EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 29 April 2026 9.30 to 11.10

Module 4A10

FLOW INSTABILITY

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A10 data sheet (two pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 For a form of convection, small-amplitude perturbations w' and θ' about a base state are to be modelled by the following system of coupled ordinary differential equations

$$\frac{dw'}{dt} = -4w' - k\theta' - 4\epsilon w' \qquad \text{and} \qquad \frac{d\theta'}{dt} = 2\theta' - kw' - \epsilon\theta' \qquad (1a,b)$$

Here, w' represents a vertical velocity, θ' a thermal anomaly, k a horizontal wavenumber, $\epsilon > 0$ a damping/diffusion parameter and t denotes time.

(a) (i) Show that there are normal modes with $w' \propto e^{st}$ and $\theta' \propto e^{st}$ for growth rate s where

$$s = -1 - \frac{5}{2}\epsilon \pm \frac{1}{2}\left[9(\epsilon + 2)^2 - 4k^2\right]^{1/2}$$ [25%]

(ii) Deduce that a state for $8 < k^2 < 9$ is stable for $\epsilon = 0$ and stable for large ϵ. [15%]

(iii) Deduce that the system is unstable if $8 < k^2 < 9$ and $(2\epsilon - 1)^2 < 9 - k^2$. [10%]

(b) Double-diffusive convection occurs when two properties that influence density diffuse at different rates. Noting that heat diffuses in water two orders of magnitude faster than salt diffuses in water, the following simple experiment is proposed to observe double-diffusive convection. A glass beaker is half filled with a layer of cold freshwater *above* which rests a layer of hot salty water. The experiment has been set up very carefully so as to minimise the creation of disturbances. As a consequence, initially both layers may be considered to be at rest and the interface between the layers to be perfectly horizontal. The density of the hot salty water is less than that of the cold freshwater, so that the liquids are initially in a hydrostatically stable equilibrium.

(i) By considering a small amplitude disturbance that locally displaces a parcel of liquid upwards (or downwards) at some position on the interface, develop a physical argument that explains the growth of the disturbance and, thereby, the onset of convective structures commonly referred to as 'salt fingers'. [20%]

(ii) Focusing on the heat transfer aspects rather than the salinity diffusion, with reference to this experiment, relate each term on the right-hand side of the evolution equation for w' and θ' (equations (1a,b)) to the motion. Be explicit concerning which terms have stabilising/destabilising roles. [30%]

2 A Newtonian liquid of density ρ, viscosity μ, and surface tension γ forms a thin film of uniform thickness h_0 coating the underside of a large, horizontal, rigid plate. The surrounding phase is a passive gas at constant pressure. Gravity acts downward with acceleration g. The film is initially flat and at rest.

(a) The film is nominally in an unstable configuration, yet small-scale disturbances can be stabilised. Explain clearly the physical origin of this stabilisation. In your answer, derive or state the relevant scalings for the capillary pressure (as a function of interface curvature) and the gravitational pressure (as a function of hydrostatic variations), and use these scalings to show how the relative importance of these effects changes with disturbance wavelength. [30%]

(b) Under an approximation in which the film thickness $h(x,t)$ varies slowly in the horizontal direction x and inertia is neglected, the evolution equation for the thickness $h(x,t)$ of the liquid film with time t is taken to be

$$\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial x} \qquad \text{where} \qquad q = \frac{\rho g}{3\mu}\left(h^3\frac{\partial h}{\partial x}\right) + \frac{\gamma}{3\mu}\left(h^3\frac{\partial^3 h}{\partial x^3}\right)$$

For small amplitude perturbations $h'(x,t)$ about the uniform base state described in part (a), show that the equation governing the linear stability of the film takes the form

$$\frac{\partial h'}{\partial t} = A\frac{\partial^2 h'}{\partial x^2} + B\frac{\partial^4 h'}{\partial x^4}$$

and identify the coefficients A and B in terms of ρ, g, γ, μ, and h_0. [20%]

(c) By seeking normal-mode solutions for $h'(x,t)$ with horizontal wavenumber k and temporal growth rate s, derive the dispersion relation $s(k)$. Comment on the physical significance of the two terms in this relation and on the role of viscosity. [20%]

(d) A thin film of water on the underside of a horizontal pane of glass is observed to break down into droplets with a spacing of 2-3 cm. Does the theory match well this observation? In your calculations, take $\rho = 1000\,\mathrm{kg/m^3}$, $\gamma = 0.072\,\mathrm{N/m}$ and $g = 9.81\,\mathrm{m/s^2}$. [30%]

(TURN OVER

3 Consider the flow through the two-dimensional channel of width H shown in Fig. 1. A segment of one of the channel walls between $x = 0$ and $x = L$ is flexible, where the width is $h(x,t)$. The fluid in the channel is assumed to be inviscid and incompressible with density ρ. In the absence of perturbations, the entire channel has uniform width H and fluid flows in the x-direction with steady velocity U.

Let $u(x,t)$ be the fluid velocity and $p(x,t)$ the pressure, where both are assumed uniform in the coordinate direction y. The flow is forced such that $p = 0$ and $u = U$ at $x = 0$. The flow obeys the one-dimensional version of volume and momentum conservation, which are simplified as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0 \tag{1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{1b}$$

The flexible wall is modelled as a damped-stretched membrane such that its motion is governed by

$$\mu\frac{\partial^2 h}{\partial t^2} + \beta\frac{\partial h}{\partial t} = T\frac{\partial^2 h}{\partial x^2} + p \qquad \text{for } 0 \le x \le L \tag{1c}$$

where μ, β, and T are constant parameters.

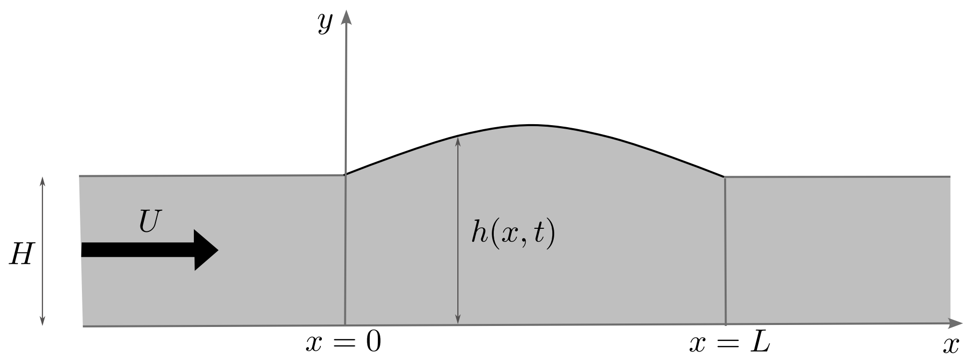

Fig. 1

(a) Show that $u(x,t) = U$ and $h(x,t) = H$ satisfy the governing equations. What is the pressure P in the channel for this situation? [10%]

(b) For small perturbation $u'(x,t) = u - U$, $h'(x,t) = h - H$ and $p'(x,t) = p - P$, determine the linearised version of equations (1a-c) that u', h' and p' satisfy. [20%]

(c) Consider the fundamental mode of vibration of the membrane

$$h'(x,t) = a(t)\sin\left(\frac{\pi x}{L}\right)$$

where $a(t)$ is its amplitude. Using the linearised version of equation (1c) derived in part (b), determine an ordinary differential equation for $a(t)$, possibly in terms of integrals of $p'(x,t)$. [20%]

(d) Using the equations governing the flow perturbations, show that $a(t)$ satisfies an equation of the form

$$m^* \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} + b^* \frac{\mathrm{d}a}{\mathrm{d}t} + k^* a = 0$$

where m^*, b^* and k^* are constants. Determine these constants. [20%]

(e) By solving the ordinary differential equation from part (d), determine the criteria for the onset of linear instability. [30%]

4 As shown in Fig. 2(a), wind blows steadily with horizontal speed U along z past a long overhead wire of mass λL suspended between two rigid posts at $x = 0$ and $x = L$. The cross-section of the wire has dimensions much smaller than the length. A tension T is maintained in the wire, which holds it taut. The weight of the wire can be ignored, so that in its unperturbed state, it lies along the x-axis. The wire can deform only by a small, purely vertical displacement, $h(x, t)$; it cannot twist. Any free oscillations of the wire decay with time proportional to $e^{-\beta t}$, for a constant β.

The cross-section of the wire has a shape shown in Fig. 2(b). The results from a two-dimensional aerodynamic characterisation of this section are shown in Fig. 2(c), where the cross-section was subject to an oncoming steady flow of speed V at an angle of attack α. The lift force per unit length is parameterised as $\frac{1}{2}\rho V^2 c\, C_L(\alpha)$, where ρ is the density of air, c the nominal width of the shape, and C_L the lift coefficient. For small α, $C_L = -m\alpha$ for $m > 0$, shown as the dashed line in Fig. 2(c). The drag force is negligible.

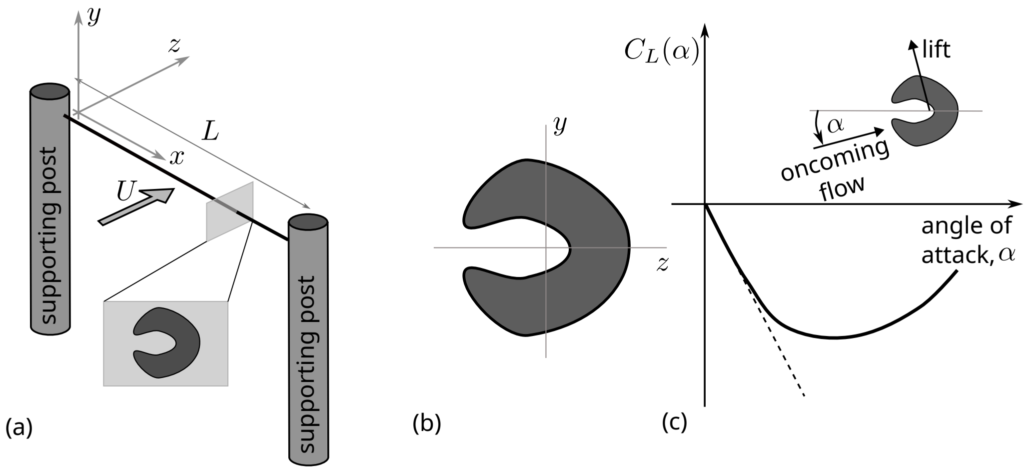

Fig. 2

(a) State the equation governing $h(x, t)$. Include terms for the inertia of the wire, the damping, the tension in the wire, and a generic vertical aerodynamic force per unit length $f(x, t)$. [10%]

(b) Accounting for the vertical speed of the wire $\partial h/\partial t$, determine the vertical component of the lift. Equating this lift to $f(x, t)$, rewrite the equation found in part (a). [20%]

(c) Perform a normal mode analysis of the equation derived in part (b). Determine the (possibly complex) frequency of oscillations of the normal modes. [50%]

(d) Based on your analysis, determine the threshold for linear instability. Which mode of wire displacement becomes unstable at the lowest wind speed? [20%]

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EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla \phi \text{ and } \nabla^2 \phi = 0$$

Bernoulli's equation

for inviscid flow $\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \frac{\partial \phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u, v, w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x, \theta, t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma \left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 \eta}{\partial \theta^2} \right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is $\begin{matrix}\text{unstable} \\ \text{stable}\end{matrix}$ to inviscid axisymmetric disturbances if Γ^2 $\begin{matrix}\text{decreases} \\ \text{increases}\end{matrix}$ with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu \left(\frac{d^2 V}{dr^2} + \frac{1}{r} \frac{dV}{dr} - \frac{V}{r^2} \right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2}=0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla\cdot\boldsymbol{u}=0$$

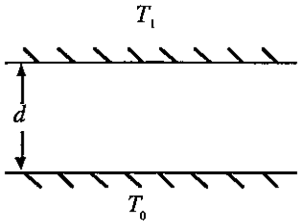
$$\frac{D\boldsymbol{u}}{Dt}=-\frac{1}{\rho_0}\nabla p+(1-\alpha(T-T_0))\boldsymbol{g}+\upsilon\nabla^2\boldsymbol{u}$$

and $$\frac{DT}{Dt}=\kappa\nabla^2 T$$

Rayleigh-Bénard convection

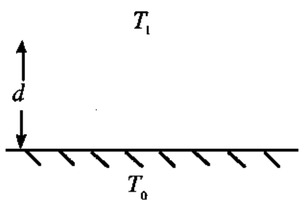
A fluid between two **rigid** plates is unstable when

Ra ≥ 1708



A liquid with a **free** upper surface is unstable when

$$\frac{\text{Ra}}{\text{Ra}_c}+\frac{\text{Ma}}{\text{Ma}_c}\geq 1$$



where

$$\text{Ra}=\frac{g\alpha(T_0-T_1)d^3}{\upsilon\kappa},\quad \text{Ma}=\frac{\chi(T_0-T_1)d}{\rho\upsilon\kappa}\quad \text{with } \chi=-\frac{d\sigma}{dT}$$

$\text{Ra}_c\approx 670 \qquad \text{Ma}_c\approx 80.$

USEFUL MATHEMATICAL FORMULA

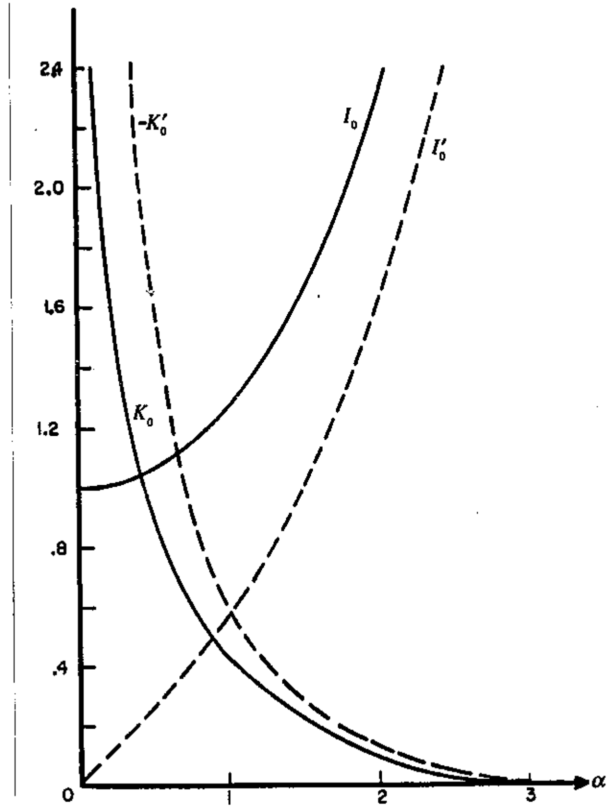
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-k^2f=0.$$

$I_0(kr)$ is finite at $r=0$ and tends to infinity as $r\to\infty$,

$K_0(kr)$ is infinite at $r=0$ and tends to zero as $r\to\infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative