EGT2
ENGINEERING TRIPOS PART IIA

Monday 11 May 2026 14.00 to 15.40

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 Finite Element Methods data sheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Summarise the step-by-step procedure for using the finite element method to solve time-independent (steady-state) differential equations. [20%]

(b) Consider the nonlinear boundary-value problem

$$-\frac{d}{dx}\left[(1+u^2)\frac{du}{dx}\right] = f(x) \quad \text{for} \quad 0 < x < L,$$

subject to the boundary conditions

$$(1+u^2)\frac{du}{dx}\bigg|_{x=0} = 0, \qquad u|_{x=L} = u_L,$$

where $f(x)$ is a given body-force density and u_L is a prescribed constant.
Derive the weak formulation of this problem. [60%]

(c) What is the simplest finite element you would use to approximate the solution of (b) in one spatial dimension? Comment on the continuity of the approximate solution and its derivative. [20%]

2 Consider a uniform elastic bar of length L, cross-sectional area A, Young's modulus E, and mass density ρ. The axial displacement $u(x,t)$ satisfies

$$\rho A\,\frac{\partial^2 u}{\partial t^2} = EA\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0.$$

The bar is fixed at $x = 0$ and attached to a linear spring of stiffness k at $x = L$. Thus, the boundary conditions are

$$u(0,t) = 0, \qquad EA\,\left.\frac{\partial u}{\partial x}\right|_{x=L} + k\,u(L,t) = 0.$$

The bar is released from rest with initial conditions

$$u(x,0) = u_0\frac{x}{L}, \qquad \frac{\partial u}{\partial t}(x,0) = 0,$$

where u_0 is a given constant.

(a) Using a single two-node linear element with a *consistent* mass matrix, derive the semi-discrete finite-element equation of motion in the form

$$\mathbf{M}\,\ddot{\mathbf{d}}(t) + \mathbf{K}\,\mathbf{d}(t) = \mathbf{0},$$

where $\mathbf{d}(t) = [u_1(t)\ u_2(t)]^T$. Clearly show how the spring modifies the stiffness matrix. Then enforce the essential boundary condition $u_1(t) = 0$ and obtain the reduced scalar equation for $u_2(t)$. [50%]

(b) Solve the reduced equation for $u_2(t)$, and hence write down the one-element finite-element displacement field $u(x,t)$ for $0 \le x \le L$. State the natural frequency of the one-element model. [50%]

3 (a) Fig. 1(a) shows a finite element mesh consisting of one single three-node triangular element, which represents an elastic plane-stress problem with Young's modulus E = 200 and Poisson's ratio v = 0. In the following, only consider the unconstrained degrees of freedom, *i.e*, movement of node 2.

(i) Write down the three shape functions of the element. [10%]

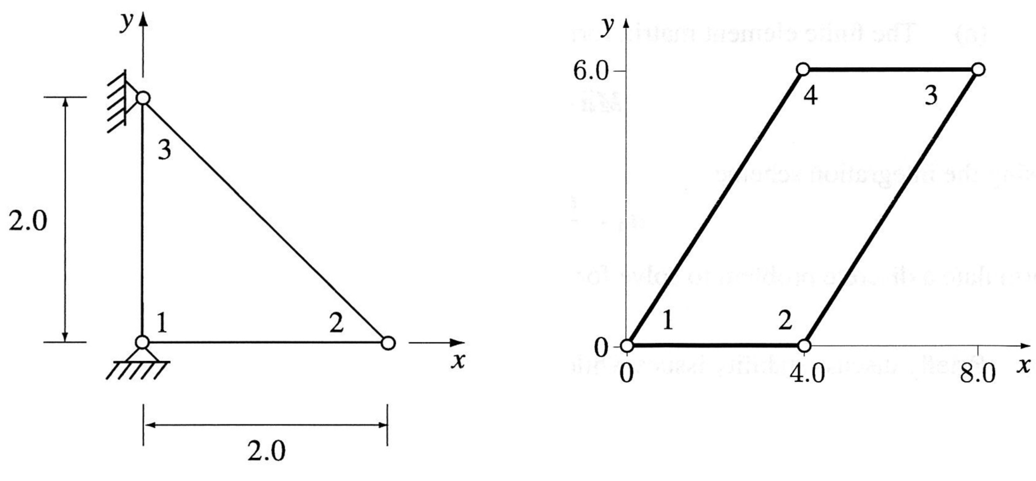
(ii) Calculate the strain-displacement matrix. [10%]

(iii) Calculate the stiffness matrix. [10%]

(b) Fig. 1(b) shows a four-node element. Use the isoparametric mapping method to answer the following questions.

