EGT3
ENGINEERING TRIPOS PART IIB

Friday 2 May 2025 9.30 to 11.10

Module 4A10

FLOW INSTABILITY

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A10 data sheet (two pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) (i) Describe a simple experiment involving the heating of a viscous liquid that would enable the observer to visualise Rayleigh-Bénard convection. [10%]

(ii) Describe the structure of the flow one would expect to observe. Use simple sketches to illustrate this structure as viewed from above and from the side. [10%]

(iii) Define the Rayleigh number in terms of the parameters that characterise your experiment and explain how you could estimate the value of the Rayleigh number which when exceeded leads to the onset of convection. What value would you expect this number to take? [10%]

(b) The following coupled linear ordinary-differential system of equations for the vertical velocity w and temperature difference θ is proposed as a model for the Rayleigh-Bénard instability

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{\nu}{d^2}w + \alpha g \theta$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\kappa}{d^2}\theta + \beta w$$

In this system, which has null solution $w = \theta = 0$, ν is the kinematic viscosity, κ the thermal diffusivity, α the coefficient of thermal expansion, g the acceleration due to gravity, d a vertical length scale, and β a dimensional constant. Take $\nu, \kappa, \alpha, \beta, d$ and g to be constants.

(i) Introduce normal modes of the form $w', \theta' \propto e^{st}$, for small amplitude perturbations w' and θ', to show that the growth rate of instability s is

$$s = -\frac{1}{2}(\nu + \kappa)d^{-2} \pm \frac{1}{2}\left[(\nu - \kappa)^2 d^{-4} + 4\alpha\beta g\right]^{1/2}$$

[40%]

(ii) Using the result in part (b)(i), comment on the stability/instability of the system in terms of the parameter R, where $R = \alpha\beta g d^4/\kappa\nu$. [10%]

(iii) Show that if there is no dissipation, then

$$\frac{\mathrm{d}^2 w'}{\mathrm{d}t^2} + N^2 w' = 0$$

where $N^2 = -\alpha\beta g$ defines a buoyancy frequency N. [20%]

2 A cloud of gas is to be modelled by the Poisson equation

$$\nabla^2 \phi = 4\pi G \rho$$

the Euler equations

$$\rho\Big(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\Big) = -\nabla p - \rho \nabla \phi$$

the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

and an equation of state

$$p = p(\rho)$$

In these equations, p denotes pressure, t time, ρ density, $\boldsymbol{u}$ the velocity vector, G a constant, and ϕ a gravitational potential. The basic state of rest is $\boldsymbol{u} = 0$, $\phi = \Phi(\boldsymbol{x})$ and $\rho = R(\boldsymbol{x})$, where $\boldsymbol{x}$ denotes the position vector. The stability of the cloud is to be assessed by considering small amplitude perturbations to this basic state of rest of the form

$$\boldsymbol{u} = 0 + \boldsymbol{u}', \quad \phi = \Phi + \phi', \quad \rho = R + \rho'$$

(a) Neglecting products of small quantities, show that the linearised equations of motion are

$$\nabla^2 \phi' = 4\pi G \rho'$$

$$R \frac{\partial \boldsymbol{u}'}{\partial t} = -c^2 \nabla \rho' - \rho' \nabla \Phi - R \nabla \phi'$$

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \big(R \boldsymbol{u}'\big) = 0$$

where $c = \left[\frac{\mathrm{d}p(R)}{\mathrm{d}R}\right]^{1/2}$ is the basic speed of sound. [50%]

(b) By further assuming that the basic state is R = constant, Φ = constant, and seeking normal mode solutions of the form $\boldsymbol{u}', \rho', \phi' \propto e^{i\boldsymbol{k}\cdot\boldsymbol{x}+st}$ for growth rate s and wavenumber vector $\boldsymbol{k}$, show that

$$s^2 = 4\pi G R - c^2 \boldsymbol{k}^2$$

[40%]

(c) Comment on the stability/instability of the system. [10%]

3 Consider a circular conduit of uniform cross-sectional area A and mass per unit length λ, transporting a frictionless fluid of density ρ. The basic situation is shown in Fig. 1. The velocity profile of the flow of the fluid across the section of the conduit is uniform with speed U. The conduit is infinite in length and perfectly flexible, i.e. has no bending rigidity. The centreline of the conduit is given by the curve $\boldsymbol{X}(s,t)$, where s is the arc-length variable along its length and t is time. At any time t, the unit vector tangent to the centreline, $\hat{\boldsymbol{t}}$, the curvature, κ, and the unit normal, $\hat{\boldsymbol{n}}$, are given by

$$\hat{\boldsymbol{t}} = \frac{\partial \boldsymbol{X}}{\partial s} \qquad \text{and} \qquad \frac{\partial \hat{\boldsymbol{t}}}{\partial s} = \kappa \hat{\boldsymbol{n}}$$

You may assume that the radius of the cross-section of the conduit is much smaller than its length, and that the centreline is confined to the two-dimensional xy-plane and is inextensible. Also assume that the medium outside the conduit does not exert any force on the conduit or influence its motion in any way. Ignore the effects of gravity.

