EGT2
ENGINEERING TRIPOS PART IIA

*** May 2026 2 to 3.40

Module 3F2

SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a)

(i) State space model:

$$\dot{x} = \cos(\theta)v$$
$$\dot{y} = \sin(\theta)v$$
$$\dot{\theta} = \omega$$

[15%]

(ii) Equilibrium control inputs: $v_{eq} = \omega_{eq} = 0$.
Any $x_{eq} \in \mathbb{R}$, $y_{eq} \in \mathbb{R}$, $\theta_{eq} \in [0, 2\pi)$ is an equilibrium point. [10%]

(iii) Linearized system:

$$\begin{bmatrix} \dot{\delta x} \\ \dot{\delta y} \\ \dot{\delta \theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta v \\ \delta \omega \end{bmatrix}$$

[15%]

(b)

(i) Unmodeled dynamics of the robot body. [10%]

(ii) The Laplace transform of the linearized system yields

$$\begin{bmatrix} \Delta X(s) \\ \Delta Y(s) \\ \Delta \Theta(s) \end{bmatrix} = \frac{1}{s} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta V(s) \\ \Delta \Omega(s) \end{bmatrix} \tag{1}$$

with $\Delta V(s)$ and $\Delta\Omega$ the Laplace transforms of δv and $\delta\omega$, respectively. Considering small deviations $\delta\rho$ and $\delta\lambda$ around $\rho_{eq} = 0$ and $\lambda_{eq} = 0$, we also have

$$\begin{bmatrix} \Delta V(s) \\ \Delta \Omega(s) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta R(s) \\ \Delta \Lambda(s) \end{bmatrix} \tag{2}$$

with ΔR and $\Delta\Lambda$ the Laplace transforms of $\delta\rho$ and $\delta\lambda$, respectively. The wheel dynamics yields in turn

$$\begin{bmatrix} \Delta R(s) \\ \Delta \Lambda(s) \end{bmatrix} = \frac{1}{s+a} \Delta T(s) \tag{3}$$

with $\Delta T(s)$ the Laplace transform of $\delta\tau$. Combining (1), (2), and (3) we obtain the transfer matrix

$$G(s) = \frac{1}{4s(s+a)} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{4}$$

[25%]

(iii) Substituting the proposed controller in $(\Delta X, \Delta Y, \Delta\Omega)^\top = G(s)\Delta T$, we obtain $\Delta X = 0$, $\Delta\Omega = 0$, and

$$\Delta Y(s) = H(s)K(s)(W(s) - \Delta Y(s))$$

where

$$H(s) = \frac{1}{4s(s+a)}$$

The controller can be designed based on the scalar equation above. The error $E(s) = W(s) - \Delta Y(s)$ is given by

$$E(s) = \frac{1}{1 + H(s)K(s)} W(s) = \frac{4s(s+a)}{4s(s+a) + K(s)} W(s)$$

A proportional controller $K(s) = k_p > 0$ ensures the closed-loop system is asymptotically stable (shown with Routh-Hurwitz or root locus), and the steady-state error is zero since

$$\lim_{s \to 0} sE(s) = 0$$

for constant input reference $W(s) = \bar{w}s^{-1}$ [25%]

2 (a) Since the input and output of $P(s)$ are given by $u - z$ and x_1, respectively, we can set

$$z = x_2 + 2x_3$$

and recover $P(s)$ from the first line of the state space equation:

$$\dot{x}_1 = -2x_1 - z(t) + u(t) \implies X_1(s) = \frac{1}{s+2}(U(s) - Z(s))$$

Since the input and output of $Q(s)$ are given by x_1 and z, respectively, we can recover $Q(s)$ by calculating the transfer function of the state space system

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & \alpha \end{bmatrix}}_{A} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} x_1$$

$$z = \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{C} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

We have

$$(sI - A)^{-1} = \frac{1}{(s+1)(s-\alpha) - 1} \begin{bmatrix} s - \alpha & 1 \\ 1 & s+1 \end{bmatrix}$$

and hence

$$C(sI - A)^{-1}B = \frac{3s - \alpha + 5}{s^2 + (1-\alpha)s - (1+\alpha)} = Q(s)$$

[25%]

