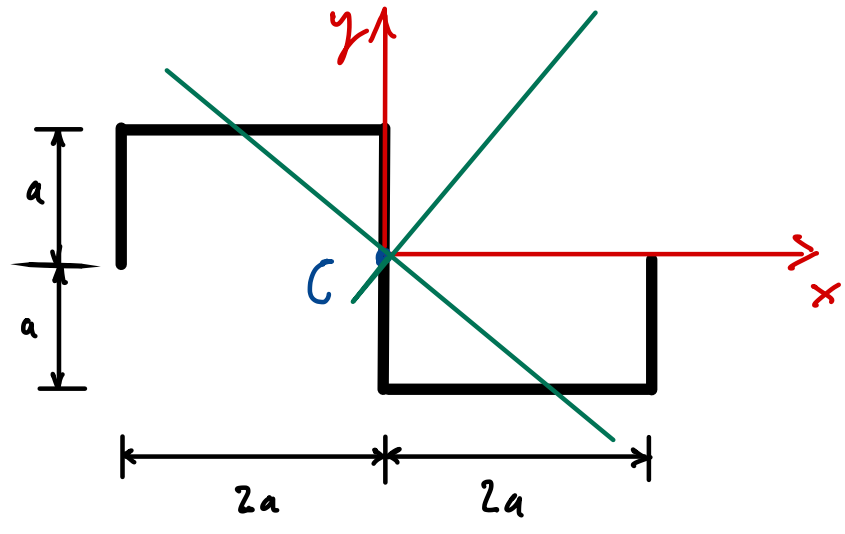
Question 1

a)



i) $I_{xx} = 2 \cdot 2at \cdot a^2 + 2\,\frac{(2a)^3 t}{12} = \frac{16}{3} a^3 t$

$I_{yy} = 2 \cdot at\,(2a)^2 + 2 \cdot \frac{(2a)^3 t}{12}$

$+ 2 \cdot 2at \cdot a^2 = \frac{40}{3} a^3 t$

$I_{xy} = -2at \cdot 2a \cdot \frac{a}{2}$

$- 2 \cdot 2at\, a \cdot a = -6\, a^3 t$

Mohrs circle:

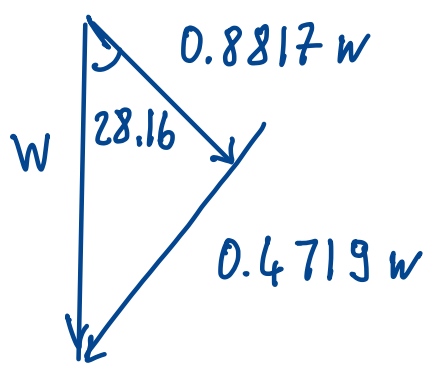
center: $\frac{1}{2}\left(\frac{16}{3} + \frac{40}{3}\right) a^3 t = \frac{28}{3}\, a^3 t$

radius: $\sqrt{(-6a^3 t)^2 + \left(\frac{40}{3} a^3 t - \frac{28}{3} a^3 t\right)^2} \approx 7.21\, a^3 t$

angle: $= 28.16°$

$\Rightarrow I_{\xi\xi} = 2.122\, a^3 t \qquad I_{\eta\eta} = 16.54\, a^3 t$

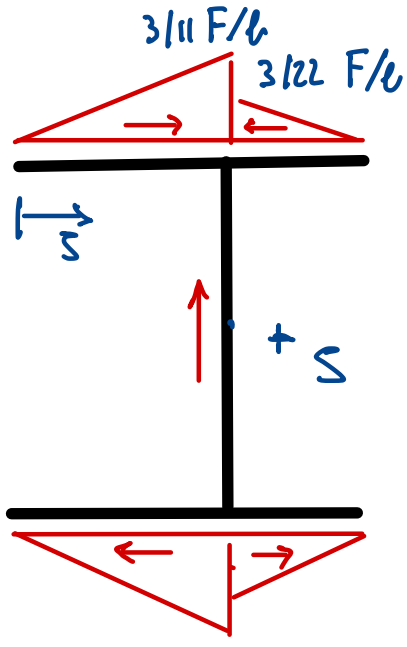
ii) $w = 8atg$



from Databook: $\frac{5wl^4}{384EI} \approx 0.0130\, \frac{wl^4}{EI}$

total deflection: $\delta = \sqrt{0.00553^2 + 0.02316^2}\, \frac{gL^4}{Ea^2} = 0.02381\, \frac{gL^4}{Ea^2}$

