EGT2
ENGINEERING TRIPOS PART IIB

Tuesday 5 May   9:30 to 11:10

Module 3F2

SYSTEMS & CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

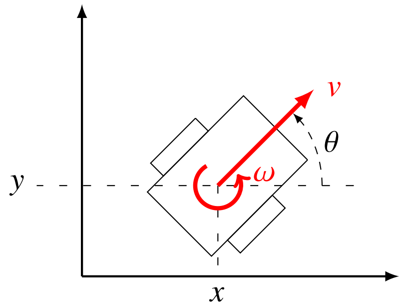
SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A two-wheeled mobile robot viewed from above is shown in the figure below. In the diagram, x and y are the Cartesian coordinates of the robot, θ is the orientation of the robot, v is the magnitude of the robot's forward velocity, and ω is the robot's angular velocity. The robot state vector is given by $(x, y, \theta)^\top$.



(a) Treating $(v, \omega)^\top$ as a control input:

(i) Derive a state space model of the robot. [15%]

(ii) What are the system's equilibrium points and corresponding inputs? [10%]

(iii) Linearize the model at an equilibrium that satisfies $\theta = 90°$. [15%]

(b) In a refined model, $(v, \omega)^\top$ is related to the robot's right and left wheel angular velocities, ρ and λ respectively, according to

$$v = \frac{1}{4}(\rho + \lambda), \quad \omega = \frac{1}{4}(\rho - \lambda)$$

The velocities ρ and λ depend in turn on the dynamics of the wheels, given by

$$J\dot{\rho} = -a\rho + \tau_1 + d_1$$
$$J\dot{\lambda} = -a\lambda + \tau_2 + d_2$$

where $a > 0$ and $J = 1$ are constant parameters, τ_1 and τ_2 are input torques, and d_1 and d_2 are load disturbances.

(i) Without equations, explain the source of the disturbance $d = (d_1, d_2)^\top$. [10%]

(ii) Consider small inputs $\tau = \delta\tau = (\delta\tau_1, \delta\tau_2)^\top$. Find the transfer matrix between $\delta\tau$ and the state $(\delta x, \delta y, \delta\theta)^\top$ of the linearized system from (a). [25%]

(iii) Let ΔY and ΔT be the Laplace transforms of δy and $\delta\tau$, respectively. Let

$$\Delta T(s) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} K(s)(W(s) - \Delta Y(s))$$

Assuming $d = 0$, propose a controller $K(s)$ that drives the state δy of the linear system to a constant reference $w(t) = \bar{w}$, $t \geq 0$, with zero steady-state error. [25%]

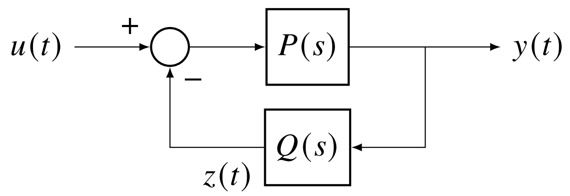
2 Consider the state space model

$$\dot{x} = \begin{bmatrix} -2 & -1 & -2 \\ 1 & -1 & 1 \\ 1 & 1 & \alpha \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

where $x = (x_1,\ x_2,\ x_3)^\top$ is the state vector, u is the input, y is the output, and α is a parameter of the model.

(a) Show that the system can be represented in terms of the feedback interconnection below,



where

$$P(s) = \frac{1}{s+2}, \qquad Q(s) = \frac{3s - \alpha + 5}{s^2 + (1-\alpha)s - (1+\alpha)}$$

[25%]

(b) Use the Routh-Hurwitz criterion to show that for $\alpha < 1$, the system is asymptotically stable. Hint: the transfer function can be obtained from $P(s)$ and $Q(s)$. [25%]

(c) A proportional controller

$$u = k(r - y)$$

is used to drive the output y towards a reference r.

(i) For $\alpha = 0.5$, sketch the root locus of the system for varying $k \geq 0$. [25%]

(ii) For $\alpha = 0.5$, explain why the system becomes unstable for a sufficiently high gain $k > 0$. [10%]

(iii) Establish a condition on α such that the resulting closed-loop system is stable for all values of $k \geq 0$. [15%]

3 A PI controller with transfer function $k(1 + \frac{1}{\tau_i s})$ is to be designed to regulate the temperature T of a thermal system modelled by a first-order transfer function $H(s) = \frac{1}{1+\tau s}$. The objective of the control design is to speed up the response time of the thermal system and to ensure good rejection of load disturbances.

(a) Draw a block diagram of the closed-loop control system, using one block for the plant, one block for the controller, and indicating the signals T (regulated temperature), T_r (reference temperature), and d (heat flow disturbance). [10%]

(b) Determine a state-space model of the closed-loop system, calculating the corresponding matrices A, B, C, D and briefly explaining the meaning of the state variables. [20%]

(c) Sketch the root locus of the closed-loop system as a function of the loop-gain parameter k and discuss how it depends on the value of the time constant $\tau_i > 0$. Considering the control design objectives, indicate on the plot your preferred location for the closed-loop poles. Justify your choice. [20%]

(d) Determine the values of the gain k and the parameter τ_i to obtain the closed-loop transfer function $T = \frac{1}{0.2\tau s+1} T_r$. Discuss the advantages and limitations of this controller design with respect to the choice you made in (c). [20%]

(e) For the choice of parameters in (d), study the controllability and the observability of the closed-loop system. Comment on your result. [20%]

(f) For the same choice of parameters as in (d), determine a minimal state-space model for the closed-loop control system. [10%]

4 (a) Define the property of controllability and provide a controllability test for a linear system with state-space realisation (A, B, C, D). [10%]

(b) The controllability Gramian W_t of a linear system is defined as

$$W_t = \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau, \ t > 0$$

Provide a characterisation of controllability in terms of the controllability Gramian. What is the benefit of this criterion with respect to the rank controllability test? [20%]

(c) Compute the controllability Gramian of a parallel bank of three RC filters with state-space model

$$\begin{aligned}\dot{x}_1 &= -a_1 x_1 + u,\\ \dot{x}_2 &= -a_2 x_2 + u,\\ \dot{x}_3 &= -a_3 x_3 + u.\end{aligned}$$

[30%]

(d) Under what conditions is the system in (c) controllable ? [20%]

(e) Suppose that $a_1 = 1$, $a_2 = 1.1$, and $a_3 = 5$. Estimate the unitary vector that requires the largest input energy to be reached starting from the origin. Justify your answer. [20%]

END OF PAPER

THIS PAGE IS BLANK