EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 6 May 2026 9.30 to 11.10

Module 3F3

STATISTICAL SIGNAL PROCESSING

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Let X_t be a random process on the integers. Starting at $X_0 = 0$, at each time the process takes a step forward with probability $\alpha > 0.5$ or a step backwards with probability $1 - \alpha$,

$$X_{t+1} = \begin{cases} X_t + 1 & \text{with probability } \alpha \\ X_t - 1 & \text{with probability } 1 - \alpha \end{cases}$$

Let $q_k(t)$ be the probability that the process reaches the point k for the first time in t steps. So, for example, $q_1(1) = \alpha$ because the process reaches 1 in the first step with probability α.

(a) Show that

(i) $q_1(2) = 0$

(ii) $q_1(3) = (1 - \alpha)\alpha^2$

Further, show that in general

$$q_1(t) = \alpha\delta_{t,1} + (1 - \alpha)q_2(t - 1)$$

[20%]

(b) Let

$$G_k(z) = \sum_{t=1}^{\infty} z^t q_k(t)$$

be the generating function for $q_k(t)$. State the formula relating $G_k(z)$ to $G_1(z)$ and justify your answer. [20%]

(c) Show that $G_1(z)$ obeys

$$G_1(z) = \alpha z + (1 - \alpha)zG_1(z)^2$$

and verify that at $z = 1$, $G_1(1) = 1$ is a solution. [30%]

(d) Using the generating functions, find the expected time it takes to progress one step forward, $\sum_{t=1}^{\infty} tq_1(t)$, and the expected time to reach k for the first time, $\sum_{t=1}^{\infty} tq_k(t)$. What happens as $\alpha \to 0.5$? [30%]

2 Consider a Markov chain on N states, labelled $1, 2, \ldots, N$. Let the matrix Q_{ij} denote the transition probability from state i to state j, and assume that the chain has a stationary distribution, $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$.

At time $t = 0$ the chain is initialised in state i with probability $p_i(0)$, and let $p_i(t)$ denote the probabity of being in state i at time t.

(a) State the:

(i) equation that relates $p_i(t+1)$ to the values $p_j(t)$;

(ii) equation that π_i must satisfy because $\boldsymbol{\pi}$ is a stationary distribution.

[20%]

(b) Let

$$d(t) = \sum_{i=1}^{N} \left| p_i(t) - \pi_i \right|$$

Show that $d(t+1) \leq d(t)$. [30%]

(c) Let f_i be any function of the states. At time t the expected value is

$$E_t[f] = \sum_{i=1}^{N} p_i(t) f_i$$

while in the stationary distribution $\boldsymbol{\pi}$ the expected value is $\mu = \sum_{i=1}^{N} \pi_i f_i$. Show that

$$\left| E_t[f] - \mu \right| \leq M d(t)$$

where M is the maximum value of $|f_i|$. [30%]

(d) Finite irreducible aperiodic Markov chains are ergodic. Comment on this fact, and contrast and compare it to the results of parts (b) and (c). [20%]

3 A wide-sense stationary discrete-time real-valued random process $\{X_n\}$ is output through a zero-order hold circuit with sampling interval T seconds so that the continuous-time output is:

$$X(t) = X_n, \;\; nT \leq t < (n+1)T\,,\; n = -\infty, ..., -1, 0, 1, ..., \infty.$$

(a) If $\{X_n\}$ has power spectrum $\mathcal{S}_X(\exp(j\Omega))$, show that the average power of $X(t)$ is

$$\frac{1}{\pi T}\int_0^{\pi} \mathcal{S}_X(\exp(j\Omega))d\Omega\,.$$

[20%]

(b) A *causal* discrete time filter with unit pulse response $\{h_n\}$, $n = 0, ..., \infty$ is applied to $\{X_n\}$. Show that the output $\{Y_n\}$ is wide-sense stationary if

$$\sum_{n=0}^{\infty} |h_n| < \infty\,.$$

[30%]

(c) Suppose that $\{X_n\}$ is passed through a digital filter satisfying the condition from part (b) above to give a new process $\{Y_n\}$. Show that the power spectrum of $\{Y_n\}$ is

$$\mathcal{S}_Y(\exp(j\Omega)) = \mathcal{S}_X(\exp(j\Omega))|G(\exp(j\Omega))|^2$$

where $G(\exp(j\Omega))$ is a function that should be defined in terms of $\{h_n\}$. [30%]

(d) The function $G(\exp(j\Omega))$ from part (c) above is now defined for $\Omega \in [-\pi, \pi]$ to be:

$$G(\exp(j\Omega)) = \begin{cases} 1, & |\Omega| \in [\Omega_l, \Omega_u) \\ 0, & \text{Otherwise} \end{cases}\,,\;\; \Omega_l < \Omega_u < \pi$$

By considering the Power Spectrum of $\{Y_n\}$ in this case, give an interpretation of the Power Spectrum as a *Power spectral-density* for discrete-time random processes. [20%]

4 A set of measurements $\{x_i\}$, $i = 1, \ldots, N$, is assumed to be drawn independently from a normal distribution, $\mathcal{N}(\mu, \sigma^2)$, but the mean, μ and variance, σ^2, are unknown and believed to take the form

$$\mu = a + bV, \quad \sigma^2 = V$$

where a and b are known constants and V is a random variable.

(a) Show that the likelihood function for V may be expressed as

$$p(\{x_i\}|V = v) = Av^p e^{-(Bv + C/v)/2}$$

and give expressions for A, B, C and p in terms of the $\{x_i\}$, N, a and b. [20%]

(b) Determine the Maximum Likelihood (ML) estimator for V. Sketch the shape of the likelihood function as a function of v, paying attention to any stationary points and the behaviour at $v = 0$. [30%]

(c) A prior distribution is assigned to V with the form

$$p(v) = \sqrt{\frac{1}{2\lambda\pi v^3}} \exp\left[-\frac{(v - c)^2}{2\lambda c^2 v}\right]$$

where c and $\lambda > 0$ are constants.

Using this prior and the likelihood function of part (a), determine the posterior density $p(v|\{x_i\})$ and find the Maximum *a posteriori* (MAP) estimator for V. You may neglect the normalising constant for the posterior density. [20%]

(d) In a new experiment, samples $\{v_i\}$, $i = 1, \ldots, M$, are drawn independently from a prior probability distribution as in part (c). Determine the ML estimators for c and λ, using only the new data $\{v_i\}$ and give an interpretation of c. [30%]

END OF PAPER

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