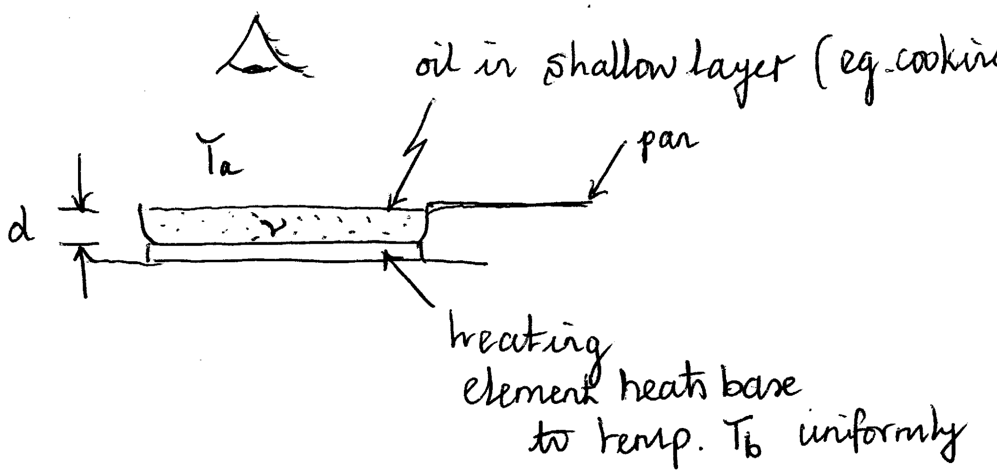
1 GRH

4A10 CRIB

(a) (i)

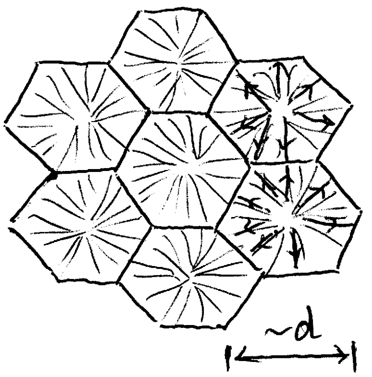


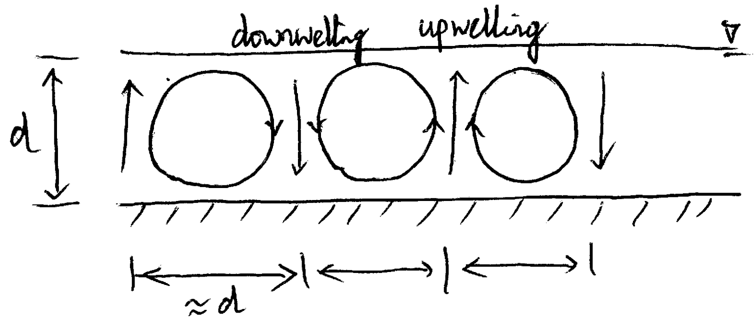
Sequentially increase T_b until structures form.
Visualise structures using eg. pepper/other powder to seed the fluid/surface

[10%]

(ii). from above: from side:

— regular cellular structure —





[10%]

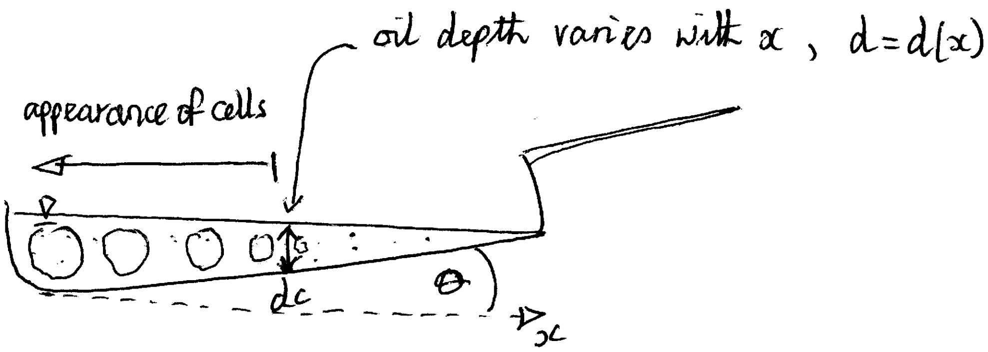
(iii).

Rayleigh number $Ra = \dfrac{g\beta(T_b - T_a)d^3}{\kappa\nu}$

β coeff. thermal expansion.
g accel^n due to gravity.
κ thermal diffusivity of cooking oil
ν kinematic viscosity of oil.

Experiment: heat an inclined pan (best option) or sequential increase T_b in arrangement in (i) (not such a good option).

1
(a) (iii) cont^d



- Consequently Rayleigh number varies with x.
- Given cellular structure (onset of convection) only appears for $Ra > Ra_c$, cells appear only for $d \geq d_c$
- Estimate critical Rayleigh no as $Ra = g\dfrac{\beta(T_b - T_a)\, d_c^3}{\kappa\nu}$
- Expect $Ra_c \approx 1710$

[10%]

(b) (i) We are given that the null solution is $W = \Theta = 0$.
Perturb system about this by letting

$$W = 0 + w' \qquad \Theta = 0 + \theta'$$

for small amplitude perturbations w', θ'

Thus $$\frac{dw'}{dt} = -\frac{\nu}{d^2} w' + \alpha g \theta' , \quad \frac{d\theta'}{dt} = \frac{-\kappa}{d^2}\theta' + \beta w'$$

We are instructed to seek normal mode sol^{ns} $w', \theta' \propto e^{st}$
hence

$$s\hat{w} = -\frac{\nu}{d^2}\hat{w} + \alpha g \hat{\theta} , \quad s\hat{\theta} = -\frac{\kappa}{d^2}\hat{\theta} + \beta \hat{w} \quad \text{—(1,2)}$$

where $w' = \hat{w} e^{st}$, $\theta' = \hat{\theta} e^{st}$ for constants $\hat{w}, \hat{\theta}$.

Now solve for growth rate s:

First, combining (1) & (2) gives

$$-\frac{\nu}{d^2} + \alpha g \lambda = -\frac{\kappa}{d^2} + \beta\lambda^{-1} \quad \text{where } \lambda = \frac{\hat{\theta}}{\hat{w}} = \text{const.}$$

$$\Rightarrow \quad (g\alpha)\lambda^2 + \left(\frac{\kappa - \nu}{d^2}\right)\lambda - \beta = 0$$

1

(b) (i) cont^d which has roots $\lambda = -\frac{1}{2\alpha g}(\kappa-\nu)d^{-2} \pm \frac{1}{2\alpha g}\left[(\kappa-\nu)^2 d^{-4} + 4\alpha g\beta\right]^{1/2}$

Finally, from (i), given $s = -\nu d^{-2} + (g\alpha)\lambda$

$$\underline{s = -\frac{1}{2}(\nu+\kappa)d^{-2} \pm \frac{1}{2}\left[(\nu-\kappa)^2 d^{-4} + 4\alpha g\beta\right]^{1/2}} \text{ as req}^d.$$

[40%]

(ii) The larger root is $s_1 = -\frac{1}{2}(\nu+\kappa)d^{-2} + \frac{1}{2}\left[(\nu-\kappa)^2 d^{-4} + 4\alpha g\beta\right]^{1/2}$

– " – smaller – " – $s_2 = -\frac{1}{2}(\nu+\kappa)d^{-2} - \frac{1}{2}\left[(\nu-\kappa)^2 d^{-4} + 4\alpha g\beta\right]^{1/2}$

Smaller root always gives $s_2 < 0$ (& clearly the root is real)

