Crib of 3F1 exam 2026

1. (a) We transform the difference equation into the z domain

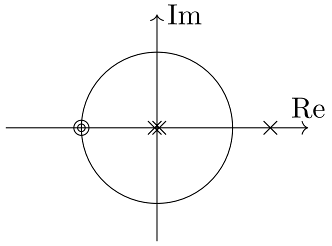
$$Y(z) = \frac{3}{2}z^{-1}Y(z) + \frac{1}{4}z^{-1}X(z) + \frac{1}{2}z^{-2}X(z) + \frac{1}{4}z^{-3}X(z)$$

hence

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} \cdot \frac{z^{-1} + 2z^{-2} + z^{-3}}{1 - \frac{3}{2}z^{-1}} = \frac{1}{4} \cdot \frac{(z+1)^2}{z^2\left(z - \frac{3}{2}\right)}.$$

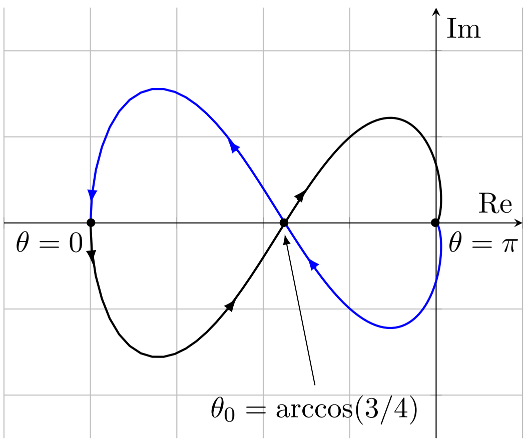
[10%]

(b) We draw the pole-zero diagram with a double-pole at the origin, a pole at $z = 3/2$, and a double-zero at $z = -1$:



[10%]

(c) (i) The complete labeled Nyquist diagram is given as



The value $\theta_0 = \arccos(3/4)$ is recovered from computing the imaginary part of the Nyquist locus and setting it to zero

$$\begin{aligned}
\operatorname{Im} G(e^{j\theta}) &= \operatorname{Im}\left[\frac{1}{4}\frac{\left(e^{j\theta/2}(e^{j\theta/2}+e^{-j\theta/2})\right)^2}{e^{2j\theta}\left(\cos\theta-\frac{3}{2}+j\sin\theta\right)}\right] \\
&= \operatorname{Im}\left[\frac{\cos^2(\theta/2)(\cos\theta-j\sin\theta)\left(\cos\theta-\frac{3}{2}-j\sin\theta\right)}{\left(\cos\theta-\frac{3}{2}\right)^2+\sin^2\theta}\right] \\
&= -\frac{\cos^2(\theta/2)}{\frac{13}{4}-4\cos\theta}\left(2\cos\theta-\frac{3}{2}\right)\sin\theta
\end{aligned}$$

which is zero either when $\sin\theta$ is zero, i.e., for $\theta = 0$ and $\theta = \pi$, or when $2\cos\theta = 3/2$, i.e., $\cos\theta = 3/4$. We compute that $G\left(e^{j\,\arccos(3/4)}\right) = -\frac{7}{8}$. [40%]

(ii) Since the open loop system has one pole at $z = 3/2$ outside the unit circle, the Nyquist stability criterion requires one anti-clockwise encirclement for the closed loop system to be stable, which is the case for $-2 < -1/K < -7/8$, i.e.,

$$\frac{1}{2} < K < \frac{8}{7}$$

. [10%]

(iii) In order for the delta response of the closed loop system to converge to a non-zero constant, it needs a pole at $z = 1$. Hence, it must be at the edge of the stability range for K, either $K = 8/7$ of $K = 1/2$. Consider the closed loop transfer function

$$F(z) = \frac{G(z)}{1+KG(z)} = \frac{(z+1)^2}{4z^2\left(z-\frac{3}{2}\right)+K(z+1)^2}$$

Evaluating the denominator for $K = 8/7$ and $z = 1$ gives $18/7$, showing that $z = 1$ is not a pole for this value of K. For $K = 1/2$ and $z = 1$, the denominator is zero, so $z = 1$ is a pole and hence $\boxed{K = 1/2}$ is the correct choice. If the input is $800\delta_k$, the output in the z domain is

$$Y(z) = 800 \cdot \frac{2(z+1)^2}{8z^3-11z^2+2z+1} = \frac{200(z+1)^2}{(z-1)\left(z^2-\frac{3}{8}z-\frac{1}{8}\right)}$$

We can use the Final Value Theorem (because the remaining 2 poles are inside the unit circle) to compute

$$\lim_{k\to\infty} y_k = \lim_{z\to 1}(z-1)Y(z)\ \boxed{=1600}$$

so the population of the captive colony will stabilise around 1600 penguins. [30%]

2. (a) We have

$$A(z) = \frac{z^2}{z^2-\alpha} = \frac{1}{1-\alpha z^{-2}} = (\alpha z^{-2})^0 + (\alpha z^{-2})^1 + (\alpha z^{-2})^2 + (\alpha z^{-2})^3 + \ldots$$
$$= 1 + \alpha z^{-2} + \alpha^2 z^{-4} + \alpha^3 z^{-6} + \ldots = \sum_{k=0}^{\infty} \alpha^k z^{-2k}$$

This corresponds to the sequence $1, 0, \alpha, 0, \alpha^2, 0, \alpha^3, 0, \ldots$, i.e., the geometric sequence $1, \alpha, \alpha^2, \alpha^3, \ldots$ interspersed with zeros at every odd position, or $a_k = \begin{cases} \alpha^{k/2} \text{ for even } k \\ 0 \text{ for odd } k \end{cases}$

[20%]

(b) We have

$$\sum_{k=0}^{\infty} |a_k| = \sum_{k=0}^{\infty} |a_{2k}| = \sum_{\ell=0}^{\infty} |\alpha^\ell|$$

hence the stability condition is the same as for a system with a regular geometric delta response $\{\alpha^k\}_{k\geq 0}$: $|\alpha| < 1$.
Considering the transfer function in the z domain, the poles are at $\sqrt{\alpha}$ and $-\sqrt{\alpha}$ if $\alpha > 0$, or $j\sqrt{-\alpha}$ and $-j\sqrt{-\alpha}$ if $\alpha < 0$. This is consistent with the conclusion above, as the system is stable if the poles are inside the unit circle, i.e., if $|\alpha| < 1$. [20%]

(c) The delta response of the system $\mathcal{B}$ can be found in the same way we found the delta response of the system $\mathcal{A}$, and gives $1, 0, 0, \beta, 0, 0, \beta^2, 0, 0, \beta^3, 0, 0, \ldots$, i.e., a geometrical response $\{\beta^k\}_{k\geq 0}$ interspersed with pairs of zeros, or $b_k = \begin{cases} \beta^{k/3} \text{ when } k \text{ is divisible by 3} \\ 0 \text{ otherwise} \end{cases}$

The system $\mathcal{C}$ obtained by multiplication $C(z) = A(z)B(z)$ in the z domain has as its delta response the discrete convolution of the delta responses of $\mathcal{A}$ and $\mathcal{B}$. Hence,

$$\begin{aligned} c_0 &= a_0 b_0 = 1 \\ c_1 &= a_0 b_1 + a_1 b_0 = 0 \\ c_7 &= a_0 b_7 + a_1 b_6 + \ldots + a_7 b_0 = a_4 b_3 = \alpha^2 \beta \\ c_{10} &= a_0 b_{10} + \ldots + a_{10} b_0 = a_4 b_6 + a_{10} b_0 = \alpha^2 \beta^2 + \alpha^5 \end{aligned}$$

[30%]

(d) i. We have

$$A(e^{j(\pi-\theta)}) = \frac{e^{2j(\pi-\theta)}}{e^{2j(\pi-\theta)} - \alpha} = \frac{e^{-2j\theta}}{e^{-2j\theta} - \alpha}$$
$$= \frac{\overline{e^{2j\theta}}}{\overline{e^{2j\theta}} - \alpha} = \frac{\overline{e^{2j\theta}}}{\overline{\overline{e^{2j\theta}} - \alpha}} = \overline{A(e^{j\theta})} \quad \text{where } \overline{z} \text{ denotes complex conjugation.}$$

Hence, $\boxed{|A(e^{j(\pi-\theta)})| = |\overline{A(e^{j\theta})}| = |A(e^{j\theta})|}$. [15%]

ii. The Bode diagram exhibits conjugate symmetry around $\theta = \pi/2$ which is the normalised frequency corresponding to $f_s/4$ where f_s is the sampling frequency. This is because the sequence $\{a_k\}$ has zeros for every odd k, looking like the discrete-time version of an impulse train with samples at half the original sampling frequency. Hence, its spectrum is periodic with period $f_s/2$ as well as f_s.

[15%]

3. (a) (i) We use the DTFT expression given in the question,

$$A(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\theta} = \sum_{k=0}^{N} e^{-jk\theta} = \frac{1 - e^{-j(N+1)\theta}}{1 - e^{-j\theta}}$$

$$= \frac{e^{-j\frac{N+1}{2}\theta}\left(e^{j\frac{N+1}{2}\theta} - e^{-j\frac{N+1}{2}\theta}\right)}{e^{-j\theta/2}\left(e^{j\theta/2} - e^{-j\theta/2}\right)} \boxed{= e^{-jN\theta/2}\frac{\sin\left(\frac{N+1}{2}\theta\right)}{\sin(\theta/2)}}$$

[10%]

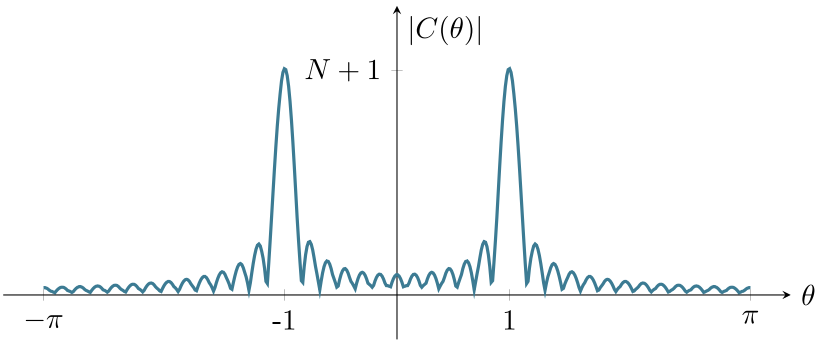
(ii) We use the inverse DTFT expression given in the question

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\theta)e^{jk\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\delta(\theta-1)+\delta(\theta+1)]e^{jk\theta}d\theta$$

$$= \frac{1}{2\pi}(e^{jk} + e^{-jk}) \boxed{= \frac{\cos k}{\pi}}$$

[10%]

(iii) Since $c_k = a_k b_k$ for $N = 100$, multiplication in the time domain is equivalent to **periodic convolution** in the DTFT domain. $A(\theta)$ is a “sinc-like function” for large N in that, for small θ, the denominator is well approximated as a linear term $\theta/2$, making the expression $|A(\theta)| \approx (N+1)\,\text{sinc}((N+1)\theta/2)$. Periodic convolution with a $\delta()$ function is the same as linear convolution, resulting in a magnitude graph roughly as follows:



[10%]

(b) $\mathbf{x}_1$ is the vector $\mathbf{x}$ element-wise multiplied by $[1,-1,1,-1,1,-1]$, or, equivalently, by $[e^{jk\pi}]_{k=0\ldots5} = [W_6^{3k}]_{k=0\ldots5}$ where $W_6 = e^{-j\frac{2\pi}{6}}$. This multiplication corresponds to a shift by 3 in the frequency domain, so the DFT of $\mathbf{x}_1$ should have the components $X_{R_6(n+3)}$, which is the casse for $\mathbf{X}_3$.

$\mathbf{x}_2$ is the time reversal $\tilde{x}_k = x_{N-k}$ (with $\tilde{x}_0 = x_N = x_0$) of $\mathbf{x}$ and its DFT is the complex conjugate of $\mathbf{X}$, which is $\mathbf{X}_6$

$\mathbf{x}_3$ is $\mathbf{x}$ shifted by 3, i.e., $\tilde{x}_k = x_{R_6(k+3)}$ which corresponds to an element-wise multiplication by $[W_6^{3k}]_{k=0\ldots6} = [1,-1,1,-1,1,-1]$ in the frequency domain. This corresponds to $\mathbf{X}_7$.

$\mathbf{x}_4$ the DFT of $\mathbf{x}_4$ is $\frac{1}{6}\mathbf{FFx}$ where $\mathbf{F}$ is the DFT matrix. Since $\mathbf{F}^{-1}$ corresponds to the rows of $\frac{1}{6}\mathbf{F}$ in reversed order $(0, N-1, \ldots, 1)$, the outcome is the time-reversed $\mathbf{x}$, which is $\mathbf{X}_5$.

$\mathbf{x}_5$ is the sum of $\mathbf{x}$ and time-reversed $\mathbf{x}$ divided by 2, which gives the sum of $\mathbf{X}$ and $\bar{\mathbf{X}}$ divided by two, or in other words $\mathrm{Re}(\mathbf{X})$, which is $\mathbf{X}_1$. It is also worth noting that $\mathbf{x}_5$ is conjugate symmetric so it is only right that its DFT should be real.

$\mathbf{x_6}$ is the same as $\mathbf{x_4}$ but shifted by 3, so its DFT should be the same as $\mathbf{X}_5$ but elementwise mutliplied by $[W_6^{3k}]_{k=0\ldots6} = [1, -1, 1, -1, 1, -1]$, which is $\mathbf{X}_4$.

$\mathbf{x_7}$ is $\mathbf{x}$ so its DFT is $\mathbf{X}$ but $[A, B, C, D, E, F]$ does not appear among the solutions. However, since we are told that $\mathbf{x}$ is real, its DFT must obey conjugate symmetry, so $F = \bar{B}$ and $E = \bar{C}$, making $\mathbf{X}_2$ the DFT of $\mathbf{x_7}$

[50%]

(c) (i) Since $N = 15 = ML = 3 \times 5$, the number of multiplications is

$$\boxed{ML^2 + N + LM^2 = ML(L + M + 1) = 15(3 + 5 + 1) = 135}$$

[10%]

(ii) The block length is $N = 45 = ML = 3 \times 15$. We can apply the same expression as for the previous part, except that for the 15-point DFT instead of using a matrix multiplication with $L^2 = 15^2$ multiplications, we can use the decomposed DFT from the previous part that needs only $\lambda = 135$ multiplications, yielding

$$\boxed{M\lambda + N + LM^2 = 3 \times 135 + 45 + 15 \times 3^2 = 585}$$

and note that it is a considerable improvement on the $N^2 = 45^2 = 2025$ multiplications needed to evaluate the DFT directly. [10%]

4. (a) Defining $\beta^2 = 2\sigma^2\omega_1\omega_2(\omega_1+\omega_2)$, we get

$$S_{XX}(\omega) = \beta^2\left[\frac{A}{\omega^2+\omega_1^2} + \frac{B}{\omega^2+\omega_2^2}\right] = \beta^2\frac{(A+B)\omega^2 + A\omega_2^2 + B\omega_1^2}{(\omega^2+\omega_1^2)(\omega^2+\omega_2^2)}$$

so $A+B=0$ and $A\omega_2^2 + B\omega_1^2 = 1$, hence

$$S_{XX}(\omega) = \frac{\beta^2}{\omega_2^2-\omega_1^2}\left[\frac{1}{\omega^2+\omega_1^2} - \frac{1}{\omega^2+\omega_2^2}\right] = \frac{\sigma^2\omega_1\omega_2}{\omega_2-\omega_1}\left[\frac{1}{\omega_1}\frac{2\omega_1}{\omega^2-\omega_1^2} - \frac{1}{\omega_2}\frac{2\omega_2}{\omega^2-\omega_2^2}\right]$$

resulting in the auto-correlation function

$$\boxed{r_{XX}(\tau) = \frac{\sigma^2\omega_1\omega_2}{\omega_2-\omega_1}\left(\frac{e^{-\omega_1|\tau|}}{\omega_1} - \frac{e^{-\omega_2|\tau|}}{\omega_2}\right).}$$

We verify that

$$r_{XX}(0) = \frac{\sigma^2\omega_1\omega_2}{\omega_2-\omega_1}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right) = \sigma^2.$$

[25%]

(b) We verify that

$$|H(j\omega)|^2 = H(j\omega)\overline{H(j\omega)} = \frac{\beta^2}{(\omega_1+j\omega)(\omega_2+j\omega)(\omega_1-j\omega)(\omega_2-j\omega)} = S_{XX}(\omega).$$

The output X of a stable linear filter $H(s)$ when the input is a signal W with power-spectral density $S_{WW}(\omega)$ is

$$S_{XX}(\omega) = |H(j\omega^2)|S_{WW}(\omega)$$

so when the input process is white, i.e., $S_{WW}(\omega) = 1$ for all ω, we get the desired power spectral density $|H(j\omega^2)|$ for the output process X.
To find the differential equation, we note that

$$H(s) = \frac{\beta}{s^2 + (\omega_1+\omega_2)s + \omega_1\omega_2}$$

and hence obtain the equation in the Laplace domain

$$s^2X(s) + (\omega_1+\omega_2)sX(s) + \omega_1\omega_2X(s) = \beta W(s)$$

which corresponds to the differential equation

$$\boxed{\frac{d^2}{dt^2}X(t) + (\omega_1+\omega_2)\frac{d}{dt}X(t) + \omega_1\omega_2X(t) = \beta W(t)}$$

where, as a reminder, $\beta = \sqrt{2\sigma^2\omega_1\omega_2(\omega_1+\omega_2)}$. [25%]

(c) (i) With a zero-order hold, the piecewise-constant signal $w_h(t) = \sum_k W_k\, p(t - kh)$ (with p a unit-height rectangle of width h) approximates white noise as $h \to 0$ provided the variance of W_k grows like $1/h$. Intuitively, as h gets smaller there are more pulses per unit time, so each pulse must carry less *energy per pulse* on average in order to keep the *power spectral density* finite.

More formally (as in the lecture derivation), for IID W_k with variance σ_W^2 one finds

$$\frac{1}{T}\mathbb{E}\{|W_h(\omega)|^2\} = \frac{\sigma_W^2}{h} \cdot \frac{2 - 2\cos(\omega h)}{\omega^2},$$

and since $2 - 2\cos(\omega h) \sim (\omega h)^2$ for small h, the right-hand side behaves like $\sigma_W^2 h$. To make the limit equal to 1 (unit PSD), we require $\sigma_W^2 h \to 1$, hence

$$\boxed{\mathrm{Var}(W_k) = \sigma_W^2 = \frac{1}{h}.}$$

[10%]

(ii) The step response of $H(s)$ in the Laplace domain is

$$G(s) = \frac{H(s)}{s} = \beta\left[\frac{1}{\omega_1\omega_2}\cdot\frac{1}{s} + \frac{1}{\omega_1(\omega_1 - \omega_2)}\frac{1}{s + \omega_1} + \frac{1}{\omega_2(\omega_2 - \omega_1)}\frac{1}{s + \omega_2}\right]$$

and hence the step response in the time domain is

$$g(t) = \frac{\beta}{\omega_1\omega_2}u(t) + \frac{\beta}{\omega_1 - \omega_2}\left(\frac{e^{-\omega_1 t}}{\omega_1} - \frac{e^{-\omega_2 t}}{\omega_2}\right)$$

where $u(t)$ is the Heaviside step function. Sampled for $k = ht$, we get the response-invariant discrete-time step response

$$g_k = \frac{\beta}{\omega_1\omega_2}u_k + \frac{\beta}{\omega_1 - \omega_2}\left(\frac{e^{-\omega_1 kh}}{\omega_1} - \frac{e^{-\omega_2 kh}}{\omega_2}\right)$$

where $\{u_k\} = \{1\}_{k\geq 0}$ is the unit step sequence. The step response in the z domain is

$$G(z) = \frac{\beta}{\omega_1\omega_2}\frac{z}{z-1} + \frac{\beta}{\omega_1 - \omega_2}\left(\frac{1}{\omega_1}\frac{z}{z - e^{-h\omega_1}} - \frac{1}{\omega_2}\frac{z}{z - e^{-h\omega_2}}\right)$$

and the discrete-time transfer function is

$$\boxed{H_h(z) = \frac{z-1}{z}G(z) = \frac{\beta}{\omega_1\omega_2} + \frac{\beta}{\omega_1 - \omega_2}\left(\frac{1}{\omega_1}\frac{z-1}{z - e^{-h\omega_1}} - \frac{1}{\omega_2}\frac{z-1}{z - e^{-h\omega_2}}\right)}$$

where $\beta = \sqrt{2\sigma^2\omega_1\omega_2(\omega_1 + \omega_2)} = \sqrt{2\omega_1\omega_2(\omega_1 + \omega_2)/h}$ [25%]

(iii) To derive the difference equation, we need to re-write the transfer function as

$$\begin{aligned} H_h(z) &= \beta\frac{\frac{1}{\omega_1\omega_2}(z - e^{-h\omega_1})(z - e^{-h\omega_2}) + \frac{z-1}{\omega_1 - \omega_2}\left(\frac{1}{\omega_1}(z - e^{-h\omega_2}) - \frac{1}{\omega_2}(z - e^{-h\omega_1})\right)}{(z - e^{-h\omega_1})(z - e^{-h\omega_2})} \\ &= \beta\frac{az^2 + bz + c}{z^2 - (e^{-h\omega_1} + e^{-h\omega_2})z + e^{-h(\omega_1 + \omega_2)}} \end{aligned}$$

where we calculate

$$\begin{cases} a &= 0 \\ b &= \frac{1}{\omega_1 - \omega_2}\left(\frac{e^{-h\omega_1}-1}{\omega_1} - \frac{e^{-h\omega_2}-1}{\omega_2}\right) \\ c &= \frac{-1}{\omega_1 - \omega_2}\left(\frac{e^{-h\omega_1}-1}{\omega_1}e^{-h\omega_2} - \frac{e^{-h\omega_2}-1}{\omega_2}e^{-h\omega_1}\right) \end{cases}$$

and we obtain the difference equation

$$\boxed{x_k - (e^{-h\omega_1} + e^{-h\omega_2})x_{k-1} + e^{-h(\omega_1+\omega_2)}x_{k-2} = \beta(bw_{k-1} + cw_{k-2})}$$

[15%]