(b) The transfer function of the full system is

$$\begin{aligned} G(s) &= \frac{P(s)}{1 + P(s)Q(s)} \\ &= \frac{s^2 + (1-\alpha)s - (1+\alpha)}{(s+2)(s^2 + (1-\alpha)s - (1+\alpha)) + (3s - \alpha + 5)} \\ &= \frac{s^2 + (1-\alpha)s - (1+\alpha)}{s^3 + \underbrace{(3-\alpha)}_{r_1} s^2 + \underbrace{(4-3\alpha)}_{r_2} s + \underbrace{(3-3\alpha)}_{r_3}} \end{aligned}$$

Apply Routh-Hurwitz:

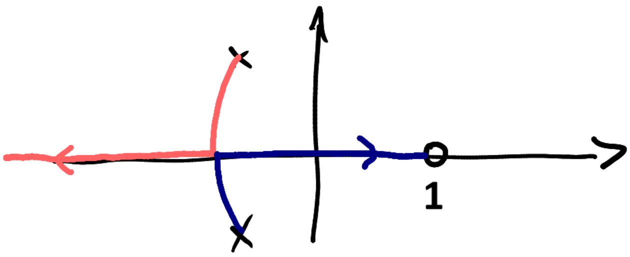
$$\begin{aligned} r_1 &= 3 - \alpha > 0 \\ r_2 &= 4 - 3\alpha > 0 \\ r_3 &= 3 - 3\alpha > 0 \\ r_1 r_2 &= (3-\alpha)(4-3\alpha) > (3-3\alpha) = r_3 \end{aligned}$$

For $\alpha < 1$ the three first conditions are immediately verified. Furthermore, by solving for the roots of

$$r_1 r_2 - r_3 = 0$$

the third condition can be verified to hold for any value of α. Hence all conditions are verified for $\alpha < 1$. [25%]

(c) (i) For $\alpha = 0.5$ the open-loop poles are $p_1 = -1.5$, $p_{2,3} = -0.5 \pm j\sqrt{0.75}$. The open-loop zeros are $z_1 = -1.5$, and $z_2 = 1$, hence there is a pole-zero cancellation. The real axis breakpoint can be found to be on $s^* = 1 - \sqrt{3}$. Resulting root locus:



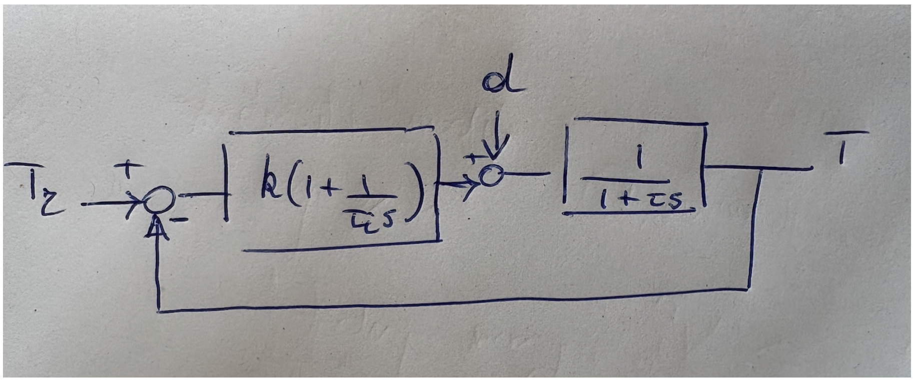
[25%]

(ii) Because of the RHP zero, one of the closed-loop poles will eventually invade the right-half complex plane. [10%]

 (cont.

(iii) We need to ensure that the two zeros remain on the left-half complex plane (LHP). The α required for this can be deduced from $G(s)$ above. Applying the Routh-Hurwitz criterion to the numerator of $G(s)$ shows that if $1 - \alpha > 0$ and $1 + \alpha < 0$ simultaneously, then the zeros will be in the LHP. Hence we require $\alpha < -1$, which also implies that the open-loop poles will remain in the LHP. [15%]

3 (a) [10%]



(b) State-space of the plant : $\dot{x}_1 = \frac{1}{\tau}(-x_1 + u),\ T = x_1$. State-space of the controller : $\dot{x}_2 = \frac{e}{\tau_i},\ u = k_p(e + x_2)$. State-space of the closed-loop system: $e = T_r - x_1$ leads to