(i) Write down the Jacobian matrix of the element. [30%]

(ii) The horizontal and vertical components of the displacement of node 4 are 0.1 and 0.2, respectively, while all other nodes do not move. Compute the strain components ε_{xx} and ε_{yy} at the four nodes. [40%]



(a) (b)

Fig. 1

4 (a) Comment on the advantages of the weak form of a differential equation in comparison with the strong form. [10%]

(b) The temperature distribution over a square domain is governed by the following equation.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + 1 = 0$$

The square domain has the side length of 1. The temperature is fixed at 0 at the bottom, left and right boundaries. Along the top boundary ($0 < x < 1$, $y = 1$), the temperature increases linearly from 0 at nodes 3 and 9 to the maximum value of 0.25 at node 6. To solve the above equation using the finite element method, the domain is discretised with four square-shaped elements as shown in Fig. 2.

(i) Derive the weak formulation of the above heat transfer equation. [10%]

(ii) Calculate the element conductance matrix and element source flux vector of the four-node element [1] in Fig. 2. [50%]

(iii) Using symmetry, assemble the relevant components of the global conductance matrix and global source flux vector so as to calculate the temperature at node 5. [30%]

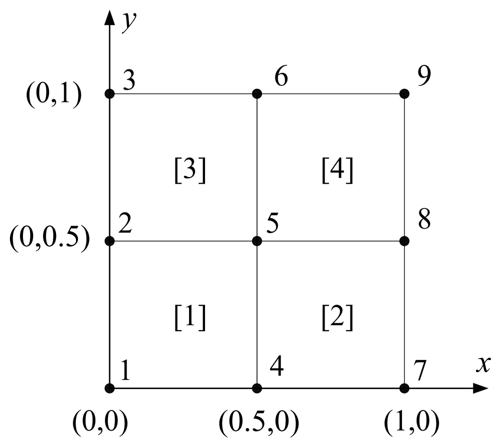


Fig. 2

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Engineering Tripos Part IIA
Module 3D7: Finite Element Methods

Data Sheet

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f} \, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI \boldsymbol{B} \, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

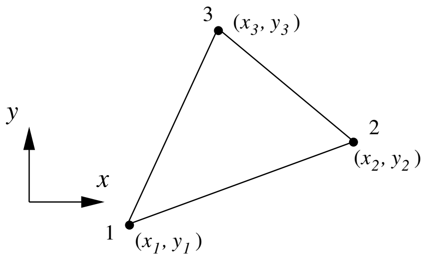
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D)

$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

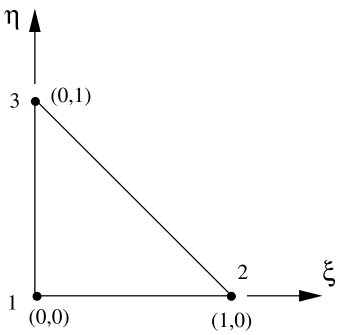
Shape functions



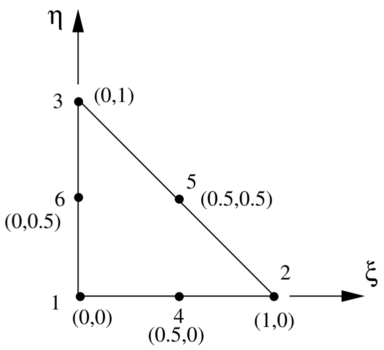
$$N_1 = \big(\big(x_2 y_3 - x_3 y_2\big) + \big(y_2 - y_3\big)\,x + (x_3 - x_2)\,y\big)/2A$$
$$N_2 = \big(\big(x_3 y_1 - x_1 y_3\big) + \big(y_3 - y_1\big)\,x + (x_1 - x_3)\,y\big)/2A$$
$$N_3 = \big(\big(x_1 y_2 - x_2 y_1\big) + \big(y_1 - y_2\big)\,x + (x_2 - x_1)\,y\big)/2A$$

$A =$ area of triangle

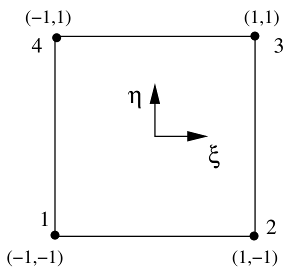
$\int_\Omega N_I = A/3,\ i = 1,\ 2,\ 3.$



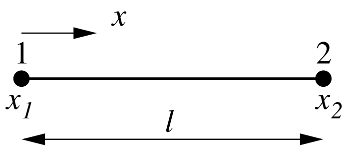
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\big(1 - \xi - \eta\big)^2 - \big(1 - \xi - \eta\big)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\big(1 - \xi - \eta\big)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\big(1 - \xi - \eta\big)$$



$$N_1 = (1 - \xi)\big(1 - \eta\big)/4$$
$$N_2 = (1 + \xi)\big(1 - \eta\big)/4$$
$$N_3 = (1 + \xi)\big(1 + \eta\big)/4$$
$$N_4 = (1 - \xi)\big(1 + \eta\big)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1, 1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$