For larger root to be negative, we require:

$$\frac{1}{2}\left[(\nu-\kappa)^2 d^{-4} + 4\alpha g\beta\right]^{1/2} < \frac{1}{2}(\nu+\kappa)d^{-2}$$

ie. $\underline{R = \frac{\alpha\beta g d^4}{\kappa\nu} < 1}$ for stability

[10%]

(iii) No dissipation requires $\nu = \kappa = 0 \Rightarrow s = \pm[\alpha g\beta]^{1/2}$

& hence $w' = \hat{w}\left(e^{+(\alpha g\beta)^{1/2}t} + e^{-(\alpha g\beta)^{1/2}t}\right)$

$$\Rightarrow \underline{\frac{d^2 w'}{dt^2} + N^2 w' = 0} \quad \text{with } N^2 = -\alpha g\beta.$$

[20%]

GMH

2

(a) Introducing $\underset{\sim}{u} = 0 + \underset{\sim}{u}'$, $\phi = \Phi + \phi'$, $\rho = R + \rho'$ and
$\underline{p = p(R) + p'}$

Poisson eq^n becomes $\nabla^2(\Phi + \phi') = \cancel{\nabla^2\Phi} + \nabla^2\phi' = \cancel{4\pi G R} + 4\pi G \rho'$

ie $\underline{\nabla^2\phi' = 4\pi G\rho'}$ -(1) (already linear)

Continuity eq^n becomes $\frac{\partial}{\partial t}(R+\rho') + \nabla\cdot([R+\rho'](\underset{\sim}{u}')) = 0$, $R = R(\underset{\sim}{x})$

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot(R\underset{\sim}{u}') + \cancel{\nabla\cdot(\rho'\underset{\sim}{u}')} = 0$$

on linearising

$$\underline{\frac{\partial \rho'}{\partial t} + \nabla\cdot(R\underset{\sim}{u}') = 0} \quad -(2)$$

Eq^n of state:

$$p = p(\rho) \longrightarrow p(R) + p' = p(R + \rho')$$

expand using Taylor series gives

$$p(R) + p' = p(R) + \frac{dp(R)}{dR}\rho' + \cancel{\frac{1}{2!}\frac{d^2p(R)}{dR^2}\rho'^2 + \ldots}$$

on linearising

so $\underline{p' = c^2\rho'}$ where $c^2 = \frac{dp(R)}{dR}$

Euler in base state reduces to (noting $\underset{\sim}{u} = 0$)

$$\underline{0 = \nabla p(R) - R\nabla\Phi}$$

Finally, perturbing Euler eq^{n}s

$$(R+\rho')\left(\frac{\partial \underset{\sim}{u}'}{\partial t} + \cancel{\underset{\sim}{u}'\cdot\nabla\underset{\sim}{u}'}\right) = -\nabla p(R) - \nabla p' - (R+\rho')\nabla(\Phi + \phi')$$

on linearising
$$R\frac{\partial \underset{\sim}{u}'}{\partial t} = \underbrace{-\nabla p(R) - R\nabla\Phi}_{=0} - \nabla p' - R\nabla\phi' - \rho'\nabla\Phi$$

& from linearised eq^n of state $\nabla p' = c^2\nabla\rho'$ gives

$$\underline{R\frac{\partial \underset{\sim}{u}'}{\partial t} = -c^2\nabla\rho' - \rho'\nabla\Phi - R\nabla\phi'} \quad (3)$$

[50%]

2

(b) Challenge is now to solve perturbed system of eq^{n}s for s on introducing normal modes.

First, taking $\partial/\partial t$ (2) and sub. for $R\,\partial \underline{u}'/\partial t$ from (3) enables eq^{n}s to be combined & expressed solely in terms of ρ':

$$\frac{\partial^2 \rho'}{\partial t^2} + \nabla.\left(R\frac{\partial \underline{u}'}{\partial t}\right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 \rho'}{\partial t^2} + \left[-c^2\nabla^2\rho' - \rho\nabla^2\Phi - R\nabla^2\phi'\right] = 0$$

From Poisson's eq^{n} $\nabla^2\Phi = 4\pi G R$ [but will also give credit if candidates acknowledge $\Phi = \text{const}$ & thus $\nabla^2\Phi = 0$ — the famous "Jeans" swindle]

& $\nabla^2\phi' = 4\pi G\rho'$

Thus
$$\frac{\partial^2 \rho'}{\partial t^2} - c^2\nabla^2\rho' - 0 - 4\pi R G\rho' = 0$$

Introducing $\rho' = \hat{\rho}\, e^{i\underline{k}.\underline{x} + st}$ gives

$$s^2 - c^2(i\underline{k})^2 - 4\pi R G = 0$$

$$\text{ie } \underline{s^2 = 4\pi G R - c^2\underline{k}^2}$$ as req^{d}.

[40%]

(c) System stable for

$$c^2\underline{k}^2 < 4\pi G R$$

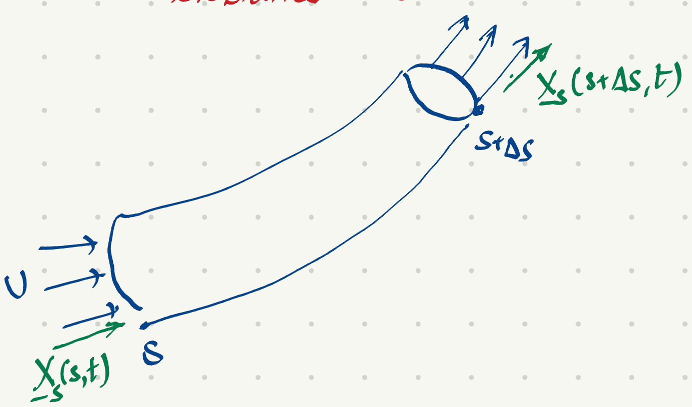
$$\underline{\underline{k} < \frac{\sqrt{4\pi G R}}{c}}.$$ ie for sufficiently long wave disturbances.

[10%]

4A10 - Flow instabilities - crib sheet.

Shreyas Mandre

Q3:



(a) To derive the centreline equation, use Newton's second law.
Consider an element of length Δs. The net force arises from the tension as

$$\text{Net force} = T\frac{\partial \underline{X}}{\partial s}\Big|_s^{s+\Delta s} \approx \Delta s \frac{\partial}{\partial s}\left(T\frac{\partial \underline{X}}{\partial s}\right) + \dots \qquad \text{(1 mark)}$$

This force equals rate of change of momentum, and because there is no accumulation of mass,

rate of change of momentum = mass of conduit × acceleration of conduit
+ mass of fluid × acceleration of fluid.

The first term is $\lambda \cdot \frac{\partial^2 \underline{X}}{\partial t^2}\Delta s$. For the second term, the fluid mass is $\rho A \Delta s$, (1 mark)
and the fluid acceleration is $\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)^2 \underline{X}$. (Here we must use the material derivative.) (2 marks)

Putting it all together, $\lambda \frac{\partial^2 \underline{X}}{\partial t^2} + \rho A\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)^2 \underline{X} = \frac{\partial}{\partial s}\left(T\frac{\partial \underline{X}}{\partial s}\right)$. (1 mark)

Expanding the material derivative yields the required result.

The left hand side is the inertia of the conduit + fluid and the right hand side is the elastic force. (1 marks)

(b) Inextensibility implies $\left|\frac{\partial \underline{X}}{\partial s}\right|^2 = 1$, 2 marks

(c) If $\underline{X} = s\hat{x}$ and $T = T_0$, then lhs $= 0$ and $\underline{X}_s = \hat{x} =$ constant,

so $(T\underline{X}_s)_s = (T_0\hat{x})_s = 0$ and $\underline{X}_{ss} = \hat{x}_s = 0$ so rhs $= 0$. 3 marks

(d) linear perturbation $\underline{X} = s\hat{x} + \underline{X}'$ and $T = T_0 + T'$. 1 mark

$$\lambda \frac{\partial^2 \underline{X}'}{\partial t^2} + \rho A\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)^2 \underline{X}' = T_0\frac{\partial^2 \underline{X}'}{\partial s^2} + T'\frac{\partial (s\hat{x})}{\partial s}$$

1 mark

$\frac{\partial^2 \underline{X}'}{\partial s^2}$ is in the direction of centre of curvature, so normal to $\hat{x}$.

Inextensibility says, $(\hat{x} + \underline{X}'_s)\cdot(\hat{x} + \underline{X}'_s) \approx 1 \Rightarrow 2\hat{x}\cdot\underline{X}'_s = 0$

$\Rightarrow$ $\underline{X}'_s$ is in the $\hat{y}$ direction. $\Rightarrow$ we can take $\underline{X}' = Y(s)\,\hat{y}$. 2marks

$$\Rightarrow 2\rho A U \frac{\partial^2 Y}{\partial t \partial s}\hat{y} + (\rho A + \lambda)\frac{\partial^2 Y}{\partial t^2}\hat{y} = T'\hat{x} + (T_0 - \rho U^2 A) Y_{ss}\,\hat{y}.$$

$\Rightarrow$ Decomposing along x and y yields $T' = 0$ and

$$2\rho A U \frac{\partial^2 Y}{\partial t \partial s} + (\rho A + \lambda)\frac{\partial^2 Y}{\partial t^2} = (T_0 - \rho U^2 A)\frac{\partial^2 Y}{\partial s^2}$$

2 marks

(e) Assuming $Y(s,t) \sim e^{i\omega t} e^{iks}$ 2 marks

$+2\rho A U \omega k + (\rho A + \lambda)\,\omega^2 = +(T_0 - \rho U^2 A)\,k^2$ ... quadratic in ω with real coeffs.
Instability when ω is complex.

$\Rightarrow$ Instability when discriminant is -ve, i.e.

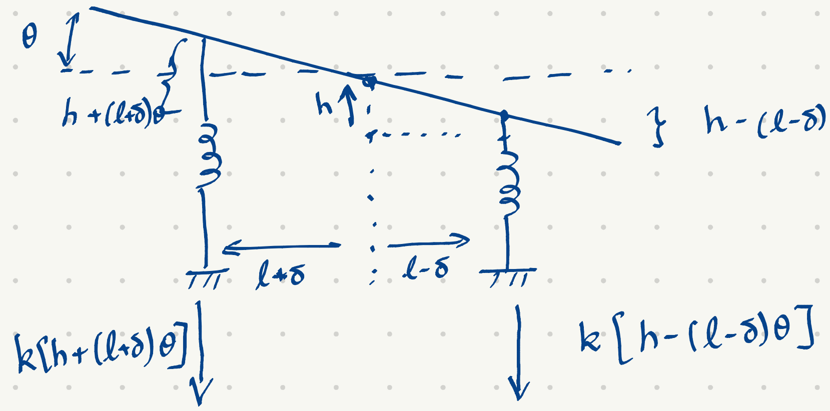
Discriminant $= 4\rho^2 A^2 U^2 k^2 + 4(\rho A + \lambda)(T_0 - \rho U^2 A)k^2$

$= \cancel{4\rho^2A^2U^2k^2} + 4(\rho A + \lambda)T_0 k^2 - \cancel{4\rho^2A^2U^2k^2} - 4\lambda\rho A U^2 k^2 = 4k^2\left[T_0(\rho A + \lambda) - \rho A \lambda U^2\right]$

$$\Rightarrow \quad \omega = \frac{-\rho A U k \pm k\sqrt{T_0(\rho A + \lambda) - \rho A \lambda U^2}}{(\rho A + \lambda)}$$ 4 marks

$$\boxed{\Rightarrow \quad U^2 > \left(\frac{\rho A + \lambda}{\rho A \lambda}\right) T_0 = T_0\left(\frac{1}{\lambda} + \frac{1}{\rho A}\right)}$$ 2marks

Q4.



(a) Net vertical force (along y)

$$= -k[h-(\ell-\delta)\theta] - k[h+(\ell+\delta)\theta]$$

$$= -2kh - 2k\delta\theta$$ 1 mark

Net clockwise torque.

$$= k[h-(\ell-\delta)\theta](\ell-\delta) - k[h+(\ell+\delta)\theta](\ell+\delta)$$

$$= -k(\ell-\delta)^2\theta - k(\ell+\delta)^2\theta - 2kh\delta$$

$$= -2k(\ell^2+\delta^2)\theta - 2kh\delta.$$ 1 mark

$\Rightarrow$ Equations of motion:

$$m\frac{d^2h}{dt^2} = -2kh - 2k\delta\theta + \gamma_\mu UL\theta$$ 1 mark

$$m(\ell^2+\delta^2)\frac{d^2\theta}{dt^2} = -2kh\delta - 2k(\ell^2+\delta^2)\theta.$$

OR.

$$m\frac{d^2\theta}{dt^2} = -2k\frac{\delta}{\ell^2+\delta^2}h - 2k\theta.$$

2 marks

(b) To nondimensionalize, say $t = \tilde{t}(m/2k)^{1/2}$, $h = \tilde{h}\delta$, 1 mark

$$\Rightarrow \frac{d^2\tilde{h}}{d\tilde{t}^2} = -\tilde{h} - \left(1 - \frac{\gamma_\mu UL}{2k\delta}\right)\theta \ldots$$ Let $\frac{\gamma_\mu UL}{2k\delta} = c$

1 mark 1 mark

$$\frac{d^2\tilde{\theta}}{d\tilde{t}^2} = -\frac{\delta^2}{\ell^2+\delta^2}\tilde{h} - \tilde{\theta}$$

1 mark

Let $d = \frac{\delta^2}{\ell^2+\delta^2}$

1 mark.

$$\Rightarrow \quad \frac{d^2}{dt^2}\begin{bmatrix}\tilde{h}\\ \tilde{\theta}\end{bmatrix} = \begin{bmatrix}-1 & (c-1)\\ -d & -1\end{bmatrix}\begin{bmatrix}\tilde{h}\\ \tilde{\theta}\end{bmatrix}$$

Therefore, only two parameters govern the dynamics, c and d.

The fluid speed is non-dimensionalized in c.

(c) Steady state $h = \theta = 0$, which corresponds to $h = \theta = 0$. 3 marks

(d) Perturbing $\theta = \theta'$, $h = h'$ $\Rightarrow$ get the same equations. 5 marks

(e) If $h, \theta \propto e^{i\omega t}$ the ω^2 is the eigenvalue of. 1 mark

$$\begin{bmatrix}1 & 1-c\\ d & 1\end{bmatrix}$$

1 marks

$$\left.\begin{aligned}&\Rightarrow (1-\omega^2)^2 - d(1-c) = 0\\ &\Rightarrow (1-\omega^2)^2 = d(c-1).\\ &\omega^2 - 1 = \pm\sqrt{d(c-1)}\\ &\text{and } \omega^2 = 1 \pm \sqrt{d(c-1)}\end{aligned}\right\}$$

3 marks.

Instability if $d(c-1) > 1$ i.e. $c-1 > 1/d$ i.e $\boxed{c > \frac{1}{d} + 1}$

i.e. $\frac{\gamma\mu UL}{2k\delta} > \frac{\ell^2+\delta^2}{\delta^2} + 1 = \frac{\ell^2}{\delta^2} + 2$ 1 marks

and $\boxed{U > \frac{2k\delta}{\gamma\mu L}\left(\frac{\ell^2}{\delta^2} + 2\right)}$ 1 marks.