Version MAG/6

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 5 May 2026   14.00 to 15.40

Module 3M1

MATHEMATICAL METHODS

Answer all the questions.

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Consider a thin metal rod extending along the x-axis. The temperature distribution $T(x)$ is modeled as the superposition of two heat sources located at $x = 1$ and $x = -1$, where the source at $x = -1$ is twice as powerful as the source at $x = 1$. The parameter $\alpha \in \mathbb{R}$ represents the thermal concentration.

$$T(x) = \exp(-\alpha(x-1)^2) + 2\exp(-\alpha(x+1)^2)$$

(a) (i) Derive the necessary condition for a point $x^* \in \mathbb{R}$ to be a stationary point of the temperature distribution for any value of α. [10%]

(ii) For the case $\alpha = 1$, show that the stationary points are defined by the solutions of the non-linear equation:

$$\frac{1-x}{2(1+x)} = e^{-4x}$$

[10%]

(iii) For the necessary condition in Part (a)(i) use a second-order Taylor expansion of $T(x)$ to derive the Newton iteration scheme to find x^*. [15%]

(b) (i) Derive and justify a sufficient condition for a point $x^* \in \mathbb{R}$ to be a **strong local maximum** (a local hot spot). [10%]

(ii) For $\alpha = 1$, explain, without solving the non-linear equation, which of the non-zero stationary points corresponds to the **global maximum** temperature. [10%]

(iii) For $\alpha = 1$, using the derived Newton iteration scheme, perform one manual iteration starting from the stronger source center to demonstrate whether the global maximum lies to the left or right of the source center, and provide a physical justification for this shift. [15%]

(c) (i) Determine the range of values for α for which the temperature distribution $T(x)$ is **convex** for all $x \in \mathbb{R}$. [20%]

(ii) State the implication of this convexity on the nature of any stationary point x^*. [10%]

2 (a) (i) Sketch the unit ball in $\mathbb{R}^2$ under the ℓ_2 and ℓ_∞ norms. [10%]

(ii) For $x \in \mathbb{C}^n$, show that

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty.$$

[10%]

(iii) Let $A \in \mathbb{C}^{n\times n}$. There exists constants $c > 0$, $C > 0$ such that

$$c\|A\|_\infty \leq \|A\|_2 \leq C\|A\|_\infty.$$

Using the result from Part (a)(ii), find expressions for c and C in terms of n. [20%]

(b) Let $A \in \mathbb{C}^{n\times n}$. The singular value decomposition (SVD) of A is

$$A = U\Sigma V^H,$$

where $U, V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{R}^{n\times n}$ is diagonal with non-negative entries (the singular values).

(i) Explain geometrically what the SVD says about the linear transformation represented by A. Your answer should refer specifically to the properties of V, Σ and U, and the action of these matrices on a vector. [20%]

(ii) The polar decomposition of A is given by

$$A = QP,$$

where $Q = UV^H$ is unitary and P is Hermitian positive-semidefinite. Give an interpretation of this decomposition. [5%]

(iii) For Q as defined in Part (b)(ii), show that

$$\|A - Q\|_F = \|\Sigma - I\|_F \quad \text{and} \quad \|A - W\|_F = \|\Sigma - U^H W V\|_F$$

for all unitary matrices $W \in \mathbb{C}^{n\times n}$. Hence, show that

$$\|A - Q\|_F \leq \|A - W\|_F$$

and give an interpretation of this result.
Note: The Frobenius norm is invariant under a unitary operation. The following equality may be useful:

$$\|B - C\|_F^2 = \|B\|_F^2 + \|C\|_F^2 - 2\,\Re\left(\mathrm{tr}\left(C^H B\right)\right),$$

where $B, C \in \mathbb{C}^{n\times n}$ and $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. [35%]

3 For an experiment it is necessary to compute the expected value of a function. The expected value, μ, can be expressed as

$$\mu = \int h(\boldsymbol{x})p(\boldsymbol{x})d\boldsymbol{x}$$

where $h(\boldsymbol{x})$ is the value of the function for point $\boldsymbol{x}$ and $p(\boldsymbol{x})$ is the probability of the point $\boldsymbol{x}$. It is possible to obtain the value for the function $h(\boldsymbol{x})$ for any value of $\boldsymbol{x}$.

(a) Initially it is assumed that it is possible to draw samples from $p(\boldsymbol{x})$. N samples are drawn from $p(\boldsymbol{x})$, $\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N)}\}$.

(i) Give the expression for the Monte-Carlo approximation of the value of μ, $\hat{\mu}_N$, in terms of the N samples, $\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N)}\}$. [10%]

(ii) Show that the estimate $\hat{\mu}_N$ using N samples is unbiased, i.e. the expected value of $\hat{\mu}_N$ is μ, and the variance of the estimate, $\text{var}(\hat{\mu}_N)$, can be expressed as

$$\text{var}(\hat{\mu}_N) = \frac{\sigma^2}{N}; \quad \text{where } \sigma^2 = \int (h(\boldsymbol{x}) - \mu)^2 \, p(\boldsymbol{x})d\boldsymbol{x}$$

Comment on this result as the dimensionality of $\boldsymbol{x}$ increases. [30%]

(b) It is actually not possible to sample from $p(\boldsymbol{x})$. Instead N samples are to be drawn from a distribution $q(\boldsymbol{x})$, $\{\tilde{\boldsymbol{x}}^{(1)}, \ldots, \tilde{\boldsymbol{x}}^{(N)}\}$. However, it is not possible to compute the normalisation terms Z_{p} and Z_{q} for $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ respectively where

$$p(\boldsymbol{x}) = \frac{1}{Z_{\text{p}}} p^{\star}(\boldsymbol{x}), \quad q(\boldsymbol{x}) = \frac{1}{Z_{\text{q}}} q^{\star}(\boldsymbol{x})$$

It is possible to compute $p^{\star}(\boldsymbol{x})$ and $q^{\star}(\boldsymbol{x})$ for any value of $\boldsymbol{x}$.

(i) Show that a new approximation for the integral, $\tilde{\mu}_N$, can be expressed as

$$\tilde{\mu}_N = \sum_{i=1}^{N} h(\tilde{\boldsymbol{x}}^{(i)}) w^{(i)} \Bigg/ \sum_{j=1}^{N} w^{(j)}$$

and give the expression for $w^{(i)}$. You should include a discussion how samples can be drawn from $q(\boldsymbol{x})$ even though Z_{q} cannot be computed. Hint: consider the expected value of the ratio of the unnormalised distributions. [35%]

(ii) What condition must be satisfied for $\tilde{\mu}_N$ to converge to μ as $N \rightarrow \infty$? [10%]

(iii) Do you expect the estimate $\tilde{\mu}_N$ to be unbiased when the condition in Part b(ii) is satisfied? Clearly motivate your answer. Hint: in general the expected value of a ratio is not equal to the ratio of the numerator and denominator expected values. [15%]

END OF PAPER