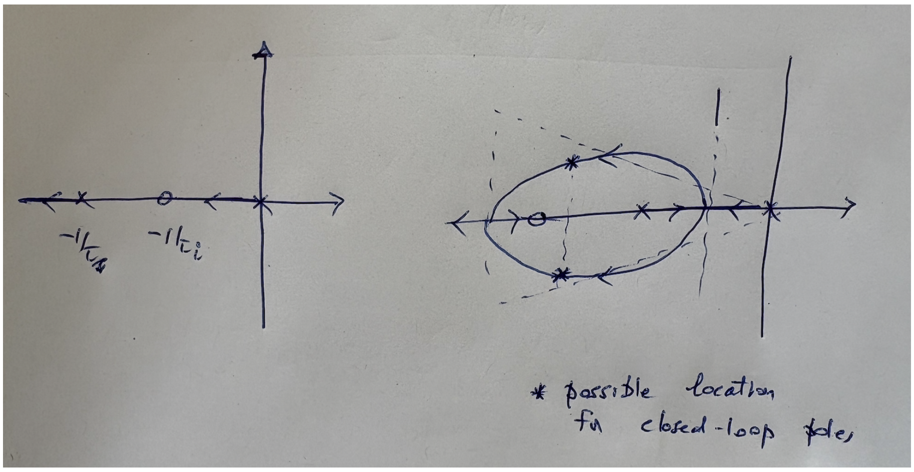
$$\begin{aligned}
\dot{x}_1 &= \frac{1}{\tau}(-x_1 - k_p(x_1 - x_2) + k_p T_r) \\
\dot{x}_2 &= \frac{1}{\tau_i}(-x_1 + T_r) \\
T &= x_1
\end{aligned}$$

Matrices are

$$A = \begin{bmatrix} -\frac{1+k_p}{\tau} & \frac{k_p}{\tau} \\ -\frac{1}{\tau_i} & 0 \end{bmatrix},\ B = \begin{bmatrix} \frac{k_p}{\tau} \\ \frac{1}{\tau_i} \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0$$

First state is output temperature T, second state is integral of output error. [20%]

(c) The two closed-loop poles should be to the left of the vertical imposed by the desired closed-loop response time, and within the cone limiting the overshoot (say to 5%). [20%]



(d) Use the controller zero $s = -\tau_i^{-1}$ to cancel the open-loop pole $s = -\tau^{-1}$, that is, choose $\tau_i = \tau$. Then the loop gain is $\frac{k_p}{\tau s}$ and the closed-loop transfer function is $\frac{k_p}{k_p + \tau s}$. Hence, choose $k_p = 5$.

This controller is a simple design that results in a first-order closed-loop step response (no overshoot). However, the disturbance rejection is slow, because the zero of the controller is a pole of the transfer function from the load disturbance to the output. [20%]

(e) Because of the pole-zero cancellation in the control design, the closed-loop system is observable but not controllable: $AB = -k_p B$. Note that the observability of the slow mode is responsible for the poor disturbance rejection, [20%]

(f) A minimal state-space realisation of the closed-loop transfer function $T = \frac{5}{5+\tau s} T_r$ is $\dot{x} = \frac{5}{\tau}(-x + T_r),\ y = T$. [10%]

4 (a) bookwork. A LTI system is controllable if for any two states x_0 and x_f, there exists a control $u(\cdot)$ defined on the time interval $[0, T]$ such that $x(0) = x_0$ and $x(T) = x_f$. A LTI system is controllable iff the controllability matrix $[B\ AB \ldots A^{n-1}B]$ has full rank. [10%]

(b) bookwork. The eigenvalues of the Gramian provide a "distance" to the loss of controllability, which is better than the "yes" or "no" answer of the rank controllability test. [20%]

(c) The matrix A is diagonal, hence $e^{A}t$ is also diagonal with elements $\exp(-a_i t)$ on the diagonal, $i = 1, 2, 3$. The matrix $BB^\top$ is the matrix filled with ones. The matrix W_t has element (i, j) equal to

$$\int_0^t e^{-(a_i+a_j)\tau} d\tau = \frac{e^{-(a_i+a_j)t} - 1}{a_i + a_j}, \ i, j = 1, 2, 3$$

[30%]

(d) The three parameters a_1, a_2, and a_3 must be different from each other for the system to be controllable. If two parameters are identical, two rows and two columns of the controllability gramian become identical. [20%]

(e) Because $a_1 \approx a_2$, the first two columns of W_t are nearly identical. This implies an eigenvalue close to zero, with corresponding eigenvector close to $[1, -1, 0]^T$. The direction that requires the largest input energy is therefore well approximated by the unit vector $\frac{1}{\sqrt{2}}[1, -1, 0]^T$. [20%]

END OF PAPER

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