b) $q = \frac{FS}{I}$



$$I = \frac{8b^3t}{12} + 2\cdot\frac{3}{2}bt\cdot b^2 = \frac{11}{3}b^3t$$

right of web:

$$S = \frac{b}{2}t\cdot b = \frac{b^2t}{2}$$

$$\Rightarrow q = \frac{Fb^2t\cdot 3}{2\cdot 11\cdot b^3t} = \frac{3}{22}\frac{F}{b}$$

left of web:

$$S = b\cdot t\cdot b = b^2t$$

$$\Rightarrow q = \frac{Fb^2t\,3}{11\,b^3t} = \frac{3}{11}\frac{F}{b}$$

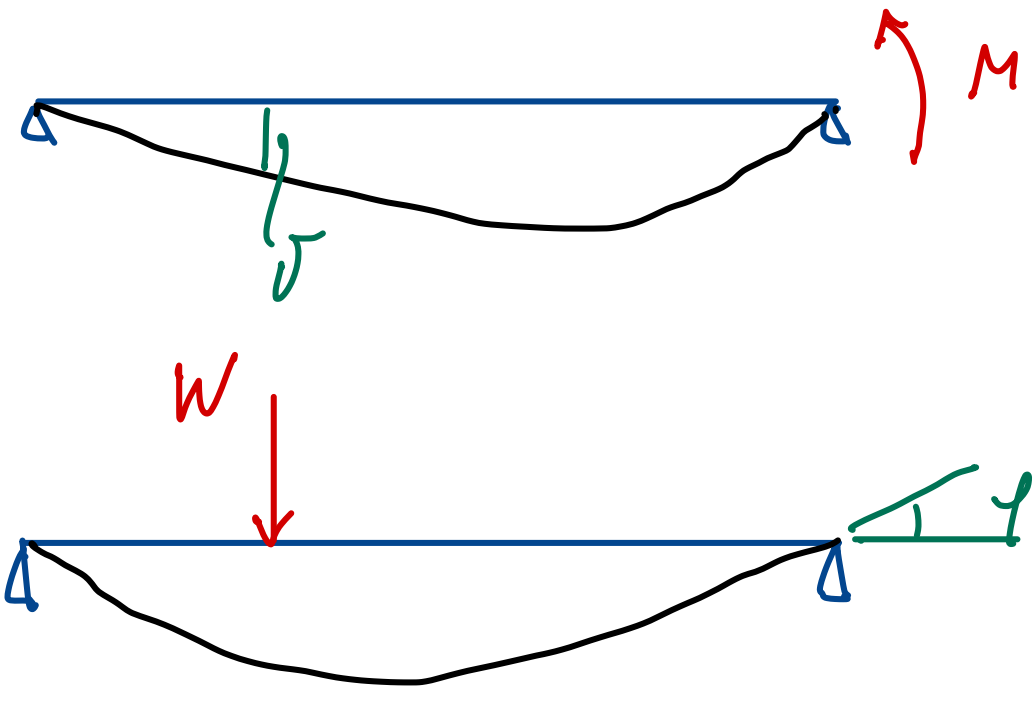
Moment equilibrium about centre of web:

$$\frac{1}{2}\left(\frac{3}{11\,b}Fb - \frac{3}{22b}F\frac{b}{2}\right)\cdot 2b = F\cdot x_s$$

$$\Rightarrow x_s = \frac{9}{44}b$$

Question 2:

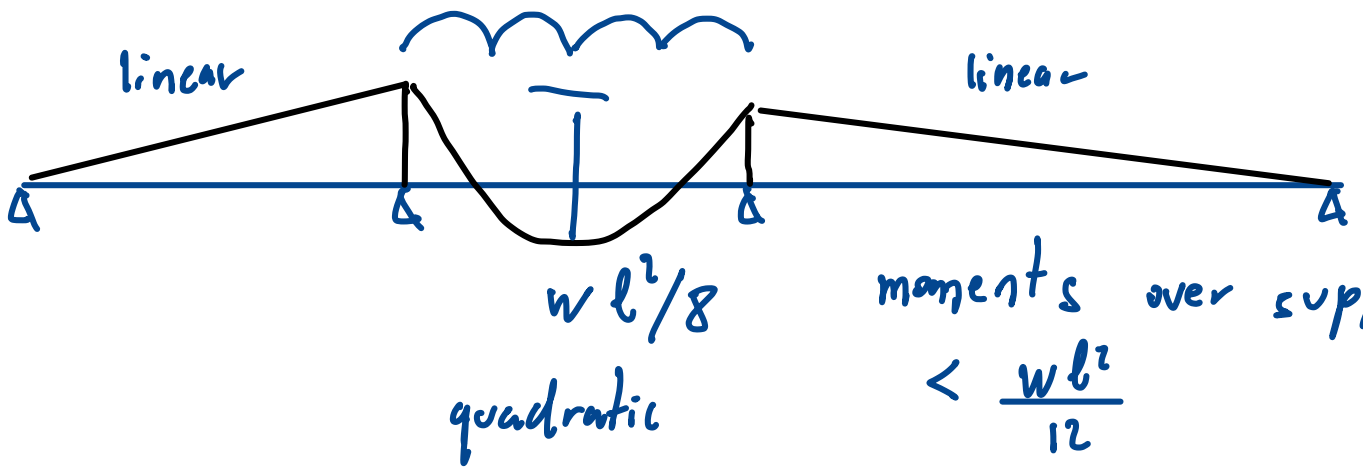
a)

