Version JS/3 after exam

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 29 April 2026   14.00 to 15.40

Module 3F1

SIGNALS & SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Macaroni penguins are a vulnerable species whose South Georgia Island population has decreased, likely due to climate change impacts on krill availability. The Cambridge-based British Antarctic Survey (BAS) operates the Bird Island Research Station on South Georgia. Suppose BAS establishes a captive breeding programme with optimal conditions (temperature control, unlimited krill, veterinary care, predator protection), enabling 50% population growth per season. However, transitioning penguins between wild and captive environments requires acclimatisation in a transition facility: 25% of penguins complete transition after 1 year, 50% further penguins after 2 years, and the remaining 25% after 3 years. If y_k is the captive colony population in year k, and x_k is the number entering transition (positive for wild $\rightarrow$ captive, negative for captive $\rightarrow$ wild), then:

$$y_k = 1.5y_{k-1} + 0.25x_{k-1} + 0.5x_{k-2} + 0.25x_{k-3}$$

(a) Write the transfer function $G(z)$ mapping the z transform $X(z)$ of the sequence $\{x_k\}_{k\geq 0}$ of penguins entering transition to the z transform $Y(z)$ of the sequence $\{y_k\}_{k\geq 0}$ of penguins in the breeding programme, assuming that $y_k = 0$ for $k \leq 0$ and $x_k = 0$ for $k < 0$. [10%]

(b) Draw a pole-zero diagram of the transfer function $G(z)$. [10%]

(c) Proportional feedback is introduced, so that the number of penguins entering transition in year k is $x_k = r_k - Ky_k$ as indicated in Fig 1. In practice, this number must be rounded to the nearest integer since there are no fractional penguins, but this can be safely neglected in your analysis.

(i) Figure 2 contains a partial Nyquist diagram of the open-loop system $G(z)$ drawn as the normalised frequency θ goes from 0 to π. Sketch the complete Nyquist diagram, labeling the points at which the diagram crosses the real axis with their corresponding normalised frequencies. [40%]

(ii) For what range of values of K is the closed-loop system stable? [10%]

(iii) For what value of K will the population of the captive colony for a single initial intake of 800 wild penguins, i.e. an input signal $\{r_k\} = \{800\delta_k\}$, tend to a positive constant, and what is that constant? [30%]

Note: $\{\delta_k\}$ is the Kronecker Delta signal $\delta_k = \begin{cases} 1 \text{ for } k = 0 \\ 0 \text{ otherwise} \end{cases}$

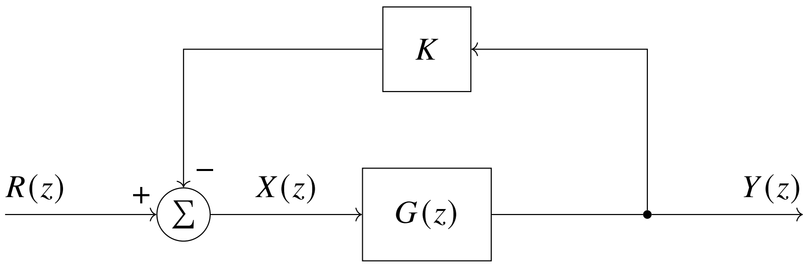

Fig. 1

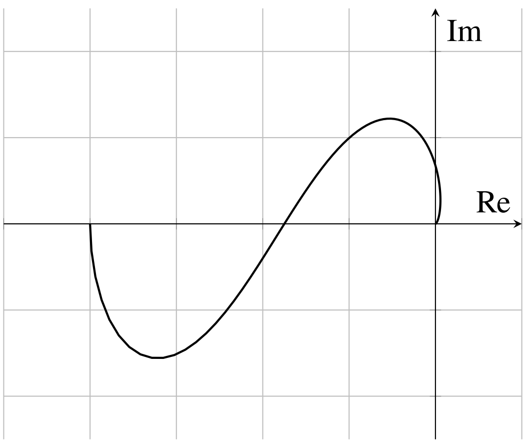

Fig. 2

2 (a) Derive from first principles the delta response $\{a_k\}_{k\geq 0}$ of the linear time-invariant system $\mathcal{A}$ with transfer function $A(z) = \frac{z^2}{z^2-\alpha}$ for some non-zero real-valued parameter α. [20%]

(b) Determine for what range of α the system $\mathcal{A}$ is stable, based on its delta response. Explain how this finding is consistent with the pole location of the transfer function in the z domain. [20%]

(c) A system $\mathcal{C}$ has transfer function $C(z) = A(z)B(z)$ where $B(z) = \frac{z^3}{z^3-\beta}$ for some real number β. Let $\{c_k\}_{k\geq 0}$ be the delta response of $\mathcal{C}$. Express c_0, c_1, c_7 and c_{10} as functions of α and β. [30%]

(d) For $\alpha = 1/2$, the Bode diagram of $\mathcal{A}$ is conjugate symmetric around $\theta = \pi/2$, i.e. $|A(e^{j(\pi-\theta)})| = |A(e^{j\theta})|$ and $\measuredangle A(e^{j(\pi-\theta)}) = -\measuredangle A(e^{j\theta})$.

(i) Verify the magnitude symmetry, i.e. show that $|A(e^{j(\pi-\theta)})| = |A(e^{j\theta})|$. [15%]

(ii) Explain conceptually why the Bode diagram exhibits this conjugate symmetry. [15%]

3 (a) The Discrete-Time Fourier Transform (DTFT) of a discrete-time signal $\{g_k\}_{-\infty<k<\infty}$ and its inverse are defined as

$$G(\theta) = \sum_{k=-\infty}^{\infty} g_k e^{-jk\theta}, \quad g_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\theta)e^{jk\theta}d\theta$$

(i) Derive the DTFT $A(\theta)$ of $a_k = \begin{cases} 1 \text{ for } 0 \leq k \leq N \text{ where } N > 0, \\ 0 \text{ otherwise} \end{cases}$ [10%]

(ii) Derive the inverse DTFT $\{b_k\}_{-\infty<k<\infty}$ of $B(\theta) = \delta(\theta - 1) + \delta(\theta + 1)$ where $\delta(.)$ is the (Dirac) impulse. [10%]

(iii) Explain qualitatively how one would obtain the DTFT $C(\theta)$ of the signal $c_k = b_k$ for $0 \leq k \leq 100$ and $c_k = 0$ otherwise. Sketch the magnitude $|C(\theta)|$ for $-\pi \leq \theta \leq \pi$ in linear (non-logarithmic) scale. [10%]

(b) Consider a real-valued vector $\mathbf{x} = [a, b, c, d, e, f]$ and its Discrete Fourier Transform (DFT) vector $\mathbf{X} = [A, B, C, D, E, F]$. We denote the complex conjugate of z by $\bar{z}$ and the real part of z by $\mathrm{Re}(z)$ in this question. Determine which of the following frequency domain vectors corresponds to which of the following time domain vectors and give a justification for your choice in each case.

Time domain	Frequency Domain
$\mathbf{x_1} = [a, -b, c, -d, e, -f]$	$\mathbf{X_1} = \mathrm{Re}([A, B, C, D, E, F])$
$\mathbf{x_2} = [a, f, e, d, c, b]$	$\mathbf{X_2} = [A, B, C, D, \bar{C}, \bar{B}]$
$\mathbf{x_3} = [d, e, f, a, b, c]$	$\mathbf{X_3} = [D, E, F, A, B, C]$
$\mathbf{x_4} = \frac{1}{6}[A, B, C, D, E, F]$	$\mathbf{X_4} = [a, -f, e, -d, c, -b]$
$\mathbf{x_5} = \frac{1}{2}[2a, b + f, c + e, 2d, e + c, f + b]$	$\mathbf{X_5} = [a, f, e, d, c, b]$
$\mathbf{x_6} = \frac{1}{6}[D, E, F, A, B, C]$	$\mathbf{X_6} = [\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}]$
$\mathbf{x_7} = [a, b, c, d, e, f]$	$\mathbf{X_7} = [A, -B, C, -D, E, -F]$

[50%]

(c) The regular DFT of length N evaluated by matrix multiplications requires N^2 multiplications. A DFT of length $N = ML$ can be evaluated using the Fast Fourier Transform (FFT) using M DFTs of length L, N twiddle factor multiplications, and L DFTs of length M.

(i) How many multiplications are needed for an FFT of length $N = 15$? [10%]

(ii) How many multiplications are needed for an FFT of length $N = 45$? [10%]

4 A drifting buoy records the vertical displacement $X(t)$ of the sea surface relative to its mean level. Over a short period the displacement may be regarded as a zero-mean Wide Sense Stationary (WSS) random process. A simple spectral model for this "sea-swell" signal is

$$S_{XX}(\omega) = \frac{2\sigma^2\,\omega_1\omega_2(\omega_1+\omega_2)}{(\omega^2+\omega_1^2)(\omega^2+\omega_2^2)}, \qquad \omega_1,\omega_2 > 0,\ \omega_1 \neq \omega_2 \tag{1}$$

where σ^2 is a positive constant.

(a) By writing (1) as a sum of two terms of the form $\dfrac{C}{\omega^2+a^2}$ and using the Fourier transform pair

$$e^{-a|\tau|} \longleftrightarrow \frac{2a}{a^2+\omega^2}, \qquad a > 0$$

obtain a closed-form expression for the autocorrelation function

$$r_{XX}(\tau) = \mathbb{E}[X(t)X(t+\tau)]$$

Verify that $r_{XX}(0) = \sigma^2$. [25%]

(b) Show that (1) can be written as $S_{XX}(\omega) = |H(j\omega)|^2$ with

$$H(s) = \frac{\sqrt{2\sigma^2\,\omega_1\omega_2(\omega_1+\omega_2)}}{(s+\omega_1)(s+\omega_2)}$$

Hence explain how $X(t)$ can be modelled as the output of a stable linear time-invariant system driven by a continuous-time white noise input $W(t)$ with two-sided PSD $S_{WW}(\omega) = 1$. Write down the corresponding linear differential equation relating $X(t)$ and $W(t)$. [25%]

(c) We wish to generate an *approximate sample path* of $X(t)$ on a computer. A D/A converter is used which holds its output constant over each sampling interval of length h. A discrete-time sequence $\{W_k\}_{k\in\mathbb{Z}}$ is used to form the piecewise-constant input

$$w_h(t) = \sum_{k\in\mathbb{Z}} W_k\, p(t-kh), \qquad p(t) = u(t) - u(t-h)$$

where $u(t)$ is the Heaviside step function.

(i) What variance should the IID samples W_k have (as a function of h) so that $w_h(t)$ behaves like unit-PSD white noise over the frequency range of interest? [10%]

(ii) Using a step-response invariant (zero-order-hold) discretisation, derive a discrete-time transfer function $H_h(z)$ which maps the input sequence $\{W_k\}$ to the sampled output sequence $x_k = X(kh)$.
Hint: If $g(t)$ denotes the continuous-time *step response* of $H(s)$ then the corresponding step-invariant digital filter satisfies

$$H_h(z) = (1 - z^{-1})\, \mathcal{Z}\{g(kh)\}$$

You may use that $\mathcal{Z}\{e^{-\alpha kh}\} = \dfrac{z}{z - e^{-\alpha h}}$ for $\alpha > 0$. [25%]

(iii) Hence write a causal difference equation which can be used to generate the sequence $\{x_k\}$ from $\{W_k\}$. Your coefficients may be left in terms of $e^{-\omega_1 h}$ and $e^{-\omega_2 h}$. [15%]

END OF PAPER

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