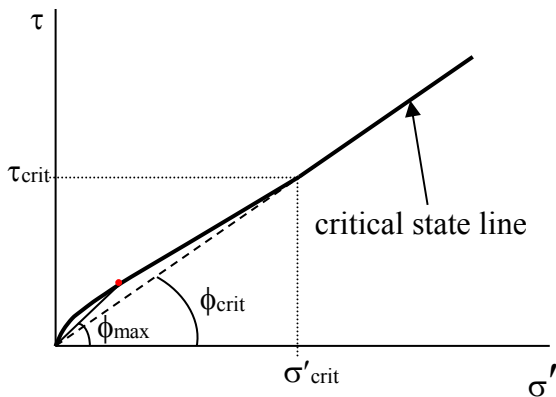


Intergranular angle of friction at sliding contacts ϕ_{μ}

Angle of dilation ψ_{max}

Angle of internal friction $\phi_{max} = \phi_{\mu} + \psi_{max}$

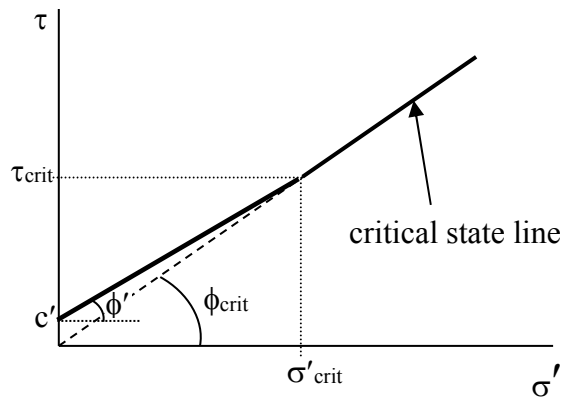
• Friction and dilatancy: secant and tangent strength parameters



Secant angle of internal friction

$$\begin{aligned}\tau &= \sigma' \tan \phi_{max} \\ \phi_{max} &= \phi_{crit} + \Delta\phi \\ \Delta\phi &= f(\sigma'_{crit}/\sigma')\end{aligned}$$

typical envelope fitting data:
power curve
 $(\tau/\tau_{crit}) = (\sigma'/\sigma'_{crit})^\alpha$
with $\alpha \approx 0.85$



Tangent angle of shearing envelope

$$\begin{aligned}\tau &= c' + \sigma' \tan \phi' \\ c' &= f(\sigma'_{crit})\end{aligned}$$

typical envelope:
straight line
 $\tan \phi' = 0.85 \tan \phi_{crit}$
 $c' = 0.15 \tau_{crit}$

• Friction and dilation: data of sands

The inter-granular friction angle of quartz grains, $\phi_\mu \approx 26^\circ$. Turbulent shearing at a critical state causes ϕ_{crit} to exceed this. The critical state angle of internal friction ϕ_{crit} is a function of the uniformity of particle sizes, their shape, and mineralogy, and is developed at large shear strains irrespective of initial conditions. Typical values of ϕ_{crit} ($\pm 2^\circ$) are:

well-graded, angular quartz or feldspar sands	40°
uniform sub-angular quartz sand	36°
uniform rounded quartz sand	32°

Relative density $I_D = \frac{(e_{\text{max}} - e)}{(e_{\text{max}} - e_{\text{min}})}$ where:

e_{max} is the maximum void ratio achievable in quick-tilt test

e_{min} is the minimum void ratio achievable by vibratory compaction

Relative crushability $I_C = \ln(\sigma_c / p')$ where:

σ_c is the aggregate crushing stress, taken to be a material constant, typical values being: 80 000 kPa for quartz silt, 20 000 kPa for quartz sand, 5 000 kPa for carbonate sand.

p' is the mean effective stress at failure which may be taken as approximately equal to the effective stress σ' normal to a shear plane.