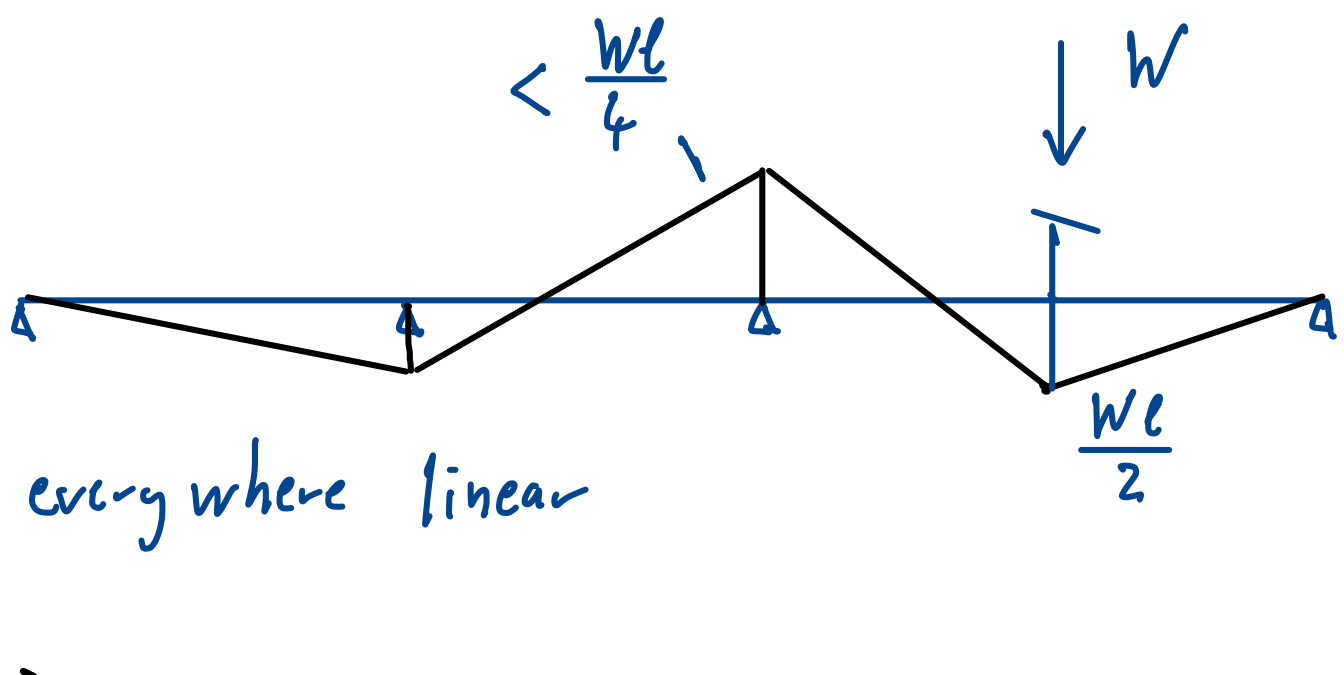


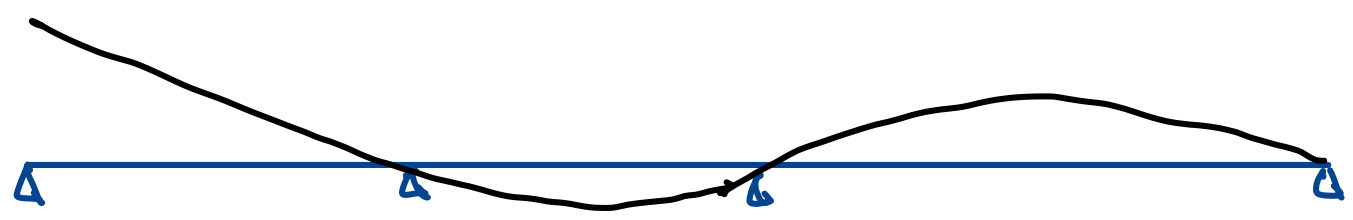
Work performed:

$M \cdot \delta = W \cdot \varphi$

b)







↓ support reaction due to w

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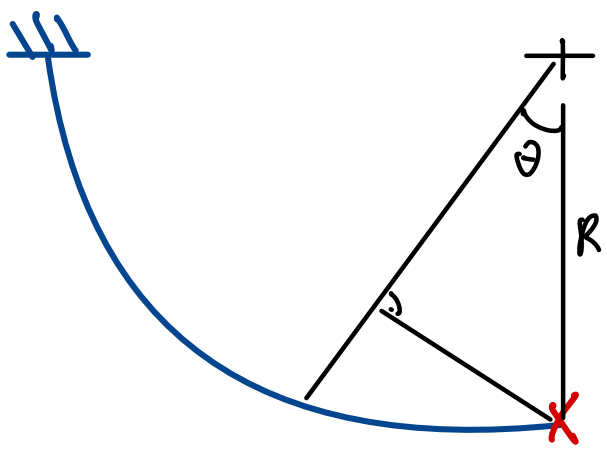
c) System statically indeterminate.



$$M = R \sin\theta$$

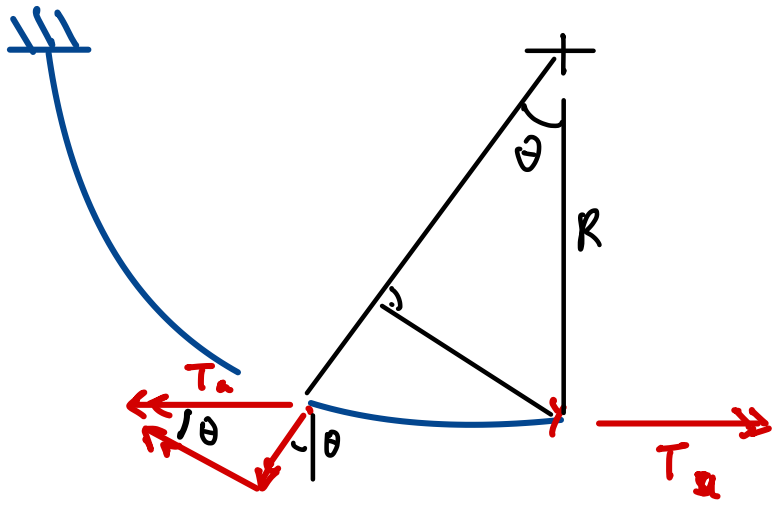
$$T = R(1-\cos\theta)$$

Tip displacement for system with unit load:

$$\delta_{11} = \int_0^{\pi/2} \frac{R^2}{EI} \sin^2\theta \, R\, d\theta + \int_0^{\pi/2} \frac{R^2}{GJ} (1-\cos\theta)^2 \, R\, d\theta$$

$$\frac{EI}{R^3}\delta_{11} = \int \frac{1}{2} - \frac{1}{2}\cos(2\theta) + \frac{1}{2}(1-2\cos\theta) + \frac{1}{4}(\cos 2\theta + 1)\, d\theta$$

$$\frac{EI}{R^3}\delta_{11} = \left[\frac{5}{4}\theta - \sin\theta - \frac{1}{8}\sin(2\theta)\right]_0^{\frac{\pi}{2}} = \frac{5\pi}{8} - 1 \approx 0.9635$$



