1.

(a) For time-independent differential equations, apply the following steps:

(a) Identify the equation of interest.
(b) Cast the equation of interest in a weak form.
(c) Select a finite element type.
(d) Construct the element matrix and vector.
(e) Assemble the global matrix and vector and apply boundary conditions.
(f) Solve the system of linear equations.

(b) Let $v(x)$ be an arbitrary test function satisfying $v(L) = 0$. Multiply the governing equation by v and integrate over $(0, L)$:

$$\int_0^L -\frac{d}{dx}\left[(1+u^2)u'\right] v\, dx = \int_0^L f\, v\, dx.$$

Integrate by parts:

$$\int_0^L (1+u^2)u'v'\, dx - \Big[(1+u^2)u'v\Big]_0^L = \int_0^L f\, v\, dx.$$

Since $v(L) = 0$ and $(1+u^2)u'\big|_{x=0} = 0$,

$$\int_0^L (1+u^2)u'v'\, dx = \int_0^L f(x)\, v\, dx, \quad \forall\, v \text{ with } v(L) = 0.$$

(c) Use the 1D two-node linear (C^0) element. Then $u_h \in C^0$ but u_h' is piecewise constant and generally discontinuous at element boundaries. Dirichlet BCs are imposed strongly at nodes; Neumann/flux BCs arise naturally from the weak form.

2.

(a)
$$\rho A\,\ddot u = EA\,u'', \quad 0<x<L, \qquad u(0,t)=0, \qquad EA\,u'(L,t)+k\,u(L,t)=0.$$

Take $v(0)=0$ and multiply/integrate:

$$\int_0^L \rho A\,\ddot u\,v\,dx = \int_0^L EA\,u''\,v\,dx.$$

Integrate by parts:

$$\int_0^L \rho A\,\ddot u\,v\,dx + \int_0^L EA\,u'v'\,dx - \big[EA\,u'v\big]_0^L = 0.$$

With $v(0)=0$ and $EA\,u'(L) = -k\,u(L)$,

$$\int_0^L \rho A\,\ddot u\,v\,dx + \int_0^L EA\,u'v'\,dx + k\,u(L,t)\,v(L) = 0,\ \forall v.$$

Use one linear element with $N_1 = 1-x/L,\ N_2 = x/L$:

$$u_h = N_1u_1 + N_2u_2, \qquad v_h = N_1\delta u_1 + N_2\delta u_2, \qquad \mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

This gives

$$\mathbf{M}\ddot{\mathbf{d}} + (\mathbf{K}_e + \mathbf{K}_s)\mathbf{d} = \mathbf{0}$$

with

$$\mathbf{M} = \int_0^L \rho A\,\mathbf{N}^T\mathbf{N}\,dx = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\mathbf{K}_e = \int_0^L EA\,\mathbf{B}^T\mathbf{B}\,dx = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -1/L & 1/L \end{bmatrix},$$

$$\mathbf{K}_s = k\,\mathbf{N}(L)^T\mathbf{N}(L) = k\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{N}(L) = [0 \ \ 1].$$

Enforce $u_1(t)=0$:

$$\frac{\rho AL}{3}\,\ddot u_2(t) + \Big(\frac{EA}{L} + k\Big)u_2(t) = 0.$$

(b) Given $u(x,0) = u_0x/L$ and $\dot u(x,0)=0$:

$$u_2(0) = u_0, \qquad \dot u_2(0) = 0.$$

Write $\ddot u_2 + \omega^2 u_2 = 0$ with

$$\omega = \sqrt{\frac{3}{\rho AL}\Big(\frac{EA}{L} + k\Big)}$$

hence

$$u_2(t) = u_0\cos(\omega t)$$

and

$$u(x,t) = N_2(x)u_2(t) = u_0\frac{x}{L}\cos(\omega t), \qquad 0\le x\le L.$$

3. (a)

i. The three shape functions of the element are

$$N_1 = 1 - \frac{x}{2} - \frac{y}{2}$$
$$N_2 = \frac{x}{2}$$
$$N_3 = \frac{y}{2}$$

ii. The strain-displacement relationship reads

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{pmatrix} = \underbrace{\begin{pmatrix} N_{2,x} & 0 \\ 0 & N_{2,y} \\ N_{2,y} & N_{2,x} \end{pmatrix}}_{\boldsymbol{B}^e} \begin{pmatrix} u_{2x} \\ u_{2y} \end{pmatrix}$$

$N_{2,x}$ means differentiation with respect to x etc.

$$\Rightarrow \boldsymbol{B}^e = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

As to be expected the $\boldsymbol{B}^e$ matrix for this linear element is constant.

iii. The stiffness-matrix can be computed without numerical integration

$$\boldsymbol{K}^e = \int_\Omega \boldsymbol{B}^{eT} \boldsymbol{D} \boldsymbol{B} = 2 \boldsymbol{B}^{eT} \boldsymbol{D} \boldsymbol{B}$$

with

$$\boldsymbol{D} = \begin{pmatrix} 200 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 100 \end{pmatrix}$$

This yields for the stiffness matrix

$$\boldsymbol{K}^e = \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix}$$

(b)

i. The Jacobian J is computed with the isoparametric mapping

$$x = \sum_i N_i x_i \qquad y = \sum_i N_i y_i$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

The four shape functions of the element

$$\begin{aligned} N_1 &= (1-\xi)(1-\eta)/4 \\ N_2 &= (1+\eta)(1-\eta)/4 \\ N_3 &= (1+\xi)(1+\eta)/4 \\ N_4 &= (1-\xi)(1+\eta)/4 \end{aligned}$$

yield the Jacobian matrix

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{1-\eta}{4}4 + \frac{1+\eta}{4}8 - \frac{1+\eta}{4}4 = 2 \\ \frac{\partial x}{\partial \eta} &= -\frac{1+\xi}{4}4 + \frac{1+\xi}{4}8 + \frac{1-\xi}{4}4 = 2 \\ \frac{\partial y}{\partial \xi} &= \frac{1+\eta}{4}6 - \frac{1+\eta}{4}6 = 0 \\ \frac{\partial y}{\partial \eta} &= \frac{1+\xi}{4}6 + \frac{1-\xi}{4}6 = 3 \end{aligned}$$

$$\Rightarrow J = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$$

ii. The strain components ϵ_{xx} and ϵ_{yy} follow from

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \epsilon_{yy} &= \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} \end{aligned}$$

The displacements are interpolated with

$$u_x = 0.1 \cdot N_4 \qquad u_y = 0.2 \cdot N_4$$

and the inverse of the Jacobian reads

$$J^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Introducing the displacement interpolation and the inverse of the Jacobian into the strain equations gives

$$\begin{aligned} \epsilon_{xx} &= -\frac{1+\eta}{80} \\ \epsilon_{yy} &= \frac{1}{60}(2+\eta-\xi) \end{aligned}$$

Substitute (-1, -1), (1, -1), (1, 1) and (-1, 1) to obtain the strains at 4 nodes.

4.

(a)

The weak form contains lower-order derivatives than the strong form, admitting less regular (reduced continuity) functions. For FEM, this means that piecewise polynomials can be used (C^0 continuity for second order PDE). In addition, the weak form allows the Neumann boundary conditions to be satisfied automatically in the formulation.

(b.i)

Let the square domain is noted as Ω and v a test function, where v=0 at ∂Ω.

$$\int_{\Omega} v\nabla^2 T dxdy = -\int_{\Omega} vdxdy$$

Integrate by part

$$\int_{\Omega} [\nabla \cdot (v\nabla T) - \nabla v \cdot \nabla T]dxdy = -\int_{\Omega} vdxdy$$

Divergence theorem

$$\int_{\Omega} \nabla \cdot (v\nabla T)dxdy = \int_{\partial\Omega} v\nabla T \cdot \boldsymbol{n}\, dl = 0$$

So, the weak form is

$$\int_{\Omega} \nabla v \cdot \nabla T dxdy = \int_{\Omega} vdxdy$$

(b.ii)

For element [1], the four nodes have coordinates: (0,0), (0.5,0), (0.5,0.5) and (0,0.5)

The four shape functions are:

$N_1^e = 4(\frac{1}{2} - x)(\frac{1}{2} - y)$

$N_2^e = 4x(\frac{1}{2} - y)$

$N_3^e = 4xy$

$N_4^e = 4(\frac{1}{2} - x)y$

$$\boldsymbol{B}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x} & \frac{\partial N_4^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_4^e}{\partial y} \end{bmatrix} = \begin{bmatrix} 4(\mathrm{y} - \frac{1}{2}) & 4(\frac{1}{2} - y) & 4y & -4y \\ 4(x - \frac{1}{2}) & -4x & 4x & 4(\frac{1}{2} - x) \end{bmatrix}$$

$$\boldsymbol{B}^{e\,T}\boldsymbol{B}^e = \begin{bmatrix} (4y-2) & (4x-2) \\ (2-4y) & -4x \\ 4y & 4x \\ -4y & (2-4x) \end{bmatrix} \begin{bmatrix} (4y-2) & (2-4y) & 4y & -4y \\ (4x-2) & -4x & 4x & (2-4x) \end{bmatrix}$$

$$= \begin{bmatrix} (4y-2)^2+(4x-2)^2 & -(4y-2)^2-4x(4x-2) & 4y(4y-2)+4x(4x-2) & -4y(4y-2)-(4x-2)^2 \\ -(4y-2)^2-4x(4x-2) & (4y-2)^2+16x^2 & -4y(4y-2)-16x^2 & 4y(4y-2)+4x(4x-2) \\ 4y(4y-2)+4x(4x-2) & -4y(4y-2)-16x^2 & 16y^2+16x^2 & -16y^2-4x(4x-2) \\ -4y(4y-2)-(4x-2)^2 & 4y(4y-2)+4x(4x-2) & -16y^2-4x(4x-2) & 16y^2+(4x-2)^2 \end{bmatrix}$$

So, the element conductance matrix is:

$$\boldsymbol{K}^e = \iint_\Omega \boldsymbol{B}^{e\,T} \boldsymbol{B}^e \, d\Omega = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

The element source flux vector is:

$$\boldsymbol{f}^e = \iint_\Omega \boldsymbol{N}^e \, d\Omega = \iint_\Omega \begin{bmatrix} 4(\frac{1}{2}-x)(\frac{1}{2}-y) \\ 4x(\frac{1}{2}-y) \\ 4xy \\ 4(\frac{1}{2}-x)y \end{bmatrix} d\Omega = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \end{bmatrix}$$

(b.iii)

There are four elements, and the only unknown is T_5. Assembling the 4 element conductance matrices and source flux vectors, the equation for node 5 (i.e. the 5th row) is:

$$4 \times \frac{2}{3} T_5 - 2 \times \frac{1}{6} T_2 - 2 \times \frac{1}{6} T_4 - 2 \times \frac{1}{6} T_6 - 2 \times \frac{1}{6} T_8 - \frac{1}{3} T_1 - \frac{1}{3} T_3 - \frac{1}{3} T_7 - \frac{1}{3} T_7 = 4 \times \frac{1}{16}$$

All the boundary-node temperatures are zero, except $T_6 = 0.25$. So,

$$4 \times \frac{2}{3} T_5 - 2 \times \frac{1}{6} \times 0.25 = 4 \times \frac{1}{16}$$

Finally, $T_5 = 1/8 = 0.125$.