Dilatancy contribution to the peak angle of internal friction is $\Delta\phi = (\phi_{\text{max}} - \phi_{\text{crit}}) = f(I_R)$

Relative dilatancy index $I_R = I_D I_C - 1$ where:

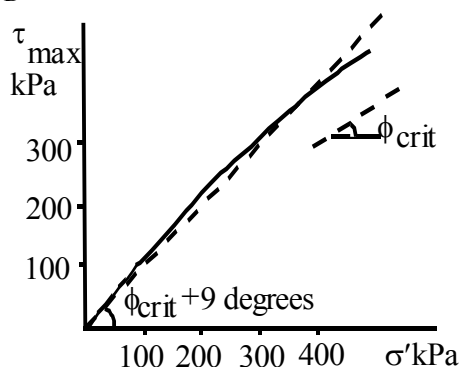
$I_R < 0$ indicates compaction, so that I_D increases and $I_R \rightarrow 0$ ultimately at a critical state

$I_R > 4$ to be limited to $I_R = 4$ unless corroborative dilatant strength data is available

The following empirical correlations are then available

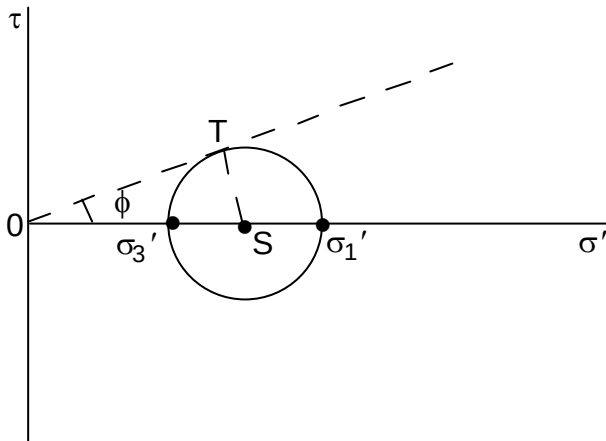
plane strain conditions	$(\phi_{\text{max}} - \phi_{\text{crit}})$	$= 0.8 \psi_{\text{max}}$	$= 5 I_R$ degrees
triaxial strain conditions	$(\phi_{\text{max}} - \phi_{\text{crit}})$	$= 3 I_R$ degrees	
all conditions	$(-\delta\varepsilon_v / \delta\varepsilon_1)_{\text{max}}$	$= 0.3 I_R$	

The resulting peak strength envelope for triaxial tests on a quartz sand at an initial relative density $I_D = 1$ is shown below for the limited stress range 10 - 400 kPa:



$$\phi_{\text{max}} > \phi_{\text{crit}} + 9^\circ \quad \text{for } I_D = 1, \sigma' = 400 \text{ kPa}$$

• Mobilised (secant) angle of shearing ϕ in the 1 – 3 plane



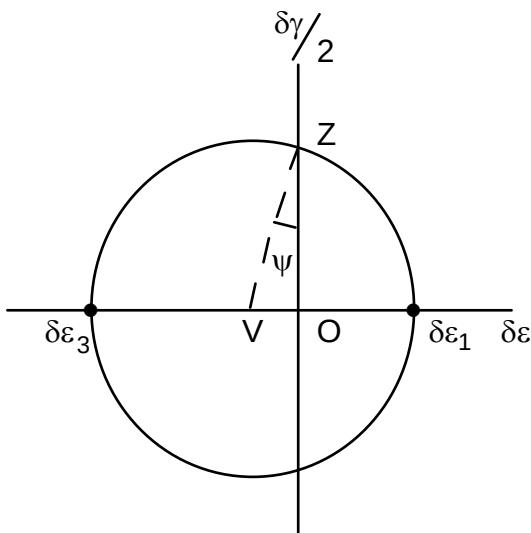
$$\begin{aligned}\sin \phi &= TS/OS \\ &= \frac{(\sigma'_1 - \sigma'_3)/2}{(\sigma'_1 + \sigma'_3)/2} \\ \left[\frac{\sigma'_1}{\sigma'_3} \right] &= \frac{(1 + \sin \phi)}{(1 - \sin \phi)}\end{aligned}$$

Angle of shearing resistance:

at peak strength $\phi_{\max}$ at $\left[\frac{\sigma'_1}{\sigma'_3} \right]_{\max}$

at critical state ϕ_{crit} after large shear strains

• Mobilised angle of dilation in plane strain ψ in the 1 – 3 plane



$$\begin{aligned}\sin \psi &= VO/VZ \\ &= -\frac{(\delta \epsilon_1 + \delta \epsilon_3)/2}{(\delta \epsilon_1 - \delta \epsilon_3)/2} \\ &= -\frac{\delta \epsilon_v}{\delta \epsilon_\gamma}\end{aligned}$$

$$\left[\frac{\delta \epsilon_1}{\delta \epsilon_3} \right] = -\frac{(1 - \sin \psi)}{(1 + \sin \psi)}$$

at peak strength $\psi = \psi_{\max}$ at $\left[\frac{\sigma'_1}{\sigma'_3} \right]_{\max}$

at critical state $\psi = 0$ since volume is constant

Plasticity: Cohesive material $\tau_{\max} = c_u$ (or s_u)

• Limiting stresses

Tresca $|\sigma_1 - \sigma_3| = q_u = 2c_u$

von Mises $(\sigma_1 - p)^2 + (\sigma_2 - p)^2 + (\sigma_3 - p)^2 = \frac{2}{3} q_u^2 = 2c_u^2$

where q_u is the undrained triaxial compression strength, and c_u is the undrained plane shear strength.

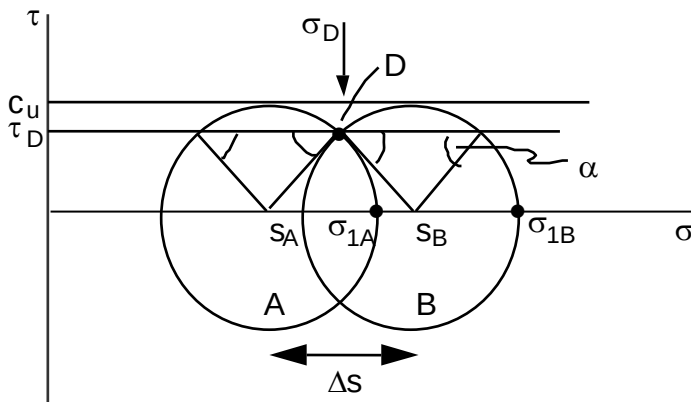
Dissipation per unit volume in plane strain deformation following either Tresca or von Mises,

$$\delta D = c_u \delta \varepsilon_\gamma$$

For a relative displacement x across a slip surface of area A mobilising shear strength c_u , this becomes

$$D = A c_u x$$

• Stress conditions across a discontinuity



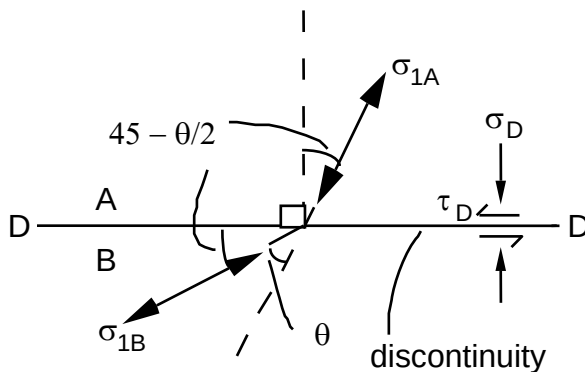
Rotation of major principal stress θ

$$s_B - s_A = \Delta s = 2c_u \sin \theta$$

$$\sigma_{1B} - \sigma_{1A} = 2c_u \sin \theta$$

In limit with $\theta \rightarrow 0$

$$ds = 2c_u d\theta$$



Useful example:

$$\theta = 30^\circ$$

$$\sigma_{1B} - \sigma_{1A} = c_u$$

$$\tau_D / c_u = 0.87$$

σ_{1A} = major principal stress in zone A

σ_{1B} = major principal stress in zone B

Plasticity: Frictional material $(\tau/\sigma')_{\max} = \tan \phi$

• Limiting stresses

$$\sin \phi = (\sigma'_{1f} - \sigma'_{3f}) / (\sigma'_{1f} + \sigma'_{3f}) = (\sigma_{1f} - \sigma_{3f}) / (\sigma_{1f} + \sigma_{3f} - 2u_s)$$

where σ'_{1f} and σ'_{3f} are the major and minor principal effective stresses at failure, σ_{1f} and σ_{3f} are the major and minor principle total stresses at failure, and u_s is the steady state pore pressure.

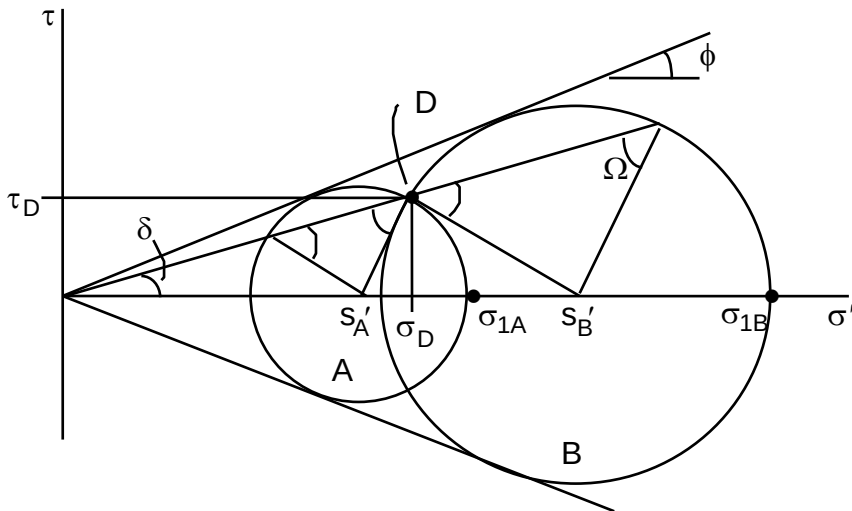
Active pressure:

$$\begin{aligned} \sigma'_v &> \sigma'_h \\ \sigma'_1 &= \sigma'_v \text{ (assuming principal stresses are horizontal and vertical)} \\ \sigma'_3 &= \sigma'_h \\ K_a &= (1 - \sin \phi) / (1 + \sin \phi) \end{aligned}$$

Passive pressure:

$$\begin{aligned} \sigma'_h &> \sigma'_v \\ \sigma'_1 &= \sigma'_h \text{ (assuming principal stresses are horizontal and vertical)} \\ \sigma'_3 &= \sigma'_v \\ K_p &= (1 + \sin \phi) / (1 - \sin \phi) = 1 / K_a \end{aligned}$$

• Stress conditions across a discontinuity



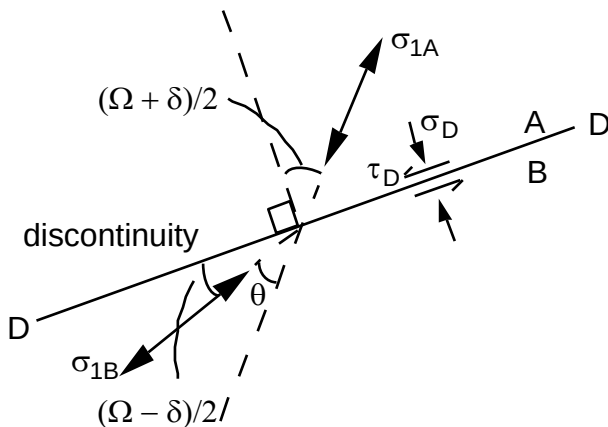
Rotation of major principal stress

$$\theta = \pi/2 - \Omega$$

σ_{1A} = major principal stress in zone A

σ_{1B} = major principal stress in zone B

$$\tan \delta = \tau_D / \sigma'_D$$



$$\sin \Omega = \sin \delta / \sin \phi$$

$$s'_B / s'_A = \sin(\Omega + \delta) / \sin(\Omega - \delta)$$

In limit, $d\theta \rightarrow 0$ and $\delta \rightarrow \phi$

$$ds' = 2s' \cdot d\theta \tan \phi$$

Integration gives $s'_B / s'_A = \exp(2\theta \tan \phi)$

Empirical earth pressure coefficients following one-dimensional strain

Coefficient of earth pressure in 1D plastic compression (normal compression)

$$K_{o,nc} = 1 - \sin \phi_{crit}$$

Coefficient of earth pressure during a 1D unloading-reloading cycle (overconsolidated soil)

$$K_o = K_{o,nc} \left[1 + \frac{(n-1)(n_{max}^\alpha - 1)}{(n_{max} - 1)} \right]$$

where n is current overconsolidation ratio (OCR) defined as $\sigma'_{v,max} / \sigma'_v$

n_{max} is maximum historic OCR defined as $\sigma'_{v,max} / \sigma'_{v,min}$

α is to be taken as $1.2 \sin \phi_{crit}$

Cylindrical cavity expansion

Expansion $\delta A = A - A_o$ caused by increase of pressure $\delta \sigma_c = \sigma_c - \sigma_o$

At radius r : small displacement $\rho = \frac{\delta A}{2\pi r}$

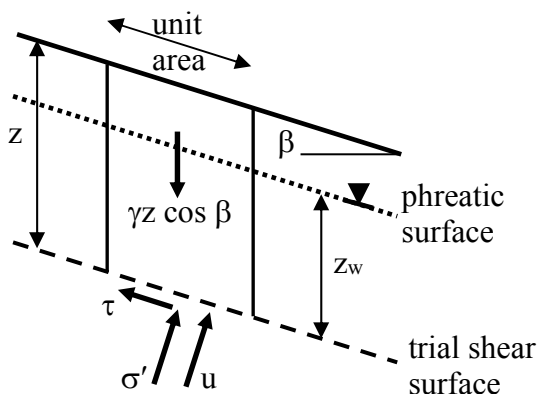
small shear strain $\gamma = \frac{2\rho}{r}$

Radial equilibrium: $r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0$

Elastic expansion (small strains) $\delta \sigma_c = G \frac{\delta A}{A}$

Undrained plastic-elastic expansion $\delta \sigma_c = s_u \left[1 + \ln \frac{G}{s_u} + \ln \frac{\delta A}{A} \right]$

Infinite slope analysis



$$\begin{aligned} u &= \gamma_w z_w \cos^2 \beta \\ \sigma &= \gamma z \cos^2 \beta \\ \sigma' &= (\gamma z - \gamma_w z_w) \cos^2 \beta \\ \tau &= \gamma z \cos \beta \sin \beta \end{aligned}$$

$$\tan \phi_{mob} = \frac{\tau}{\sigma'} = \frac{\tan \beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}$$

Shallow foundation design

Tresca soil, with undrained strength s_u

Vertical loading

The vertical bearing capacity, q_f , of a shallow foundation for undrained loading (Tresca soil) is:

$$\frac{V_{ult}}{A} = q_f = s_c d_c N_c s_u + \gamma h$$

V_{ult} and A are the ultimate vertical load and the foundation area, respectively. h is the embedment of the foundation base and γ (or γ') is the appropriate density of the overburden.

The exact bearing capacity factor N_c for a plane strain surface foundation (zero embedment) on uniform soil is:

$$N_c = 2 + \pi \quad (\text{Prandtl, 1921})$$

Shape correction factor:

For a rectangular footing of length L and breadth B (Eurocode 7):

$$s_c = 1 + 0.2 B / L$$

The exact solution for a rough circular foundation ($D = B = L$) is $q_f = 6.05 s_u$, hence $s_c = 1.18 \sim 1.2$.

Embedment correction factor:

A fit to Skempton's (1951) embedment correction factors, for an embedment of h , is:

$$d_c = 1 + 0.33 \tan^{-1} (h/B) \quad (\text{or } h/D \text{ for a circular foundation})$$

Combined V-H loading

A curve fit to Green's lower bound plasticity solution for V-H loading is:

$$\text{If } V/V_{ult} > 0.5: \quad \frac{V}{V_{ult}} = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{H}{H_{ult}}} \quad \text{or} \quad \frac{H}{H_{ult}} = 1 - \left(2 \frac{V}{V_{ult}} - 1 \right)^2$$

$$\text{If } V/V_{ult} < 0.5: \quad H = H_{ult} = B s_u$$

Combined V-H-M loading

With lift-off: combined Green-Meyerhof

$$\text{Without lift-off:} \quad \left(\frac{V}{V_{ult}} \right)^2 + \left[\frac{M}{M_{ult}} \left(1 - 0.3 \frac{H}{H_{ult}} \right) \right]^2 + \left| \left(\frac{H}{H_{ult}} \right)^3 \right| - 1 = 0 \quad (\text{Taiebet \& Carter 2000})$$

Frictional (Coulomb) soil, with friction angle ϕ

Vertical loading

The vertical bearing capacity, q_f , of a shallow foundation under drained loading (Coulomb soil) is:

$$\frac{V_{ult}}{A} = q_f = s_q N_q \sigma'_{v0} + s_\gamma N_\gamma \frac{\gamma' B}{2}$$

The bearing capacity factors N_q and N_γ account for the capacity arising from surcharge and self-weight of the foundation soil respectively. σ'_{v0} is the in situ effective stress acting at the level of the foundation base.

For a strip footing on weightless soil, the exact solution for N_q is:

$$N_q = \tan^2(\pi/4 + \phi/2) e^{(\pi \tan \phi)} \quad (\text{Prandtl 1921})$$

An empirical relationship to estimate N_γ from N_q is (Eurocode 7):

$$N_\gamma = 2 (N_q - 1) \tan \phi$$

Curve fits to exact solutions for $N_\gamma = f(\phi)$ are (Davis & Booker 1971):

$$\text{Rough base:} \quad N_\gamma = 0.1054 e^{9.6\phi}$$

$$\text{Smooth base:} \quad N_\gamma = 0.0663 e^{9.3\phi}$$

Shape correction factors:

For a rectangular footing of length L and breadth B (Eurocode 7):

$$s_q = 1 + (B \sin \phi) / L$$

$$s_\gamma = 1 - 0.3 B / L$$

For circular footings take $L = B$.

Combined V-H loading

The Green/Sokolovski lower bound solution gives a V-H failure surface.

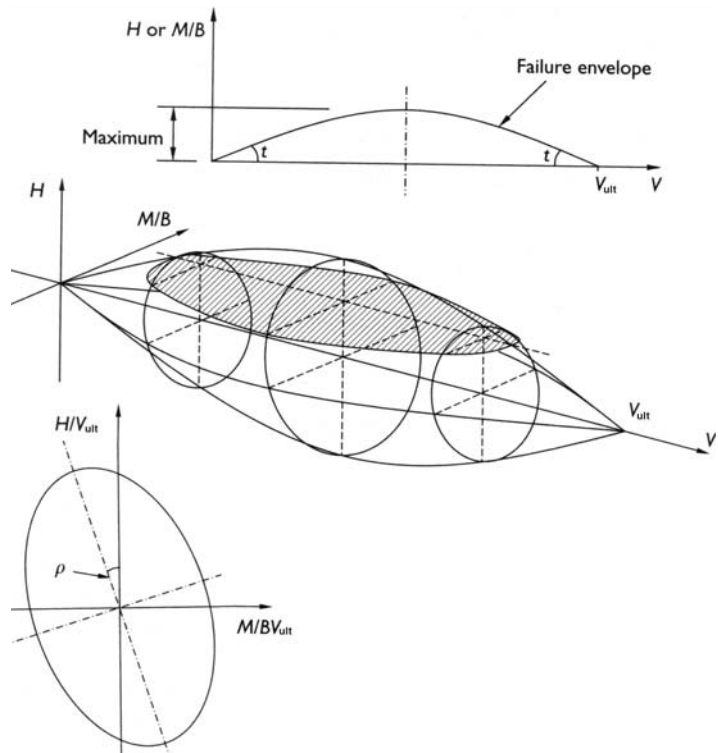
Combined V-H-M loading

With lift-off- drained conditions - use Butterfield & Gottardi (1994) failure surface shown above

$$\left[\frac{H/V_{ult}}{t_h} \right]^2 + \left[\frac{M/BV_{ult}}{t_m} \right]^2 + \left[\frac{2C(M/BV_{ult})(H/V_{ult})}{t_h t_m} \right] = \left[\frac{V}{V_{ult}} \left(1 - \frac{V}{V_{ult}} \right) \right]^2$$

where $C = \tan \left(\frac{2\rho(t_h - t_m)(t_h + t_m)}{2t_h t_m} \right)$ (Butterfield & Gottardi, 1994)

Typically, $t_h \sim 0.5$, $t_m \sim 0.4$ and $\rho \sim 15^\circ$. Note that t_h is the friction coefficient, $H/V = \tan \phi$, during sliding.



Settlement of Shallow Foundations

Elastic stress distributions below point, strip and circular loads

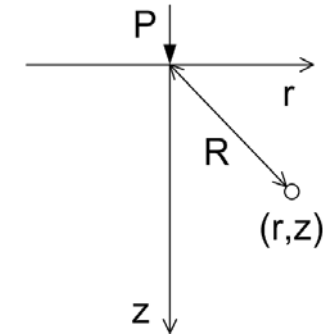
Point loading (Boussinesq solution)

Vertical stress $\sigma_z = \frac{3Pz^3}{2\pi R^5}$

Radial stress $\sigma_r = \frac{P}{2\pi R^2} \left[\frac{3r^2z}{R^3} - \frac{(1-2\nu)R}{R+z} \right]$

Tangential stress $\sigma_\theta = \frac{P(1-2\nu)}{2\pi R^2} \left[\frac{R}{R+z} - \frac{z}{R} \right]$

Shear stress $\tau_{rz} = \frac{3Prz^2}{2\pi R^5}$



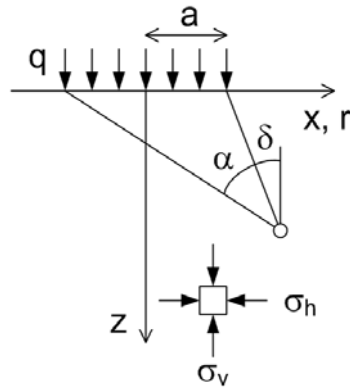
Uniformly-loaded strip

Vertical stress $\sigma_v = \frac{q}{\pi} [\alpha + \sin \alpha \cos(\alpha + 2\delta)]$

Horizontal stress $\sigma_h = \frac{q}{\pi} [\alpha - \sin \alpha \cos(\alpha + 2\delta)]$

Shear stress $\tau_{vh} = \frac{q}{\pi} \sin \alpha \sin(\alpha + 2\delta)$

Principal stresses $\sigma_1 = \frac{q}{\pi} (\alpha + \sin \alpha) \quad \sigma_3 = \frac{q}{\pi} (\alpha - \sin \alpha)$

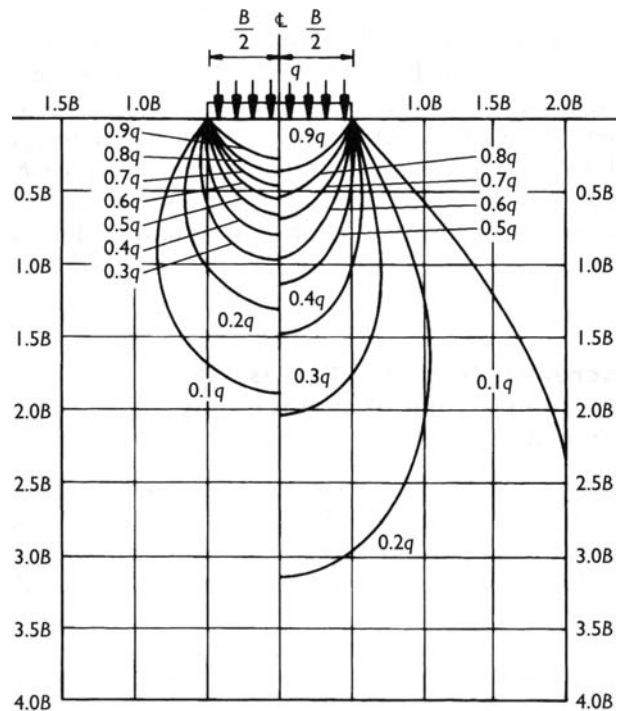


Uniformly-loaded circle radius a (on centerline, r=0)

Vertical stress $\sigma_v = q \left[1 - \left(\frac{1}{1 + (a/z)^2} \right)^{\frac{3}{2}} \right]$

Horizontal stress

$\sigma_h = \frac{q}{2} \left[(1 + 2\nu) - \frac{2(1 + \nu)z}{(a^2 + z^2)^{1/2}} + \frac{z^3}{(a^2 + z^2)^{3/2}} \right]$



Contours of vertical stress below uniformly-loaded circular (left) and strip footings (right)

