CRIB for Year 2025-2026 Part IIA Paper 3C9

Fracture mechanics of materials and structures

1. (a) The end displacement is $u = \dfrac{P\left(h + l)\right)}{\pi a^2 E}$.

Hence, the compliance is $C = \dfrac{u}{P} = \dfrac{\left(h + l)\right)}{\pi a^2 E}$

(b) The potential energy Ψ for a prescribed load P is given by the strain energy $SE = \dfrac{1}{2} Pu$ and the potential energy $-Pu$ of the loading system, such that

$$\Psi = \frac{1}{2} Pu - Pu = -\frac{1}{2} Pu = -\frac{1}{2} CP^2$$

Now advance the crack by an area of $\delta A = 2\pi a \left(\delta l\right)$.

Then, the energy released is

$$G\left(\delta A\right) = -\left(\delta \Psi\right), \text{ to give } \qquad G 2\pi a \left(\delta l\right) = \frac{1}{2} P^2 \left(\delta C\right)$$

And thereby $G = \dfrac{P^2}{4\pi a} \dfrac{\partial C}{\partial l}$

(c) Now make use of the expression for C as derived in part (b) to get

$$G = \frac{P^2}{4\pi a} \frac{\partial C}{\partial l} = \frac{P^2}{4\pi a} \frac{1}{\pi a^2 E} = \frac{P^2}{4\pi^2 a^3 E}$$

(d) In the absence of friction the debond load is $P = \left(4\pi^2 a^3 E G_c\right)^{1/2}$, and is independent of the extension Δl. If friction is present, an increase in the load by $2\pi a \tau_f \left(\Delta l\right)$ is required such that the tensile load is

$$P = \left(4\pi^2 a^3 E G_c\right)^{1/2} + 2\pi a \tau_f \left(\Delta l\right)$$

2. (a) The Coffin-Manson Law according to the Materials Data-book is

$$\Delta \varepsilon^{\mathrm{p}} N_f{}^{\beta} = C_2$$

where β and C_2 are constants. From given data:

$$0.02(500)^{\beta} = C_2 \quad \text{and} \quad 0.005(2000)^{\beta} = C_2$$

Subtract one equation by the other to get: $0.02(500)^{\beta} = 0.005(2000)^{\beta}$

Hence, $\beta = 1$ and consequently $C_2 = 10$

(b) The cyclic stress strain curve is sketched below, for reference. The remote field is elastic, but yield is exceeded at the notch root since $k_T \dfrac{\Delta\sigma}{2} > \sigma_Y$. This is evident from:

$k_T \dfrac{\Delta\sigma}{2} = 3 \times 250\text{MPa} = 750\text{MPa}$ and $\sigma_Y = 300\text{MPa}$.

Consequently, apply Neuber's rule: $k_\sigma k_\varepsilon = k_T{}^2$ where $k_\sigma = \Delta\sigma / \Delta\sigma^\infty$, $k_\varepsilon = \Delta\varepsilon E / \Delta\sigma^\infty$ and $k_T = 3$.

Rephrase $\dfrac{\Delta\varepsilon}{2} = \dfrac{\sigma_Y}{E} + \dfrac{1}{E_T}\left(\dfrac{\Delta\sigma}{2} - \sigma_Y\right)$ into the form $\dfrac{\Delta\sigma}{2} = \sigma_Y - \dfrac{E_T\sigma_Y}{E} + E_T\dfrac{\Delta\varepsilon}{2}$

and introduce $\sigma^* = \left(\dfrac{E - E_T}{E}\right)\sigma_Y$ so that $\dfrac{\Delta\sigma}{2} = \sigma^* + E_T\dfrac{\Delta\varepsilon}{2}$

Now substitute into Neuber's rule $k_\sigma k_\varepsilon = k_T{}^2$

where $k_\sigma = \dfrac{\Delta\sigma}{\Delta\sigma^\infty} = \dfrac{2\sigma^*}{\Delta\sigma^\infty} + \dfrac{E_T\Delta\varepsilon}{\Delta\sigma^\infty}$ and $k_\varepsilon = \Delta\varepsilon E / \Delta\sigma^\infty$. Thus,

$\left(\dfrac{2\sigma^*}{E_T} + \Delta\varepsilon\right)\Delta\varepsilon = 9\dfrac{\left(\Delta\sigma^\infty\right)^2}{EE_T}$ which can be rephrased to $\left(\dfrac{\sigma^*}{E_T} + \Delta\varepsilon\right)^2 = 9\dfrac{\left(\Delta\sigma^\infty\right)^2}{EE_T} + \left(\dfrac{\sigma^*}{E_T}\right)^2$, and consequently

$$\Delta\varepsilon = -\frac{\sigma^*}{E_T} + \left[\left(\frac{\sigma^*}{E_T}\right)^2 + 9\frac{\left(\Delta\sigma^\infty\right)^2}{EE_T}\right]^{1/2}$$

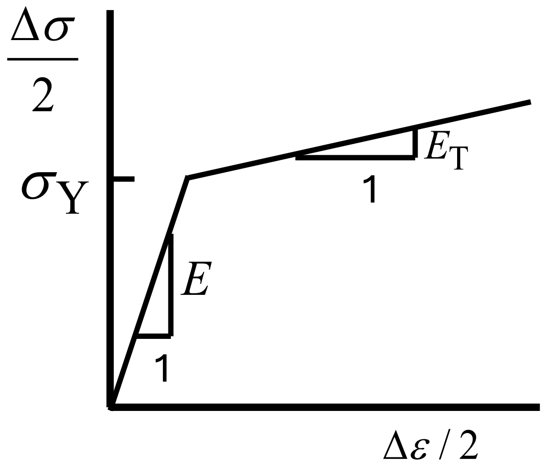
Now substitute values:

$\sigma^* = \left(\dfrac{E - E_T}{E}\right)\sigma_Y = \left(\dfrac{70-7}{70}\right)300\text{ MPa} = 270\text{ MPa}$, and so

$$\Delta\varepsilon = -\frac{270}{7000} + \left[\left(\frac{270}{7000}\right)^2 + 9\frac{(500)^2}{70000\text{x}7000}\right]^{1/2} = 0.0394$$

The corresponding stress range at the notch root is $\Delta\sigma = 2\sigma^* + E_T\Delta\varepsilon$ = 816 MPa and so the plastic strain range is $\Delta\varepsilon^p = \Delta\varepsilon - (\Delta\sigma / E)$ = 0.02774.

Now substitute this value of plastic strain range into the Coffin-Manson law $\Delta\varepsilon^p N_f{}^{\beta} = C_2$ to obtain a fatigue life N_f of $0.02774 N_f{}^1 = 10$, thereby N_f = 360 cycles



(c) Cyclic plasticity at the notch root leads to relaxation of mean stress by the phenomenon of ratchetting. There is a progressive drift in mean strain of the same sign at that of the mean stress. Thus, the initial stress cycle at the notch root has a compressive mean value, but ratchetting of the plastic strain at the notch root progressive reduces this mean stress to zero. Consequently, the Coffin-Manson law, as derived for plastic cycling at zero mean stress, suffices for determining the initiation life, with no need to correct for mean stress effects.

3. (a) Welded pressure vessels contain crack-like defects that can grow by fatigue. The steels have a high toughness and only moderate strength. Consequently, the crack length *a* remains much below the transition flaw size at fracture a_T where

$$a_T = \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma_Y}\right)^2$$

The extent of plasticity ahead of the crack tip at fracture is sufficient for net section yielding to occur and for no elastic *K* field to exist. However, the stress and strain fields at the crack tip can be characterized in terms of a unique singular field of magnitude given by the *J*-integral. Thus, failure occurs when *J* attains a critical toughness value J_{IC}. The value of *J* scales with the remote applied stress σ and remote applied strain ε approximately as $J = \sigma\,\varepsilon\,a$. The toughness J_{IC} is related to the fracture toughness K_{IC} by the usual Irwin relation, $EJ_{IC} = K_{IC}{}^2$, where *E* is Young's modulus. Note that the toughness J_{IC} drops with increasing radiation and is also temperature sensitive for ferritic steels, undergoing a ductile-brittle transition.

(b) There is a major physical difference between strength and toughness.

Metallic alloys contain dislocations that move and give rsie to plastic flow at a yield strength much below the ideal cohesive strength. The crack tip of a metallic alloy blunts by dislocation emission at the crack tip, and this prevents cleavage from occurring. Crack advance involves void growth in the plastic zone from pre-existing inclusions, and so the cleaner is the steel, the higher is the toughness.

In contrast, freshly drawn silica glass is amorphous and so contains no dislocations. In the freshly drawn state it has only defects such as cracks on the atomic scale. Consequently, it fails by cleavage when the applied tensile strength attains the ideal cohesive strength. No dislocation emission occurs from pre-existing defects and so failure is by cleavage.

(c) The stress concentration factor is 3 at the edge of a hole, and is independent of hole diameter *D*. This stress concentration factor may be sufficient to induce cyclic plasticity in a small region at the edge of the hole, and consequently fatigue crack initiation. As the fatigue crack grows it enters the outer elastic region from the edge of the hole, and a cyclic K-field then exists at the crack tip. The crack will grow when $\Delta K > \Delta K_{TH}$, the fatigue threshold stress intensity range. Note that $\Delta K > \Delta K_{TH}$ scales with the remote cyclic stress $\Delta\sigma^{\infty}$ according to $\Delta K = \Delta\sigma^{\infty}\sqrt{\pi D/2}$, and so the crack will arrest and become non-propagating when

$$\Delta\sigma^{\infty} < \frac{\Delta K_{TH}}{\sqrt{\pi D/2}}$$

Thus, small holes are much more likely to give rise to non-propagating cracks than large holes.

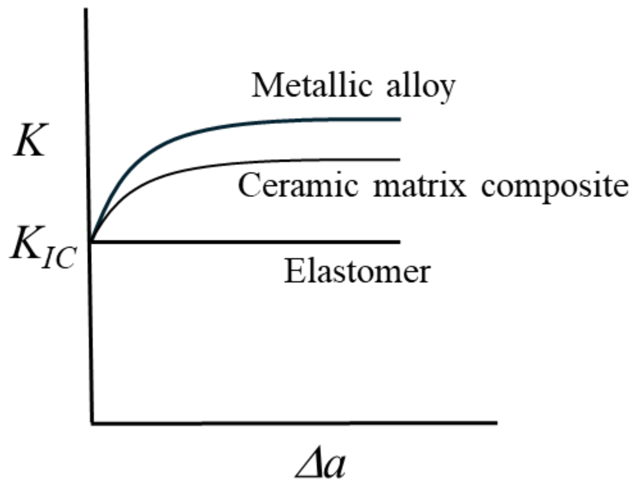
(d) Consider a long crack under an imposed K-field that scales with the remote stress. A random overload in remote stress gives rise to a peak overload K_{OL}. Upon unloading from this peak overload, a reversed plastic zone forms of extent $d = \frac{1}{4\pi}\left(K_{OL}/\sigma_Y\right)^2$, where σ_Y is the yield strength. This reversed plastic zone imposes a compressive residual stress of yield strength magnitude on the crack flanks as the crack tip grows into it, and consequently a negative value of stress intensity factor is generated, $K_{res} < 0$, by this compressive plastic zone.

The stress intensity factor *K* at the crack tip due to the remote stress ranges from $K_{\min}^{\infty}$ to $K_{\max}^{\infty}$, and upon including the effect of the residual compressive stress the net value of *K* ranges from $\left(K_{\min}^{\infty}+K_{res}\right)$ to $\left(K_{\max}^{\infty}+K_{res}\right)$. If $\left(K_{\min}^{\infty}+K_{res}\right)$<0 then the nett stress intensity factor ranges from 0 to $\left(K_{\max}^{\infty}+K_{res}\right)$. Consequently, crack growth retardation occurs until the crack grows a distance beyond the reversed plastic zone associated with the overload.

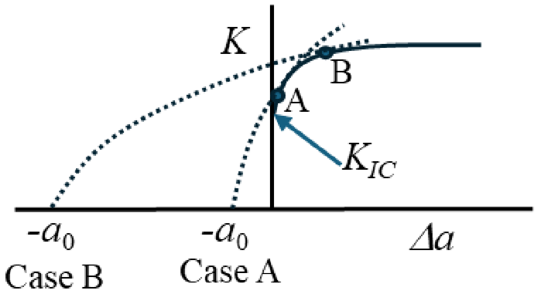
4. (a) The fracture toughness of an elastomer is dictated by the surface energy of the solid along with crack tip blunting. There are no crack-bridging mechanisms or additional damage mechanisms near the crack tip that can lead to an R-curve, see the sketch below.

In contrast, a ceramic-matric composite may contain particles or fibres. When a crack advances in the ceramic matrix, the particles or fibres may survive and thereby bridge the crack. The tensile tractions in the bridging zone is balanced by an increase in the remote K. Consequently, an R-curve is observed. The steady-state length of the bridging zone is dictated by failure of the bridging fibres/particles. The R-curve saturates when the bridging zone has achieved its steady-state length.

Metallic alloys fail by microvoid growth and/or cleavage in the vicinity of the crack tip, with an outer plastic zone. Non-proportional loading leads to plastic hysteresis in the plastic zone and leads to the R-curve.



(b) Now consider a centre-crack of initial crack length 2a_0, and current crack length $2\left(a_0+\Delta a\right)$ subjected to a remote stress σ^{∞}. The stress intensity factor is $K=\sigma^{\infty}\sqrt{\pi\left(a_0+\Delta a\right)}$. Write $K_R\left(\Delta a\right)$ as the R-curve of the material. Then, under increasing remote stress crack advance occurs by writing $K=\sigma^{\infty}\sqrt{\pi\left(a_0+\Delta a\right)}=K_R\left(\Delta a\right)$. The peak value of remote stress is achieved when the curve $K=\sigma^{\infty}\sqrt{\pi\left(a_0+\Delta a\right)}$ no longer intersects the $K_R\left(\Delta a\right)$ curve, and this occurs when $\partial K/\partial\left(\Delta a\right)=\partial K_R\left(\Delta a\right)$, the so-called point of tangency. Now consider the 2 cases of a short initial crack, case A, and a long initial crack, case B as shown below. The *K*-value at the point of tangency of the $K = \sigma^{\infty}\sqrt{\pi\left(a_0 + \Delta a\right)}$ curve and the R-curve $K_R\left(\Delta a\right)$ is marked as A for case A and B for case B in the figure. The K-value for the short crack, case A, is lower than that for the long crack, case B.



(c) At the root of the cantilever the tensile stress at the outermost fibre is $\sigma_{\max} = P\ell t/\left(2EI\right)$.

Also, for a beam of square cross-section t x t, the second moment area is $I = t^4/12$

Hence, $\sigma_{\max} = 6P\ell/\left(Et^3\right)$.

Thus, the stress range is $\Delta\sigma = \sigma_{\max}$ and the mean stress is $\sigma_{\mathrm{m}} = \sigma_{\max}/2$.

Now make use of Goodman's law as stated in the databook:

$\Delta\sigma_0 = \Delta\sigma\left(1 - \dfrac{\sigma_m}{\sigma_{ts}}\right)^{-1}$ where σ_{ts} is the ultimate tensile strength of the solid,

and $\Delta\sigma_0$ is the stress range for fully reversed loading that gives the same fatigue life.

Thus, $\Delta\sigma_0 = \sigma_{\max}\left(1 - \dfrac{\sigma_{\max}}{2\sigma_{ts}}\right)^{-1}$ where $\sigma_{\max} = 6P\ell/\left(Et^3\right)$

Finally, Basquin's law states: $N_f = \left(C_1/\Delta\sigma_0\right)^{1/\alpha}$ which can be rephrased to

$N_f = \left(\left(1 - \dfrac{\sigma_{\max}}{2\sigma_{ts}}\right)C_1/\sigma_{\max}\right)^{1/\alpha}$ in terms of $\sigma_{\max} = 6P\ell/\left(Et^3\right)$.