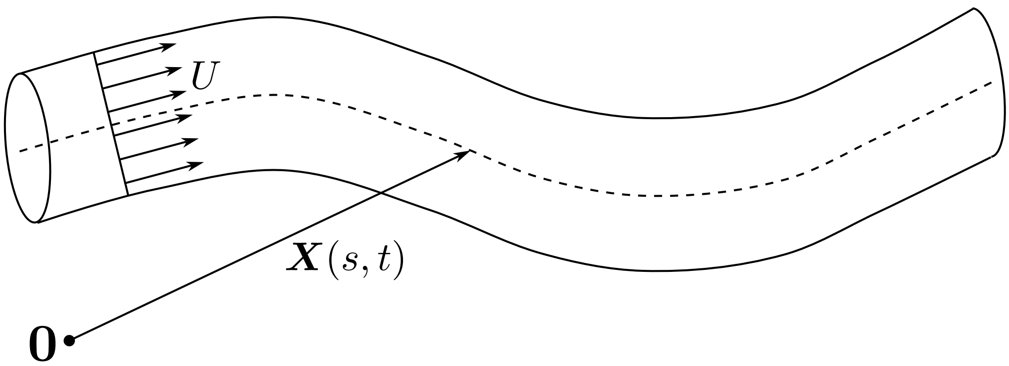

Fig. 1

(a) Show that the centreline motion is governed by the equation

$$(\lambda + \rho A)\frac{\partial^2 \boldsymbol{X}}{\partial t^2} + 2\rho A U \frac{\partial^2 \boldsymbol{X}}{\partial s \partial t} = \frac{\partial}{\partial s}\left(T\frac{\partial \boldsymbol{X}}{\partial s}\right) - \rho U^2 A \frac{\partial^2 \boldsymbol{X}}{\partial s^2} \tag{1}$$

where $T(s)$ is the tension along the conduit. Explain the physical origin of each of the terms in equation (1) and the tension T. [30%]

(b) Express the inextensibility constraint in terms of $\boldsymbol{X}(s,t)$. [10%]

(c) Let $\hat{\boldsymbol{x}}$ be the unit vector along the positive x-axis. Show that $\boldsymbol{X} = s\hat{\boldsymbol{x}}$ and $T = T_0 =$ constant is a steady-state solution of equation (1) that satisfies the inextensibility constraint. [10%]

(d) Linearly perturb the steady state given in part (c) and determine the equations that govern the perturbation. [25%]

(e) Assuming sinusoidal normal modes of wavenumber k, determine the dispersion relationship for the exponential growth rate of the perturbations. If there is a transition from stability to instability, determine the stability threshold. [25%]

4 Consider a slender rod of length L, mass m and moment of inertia $m(l^2+\delta^2)$ that is suspended asymmetrically by two massless springs. The rod is shown at a general time t in Fig. 2 together with the lengths l and δ. The springs are a distance $2l$ apart and attached to the rod at positions A and B. The position C indicates the centre of mass of the rod at time t. Each spring has a spring constant k. When the system is in static equilibrium, the rod is aligned along the x-direction and the centre of mass of the rod is at position C_0, where the vertical displacement $h(t)$ is zero. The point C_0 is offset from the midpoint of AB by a distance δ. The angular displacement of the rod, $\theta(t)$, is small, and thus the motion of points A, B and C should be regarded as purely in the y-direction. The rod is immersed in a viscous fluid of viscosity μ, flowing with uniform velocity U along the x-direction, such that the hydrodynamic force in the y-direction is

$$F_y = \gamma\mu U L\theta$$

for a dimensionless constant γ.

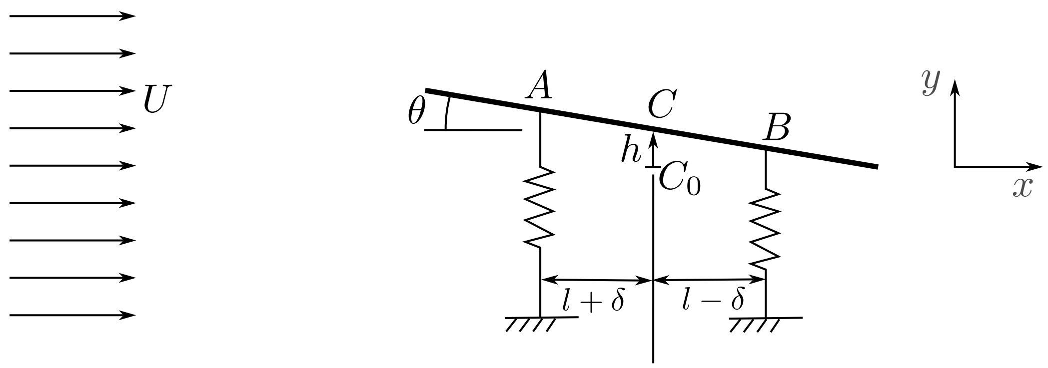

Fig. 2.