$$M = T_a \sin\theta$$

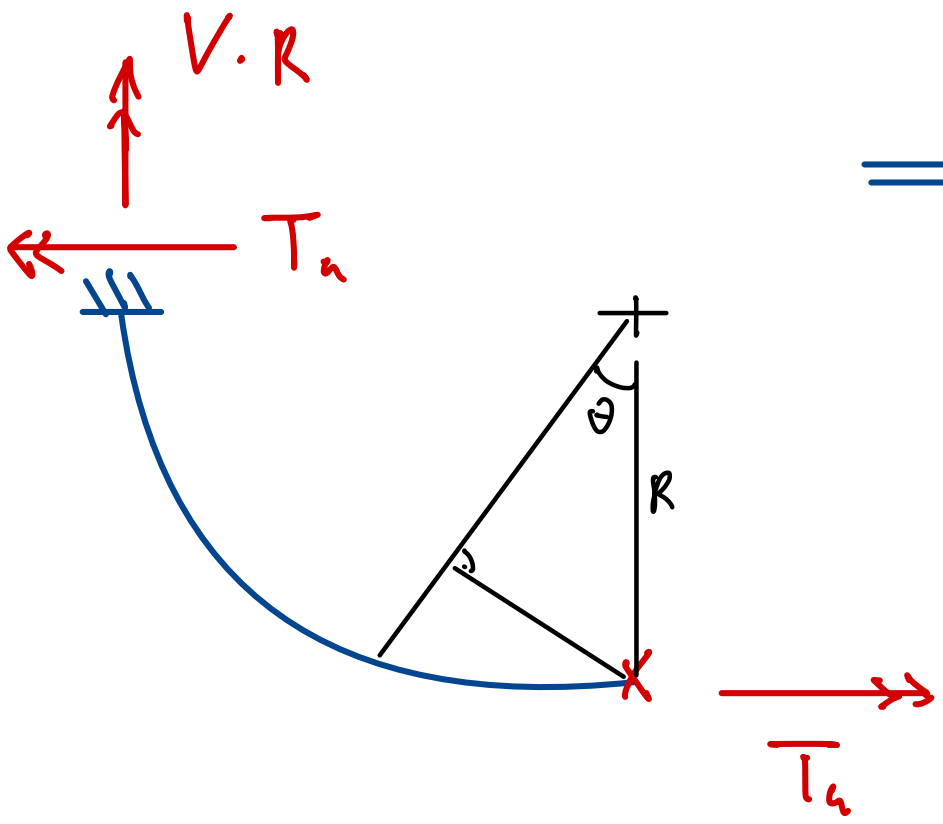
$$T = T_a \cos\theta$$

$$\delta_{10} = \int_0^{\pi/2} \frac{R}{EI} \sin^2\theta \, T_a \, R \, d\theta + \int_0^{\pi/2} \frac{R}{GJ} (1-\cos\theta) \, T_a \cos\theta \, R \, d\theta$$

$$\frac{EI}{T_a R^2}\delta_{10} = \int \frac{1}{2} - \frac{1}{2}\cos(2\theta) + \frac{1}{2}\left(\cos\theta - \frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \int \frac{1}{4} + \frac{1}{2}\cos\theta - \frac{3}{4}\cos 2\theta \, d\theta = \left[\frac{\theta}{4} + \frac{\sin\theta}{2} - \frac{3}{8}\sin(2\theta)\right]$$

$$= \frac{1}{2} + \frac{\pi}{8} \approx 0{,}8927$$

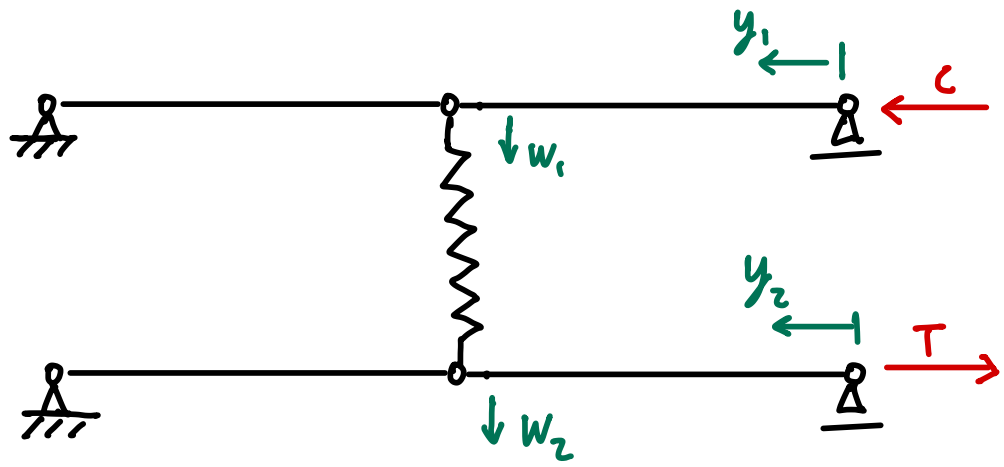


$\Longrightarrow$ support force

$$V = \frac{0{,}8927\,T_a}{0.9635R} \approx 0.9265\,\frac{T_a}{R}$$

Question 3:

a) i)



Total potential energy

$$\pi(w) = \frac{1}{2} k (w_2 - w_1)^2 - C y_1 + T y_2$$

$$y_1 = 2L - 2L\cos\theta_1 = 2L - 2L\left(1 - \frac{w_1^2}{2L^2} + \ldots\right) \approx \frac{w_1^2}{L}$$

Similarly $y_2 = \frac{w_2^2}{L}$ so that:

$$\pi(w) = \frac{1}{2} k (w_2 - w_1)^2 - C\frac{w_1^2}{L} + T\frac{w_2^2}{L}$$

$$= \frac{1}{2}\begin{pmatrix} w_1 & w_2 \end{pmatrix}\begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} w_1 & w_2 \end{pmatrix}\begin{pmatrix} -\frac{2C}{L} & 0 \\ 0 & \frac{2T}{L} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= \frac{1}{2} \underline{w}^T \underline{\underline{K}}\, \underline{w} \quad \text{with} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \underline{\underline{K}} = \begin{pmatrix} k - 2C/L & -k \\ -k & k + 2TL \end{pmatrix}$$

ii) Instability when $\det \underline{\underline{K}} = 0$

$$\left(k - \frac{2C}{L}\right)\left(k + \frac{2T}{L}\right) - k^2 = 0$$

when $T = \frac{kL}{2} \Rightarrow \left(k - \frac{2C}{L}\right)(2k) - k^2 = 0$

$$= 2k\left(k - \frac{2C}{L} - \frac{k}{2}\right) = 2k\left(\frac{k}{2} - \frac{2C}{L}\right)$$

so buckles when $k = 0$ or $C = \frac{kL}{4}$ $\left(= \frac{T}{2}\right)$

b) i)

Warping constraint still relevant away from ends because LTB implies different flange curvatures, hence different flange moments, hence different shears, hence restoring torque due to warping restraint.

ii) $$M_{cr} = \frac{\pi}{L}\sqrt{GJEI_{minor}}\left(1+\frac{\pi^2 E\Gamma}{L^2 GJ}\right)^{\frac{1}{2}}$$

$457\times191\times98$ UB $\qquad L = 8m$

$I_{minor} = 2347\times10^{-8}m^4 \qquad J = 121\times10^{-8}m^4$

$D = 467.2 - 19.6 = 447.6\ mm = 0.4476\ m$

$$\Gamma = \frac{ID^2}{4} = 2347\times10^{-8}\,\frac{0.4476^2}{4} = 117.6\times10^{-8}\ m^4$$

$$G = \frac{E}{2.6} \text{ for } \nu = 0.3$$

$$\Rightarrow \frac{\pi^2}{L^2}\frac{E}{G}\frac{\Gamma}{J} = \frac{\pi^2}{64}\,2.6\,\frac{117.6}{121} = 0.3895$$

Warping restraint factor: $\sqrt{1+0.3895} = 1.1788$

$$M_{basic} = \frac{\pi}{L}\sqrt{GJEI_{minor}} = \frac{\pi}{L}E\sqrt{\frac{JI_{minor}}{2.6}}$$

$$= \frac{\pi}{L}\left(210\times10^9\,\frac{N}{m^2}\right)\left(10^{-8}m^4\right)\sqrt{\frac{121\cdot2347}{2.6}}$$

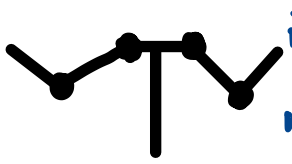
$$= 272.5\ kNm$$

$$M_{LTB} = 272.4\times1.1788 = 321.28\ kNm$$

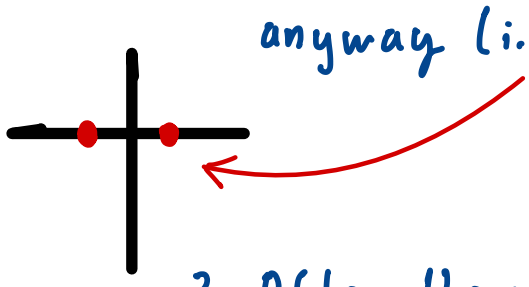
Question 4

a) Effective length of a column of length is the length of a perfect Euler strut which buckles at the same load.
So the buckling load of a column is $\frac{\pi^2 E I}{l_e^2}$

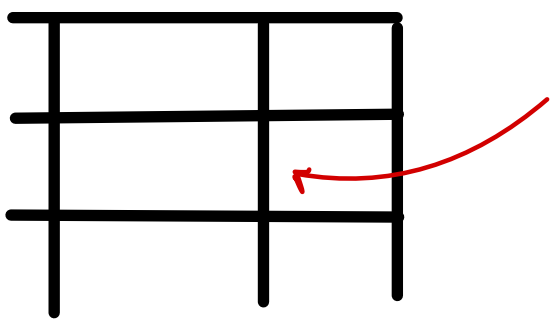
b) 1. Often as one approaches ultimate limit state, plastic hinges may form in the beams adjacent to the columns $\Rightarrow$ less ability to provide rotational stiffness.



2. Modern construction techniques tend to use simply-supported beams anyway (i.e. bolted connections $\approx$ pins)

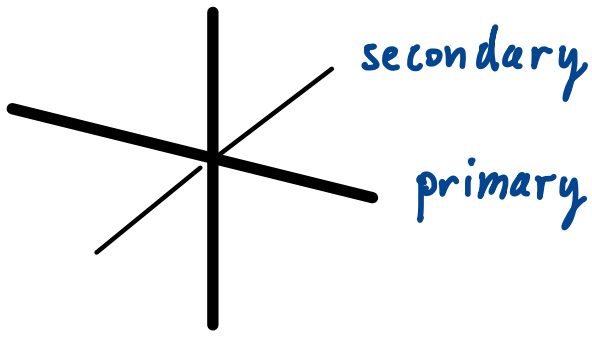


3. Often the main buckling (minor axis) is out-of-the-plane of the paper

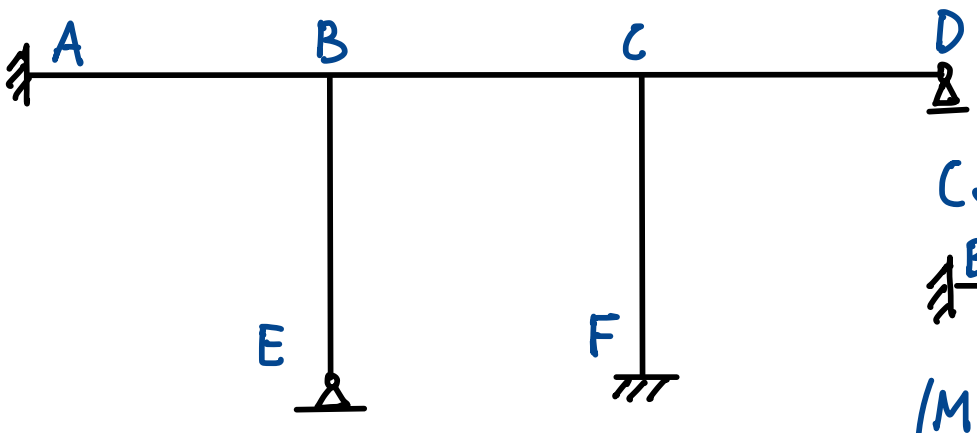


buckles out of plane

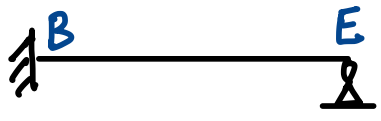
$\Rightarrow$ the beams framing are the small secondary beams out-of-the-plane



c) i)



Column BE:



$$\begin{pmatrix} M_{BE} \\ M_{EB} \end{pmatrix} = \frac{EI}{L} \begin{pmatrix} s & sc \\ sc & s \end{pmatrix} \begin{pmatrix} \theta_B \\ \theta_E \end{pmatrix}$$

But, $M_{EB} = 0 \Rightarrow sc\,\theta_B + s\,\theta_E = 0 \Rightarrow \theta_E = -c\,\theta_B$

$$\Rightarrow M_{BE} = \frac{EI}{L}\left(s\theta_B + sc(-c\theta_B)\right) = \frac{EI}{L} s(1-c^2)\theta_B$$

Column CF:

$$\begin{pmatrix} M_{CF} \\ M_{FC} \end{pmatrix} = \frac{EI}{L} \begin{pmatrix} s & sc \\ sc & s \end{pmatrix} \begin{pmatrix} \theta_C \\ \theta_F \end{pmatrix} \quad \Rightarrow M_{CF} = \frac{EI}{L} s\theta_C$$

Beam BC:

$$\begin{pmatrix} M_{BC} \\ M_{CB} \end{pmatrix} = \frac{EI}{L} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \theta_B \\ \theta_C \end{pmatrix}$$

Beam BA:

$$M_{BA} = 4 \frac{EI}{L} \theta_B$$

Beam CD:

$$M_{CD} = 3 \frac{EI}{L} \theta_C$$

Combine the element stiffness matrices to obtain:

$$\begin{pmatrix} M_B \\ M_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{EI}{L} \begin{pmatrix} 8+s(1-c^2) & 2 \\ 2 & 7+s \end{pmatrix} \begin{pmatrix} \theta_B \\ \theta_C \end{pmatrix}$$

↖ external applied moment

ii) Instability when $\det K = 0$

$$\begin{vmatrix} 8+s(1-c^2) & 2 \\ 2 & 7+s \end{vmatrix} = 0 \quad \Rightarrow 52+8s+7s(1-c^2)+s^2(1-c^2)=0$$
$$52+15s-7sc^2+s^2(1-c^2)=0$$

P/P_E	$15s-7sc^2+s^2(1-c^2)$
1.0	+20
1.2	+6
1.6	−39.8
1.8	−100.6

Linear interpolation

$$P/P_E \approx (5 \cdot 1.6 + 1 \cdot 1.8)/6 = 1.633$$

$$P = 1.633 \frac{\pi^2 EI}{L^2} = 16.12 \frac{EI}{L^2}$$

iii)

$s = (5 \times 1.2240 + 0.7170)/6 = 1.1395$

$c = (5 \times 2.348 + 4.4969)/6 = 2.7785$

Buckling mode (i.e. eigenvector)

$8 + s(1 - c^2) = 0.3246$

$7 + s = 8.1395$

$$\begin{vmatrix} 0.3426 - \lambda & 2 \\ 2 & 8.1395 - \lambda \end{vmatrix} = 0$$

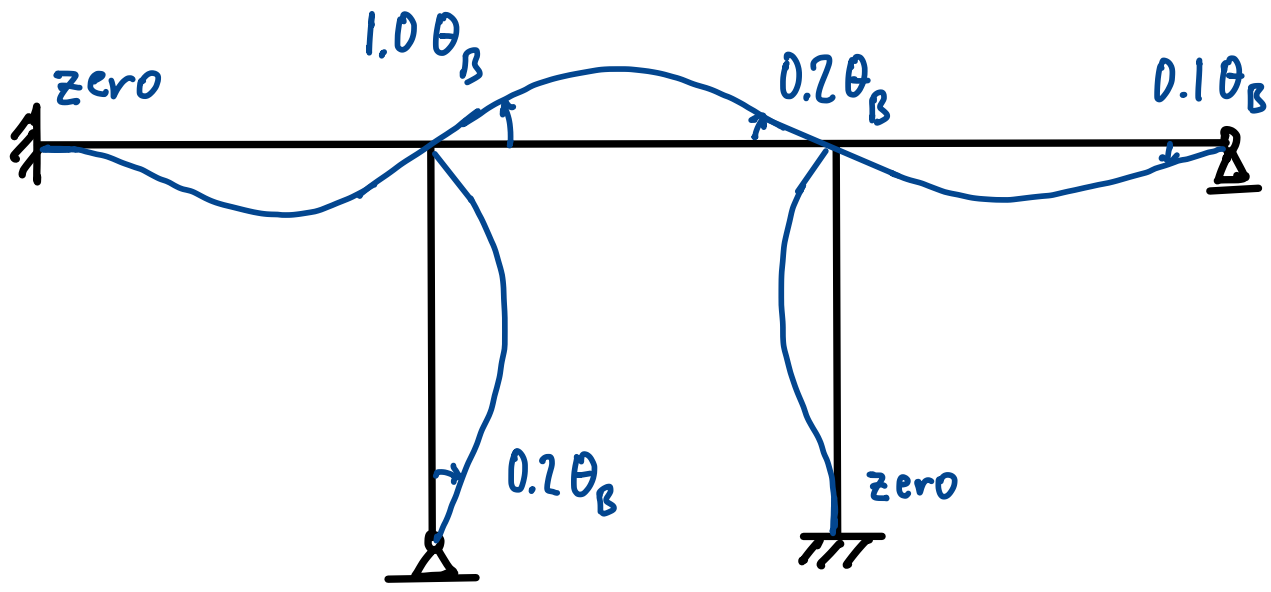
should give $\lambda \approx 0$ (buckling)
i.e. need

$0.3426 \times 8.1395 - 4 \neq 0$!

Top equation $0.3426\,\theta_B + 2\theta_C = 0$

$\Rightarrow \theta_C = -0.171\,\theta_B$

Bottom equation $2\,\theta_B + 8.1395 = 0$

$\Rightarrow \theta_C = -0.2457\,\theta_B$

The true value is somewhere in between, hence assume $\theta_C \approx -0.2\,\theta_B$