(a) Show that the equations governing the motion of the rod are

$$m\frac{d^2h}{dt^2} = -(2k)h + (\gamma\mu UL - 2k\delta)\theta \qquad (2)$$

$$m(l^2+\delta^2)\frac{d^2\theta}{dt^2} = -(2k\delta)h - 2k(l^2+\delta^2)\theta \qquad (3)$$

[20%]

(b) Non-dimensionalise the equations (2) and (3) to determine the minimum number of parameters that govern the dynamics of the rod. [20%]

(c) Find a steady-state solution of equations (2) and (3). [10%]

(d) Determine the dimensionless equations that govern the perturbations about the steady state. [10%]

(e) Determine whether there exists a linear instability as the fluid speed exceeds a threshold and, if so, determine the stability criteria. [40%]

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Module 4A10 Data Card

EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla \phi \quad \text{and} \quad \nabla^2 \phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u,v,w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x,\theta,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma \left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 \eta}{\partial \theta^2} \right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is $\begin{matrix}\text{unstable}\\ \text{stable}\end{matrix}$ to inviscid axisymmetric disturbances if Γ^2 $\begin{matrix}\text{decreases}\\ \text{increases}\end{matrix}$ with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu \left(\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} \right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2} = 0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + (1 - \alpha(T - T_0))\boldsymbol{g} + \upsilon\nabla^2\boldsymbol{u}$$

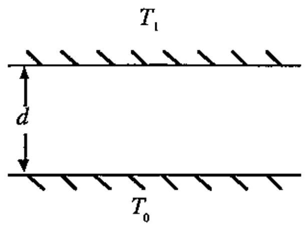
and

$$\frac{DT}{Dt} = \kappa\nabla^2 T$$

Rayleigh-Bénard convection

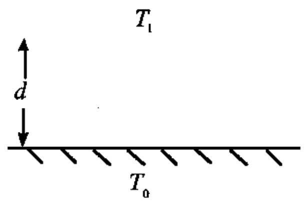
A fluid between two **rigid** plates is unstable when

$$\mathrm{Ra} \geq 1708$$



A liquid with a **free** upper surface is unstable when

$$\frac{\mathrm{Ra}}{\mathrm{Ra_c}} + \frac{\mathrm{Ma}}{\mathrm{Ma_c}} \geq 1$$



where

$$\mathrm{Ra} = \frac{g\alpha(T_0 - T_1)d^3}{\upsilon\kappa}, \quad \mathrm{Ma} = \frac{\chi(T_0 - T_1)d}{\rho\upsilon\kappa} \quad \text{with } \chi = -\frac{d\sigma}{dT}$$

$\mathrm{Ra_c} \approx 670 \qquad \mathrm{Ma_c} \approx 80.$

USEFUL MATHEMATICAL FORMULA

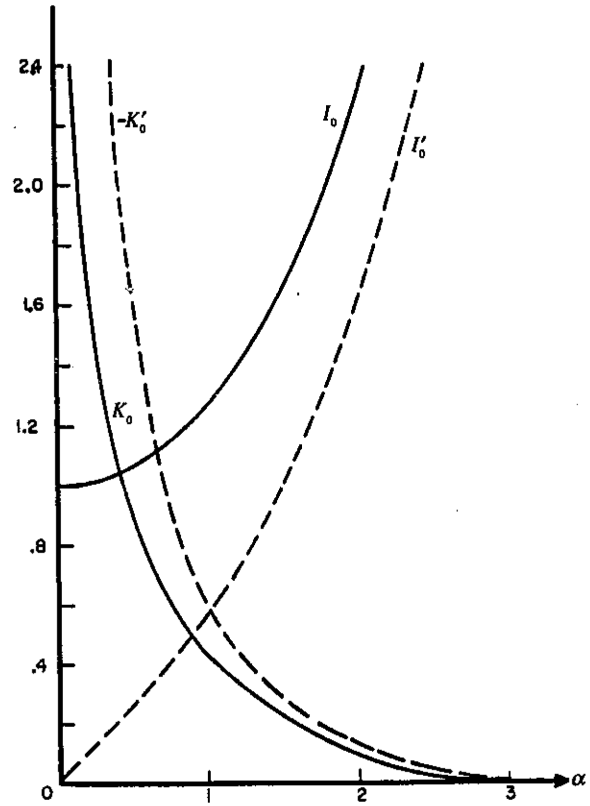
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - k^2 f = 0.$$

$I_0(kr)$ is finite at $r = 0$ and tends to infinity as $r \to \infty$,

$K_0(kr)$ is infinite at $r = 0$ and tends to zero as $r \to \infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative