4A10 Flow Instability 2026 Crib

SOLUTION to QUESTION 1

SECTION 1(a)(i) Deriving dispersion relationship

Given the linear system:

$$\frac{dw'}{dt} = -4w' - k\theta' - 4\epsilon w' \quad \text{(Eq.1)} \qquad \& \qquad \frac{d\theta'}{dt} = 2\theta' - kw' - \epsilon\theta' \quad \text{(Eq.2)}$$

for growth rate s and constants $\hat{w}, \hat{\theta}$, we seek normal modes of the form:

$$w'(t) = \hat{w}e^{st}, \qquad \theta'(t) = \hat{\theta}e^{st}$$

Substituting into Eq.1 and Eq.2 gives:

$$s\hat{w} = -4\hat{w} - k\hat{\theta} - 4\epsilon\hat{w} \qquad \Longrightarrow \qquad (s + 4 + 4\epsilon)\hat{w} + k\hat{\theta} = 0$$

$$s\hat{\theta} = 2\hat{\theta} - k\hat{w} - \epsilon\hat{\theta} \qquad \Longrightarrow \qquad k\hat{w} + (s - 2 + \epsilon)\hat{\theta} = 0$$

In matrix form:

$$\begin{pmatrix} s + 4 + 4\epsilon & k \\ k & s - 2 + \epsilon \end{pmatrix} \begin{pmatrix} \hat{w} \\ \hat{\theta} \end{pmatrix} = \mathbf{0}.$$

For nontrivial solutions $(\hat{w}, \hat{\theta}) \neq (0, 0)$, the determinant must vanish:

$$\det \begin{pmatrix} s + 4 + 4\epsilon & k \\ k & s - 2 + \epsilon \end{pmatrix} = 0.$$

Thus:

$$(s + 4 + 4\epsilon)(s - 2 + \epsilon) - k^2 = 0 \qquad \Longrightarrow \qquad s^2 + (2 + 5\epsilon)s + \left(4\epsilon^2 - 4\epsilon - 8 + k^2\right) = 0$$

This quadratic in s yields the solution requested:

$$\boxed{s = -1 - \frac{5}{2}\epsilon \pm \frac{1}{2}\sqrt{9(\epsilon + 2)^2 - 4k^2}}$$

25%

SECTION 1(a)(ii) Wavenumber ranges for stability

First, set $\epsilon = 0$ $\implies$ $s(\epsilon = 0) = -1 \pm \sqrt{9 - k^2}$
For $8 < k^2 < 9$, we have $0 < 9 - k^2 < 1$, so

$$s_+ = -1 + \sqrt{9 - k^2} < 0, \qquad s_- = -1 - \sqrt{9 - k^2} < 0$$

Thus, s_+ and s_- are real and $\{s_+, s_-\} < 0$ so perturbations decay, i.e. the system is stable.

Second, consider limit as $\epsilon \to \infty$. For large ϵ, the dominant terms (on noting we are given that k^2 is finite with $8 < k^2 < 9$)

$$s(\epsilon \to \infty) \approx -\frac{5}{2}\epsilon \pm \frac{3}{2}\epsilon \qquad \text{so} \quad s_+ \approx -\epsilon < 0, \quad s_- \approx -4\epsilon < 0$$

so

$$s_+ \approx -\epsilon < 0, \qquad s_- \approx -4\epsilon < 0 \qquad \implies \text{the system is again stable.}$$

15%

SECTION 1(a)(iii) Wavenumber range for instability

Instability requires the larger root to be positive so that $Re(s_+) > 0$, i.e.

$$s_+ = -1 - \frac{5}{2}\epsilon + \frac{1}{2}\sqrt{9(\epsilon + 2)^2 - 4k^2} > 0$$

Rearranging and squaring gives

$$(2\epsilon - 1)^2 < 9 - k^2$$

Thus, system is unstable for $8 < k^2 < 9$ and $(2\epsilon - 1)^2 < 9 - k^2$.

10%

SECTION 1(b)(i) Reasoning for onset of instability

Consider a small amplitude downward displacement of a warm(salty) parcel of fluid from the upper layer:
- it enters the cold(fresh) layer and rapidly loses heat (due to fast diffusion of heat) thereby thermally equilibrating;
- however, parcel essentially retains its original salinity (due to slow diffusion of salt);
- parcel therefore is now cold(salty);
- this cold(salty) parcel is now negatively-buoyant relative to cold(fresh) surroundings and so continues downwards, i.e. perturbation grows.

Similarly, an upward-displaced cold(fresh) parcel from below rapidly warms but retains its original freshness. This now warm(fresh) parcel, having lost the original temperature difference, is positively-buoyant in the warm(salty) layer and so continues rising, i.e. perturbation grows. This produces narrow rising and sinking convective elements, or plumes, called *salt fingers*.

20%

SECTION 1(b)(ii) Meaning of terms

Terms in w' equation

$-4w'$: stabilising linear damping of vertical motion; acts to suppress vertical motion

$-k\theta'$: buoyancy coupling between scalar perturbations and vertical motion; can be destabilising

$-4\epsilon w'$: diffusive damping; stabilising; increasing ϵ strengthens damping

Terms in θ' equation

$+2\theta'$: destabilising; represents growth of scalar (temperature/salinity) perturbations due to background stratification

$-kw'$: advection of background stratification; couples θ' to w'; can be stabilising or destabilising depending on the sign of $w'\theta'$; essential for buoyancy-driven feedback

$-\epsilon\theta'$: diffusive damping; stabilising; larger ϵ suppresses θ' more strongly 30%

SOLUTION to QUESTION 2

SECTION 2(a) Physical reasoning for stability/instability

-A liquid film hanging from the underside of a plate is a gravitationally unstable configuration given the denser fluid (the liquid) lies above a less dense fluid (the gas). However, this configuration can potentially be stabilised by the action of capillary forces resulting from surface tension which generate pressure jumps of order $\gamma\kappa$, where κ is the interface curvature.
-Gravity is destabilising as it acts to pull the liquid downward. If the film is perfectly flat, this downward pull is uniform, but if the film is perturbed slightly so that some regions are thicker ("bulges") and others thinner ("dips"), gravity amplifies these variations. Gravity tends to pull more liquid into downward bulges amplifying the thickness variation and lowering gravitational potential energy. The associated hydrostatic pressure variations scale as $\rho g h'$, where h' is the local film thickness perturbation.
-Surface tension acts to minimise the surface energy (equivalently, minimise the surface area) and, therefore, resists curvatures caused by disturbances. It has a stabilising effect because the capillary pressure $\Delta p_{\text{cap}} \sim \gamma\kappa$ opposes deformation of the interface.
-The shorter the wavelength of the disturbance, the larger the curvature for a given amplitude. Surface tension generates a capillary pressure proportional to curvature, which tends to smooth out the interface and restore it toward a flat state. For a disturbance of wavelength λ, the curvature scales as $\kappa \sim a/\lambda^2$ (for amplitude a), so $\Delta p_{\text{cap}} \sim \gamma a/\lambda^2$. In contrast, the gravitational pressure variation scales as $\Delta p_g \sim \rho g a$.
- Short-wavelength disturbances - if the curvature is high (waves are short), surface tension generates strong restoring forces that smooth out the interface. In other words, the restoring surface tension force due to capillary pressures can overcome/dominate the destabilising gravitational force resulting from the hydrostatic pressure variations, damping out the perturbations. Hence, short waves can be stabilised by surface tension. This corresponds to $\Delta p_{\text{cap}} \gg \Delta p_g$, i.e. $\gamma/\lambda^2 \gg \rho g$.
- Long-wavelength disturbances have small curvatures so restoring forces by surface tension are weak and so gravity dominates, leading to the growth of perturbations. For long waves, λ is large so $\Delta p_{\text{cap}} \sim \gamma a/\lambda^2$ becomes small compared with $\Delta p_g \sim \rho g a$, giving gravitational destabilisation.
Thus, gravity destabilises long waves, while surface tension stabilises short waves. The competition between these effects leads to a band of unstable wavelengths, characteristic of Rayleigh–Taylor-type instabilities in thin films. The transition occurs when $\gamma/\lambda^2 \sim \rho g$, defining a critical wavelength separating stable (short) and unstable (long) modes.

30%

SECTION 2(b) Linearised eqn for perturbations

Consider small amplitude perturbations h' about the uniform base state:

$$h(x,t) = h_0 + h'(x,t), \qquad |h'(x,t)| \ll h_0.$$

Substitute into evolution equation and retain only terms up to first order in h'.
First, since h_0 is constant:

$$\frac{\partial h}{\partial t} = \frac{\partial h'}{\partial t} \quad \text{and} \quad \frac{\partial h}{\partial x} = \frac{\partial h'}{\partial x}, \qquad \frac{\partial^3 h}{\partial x^3} = \frac{\partial^3 h'}{\partial x^3}.$$

Second, expanding h^3:

$$h^3(x,t) = \left(h_0 + h'(x,t)\right)^3 = h_0^3 + 3h_0^2 h'(x,t) + O(h'^2).$$

In the flux term q, h^3 multiplies derivatives of h. To linear order, products such as $h' \frac{\partial h'}{\partial x}$ are second order in h' and can be neglected. Therefore, to first order

$$h^3(x,t) \approx h_0^3.$$

Substituting the above into the evolution equation, we obtain

$$\frac{\partial h'}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{\rho g h_0^3}{3\mu}\frac{\partial h'}{\partial x} + \frac{\gamma h_0^3}{3\mu}\frac{\partial^3 h'}{\partial x^3}\right] \quad = -\left(\frac{\rho g h_0^3}{3\mu}\right)\frac{\partial^2 h'}{\partial x^2} - \left(\frac{\gamma h_0^3}{3\mu}\right)\frac{\partial^4 h'}{\partial x^4}.$$

Thus, the linearised equation is in the required form

$$\boxed{\frac{\partial h'}{\partial t} = A\frac{\partial^2 h'}{\partial x^2} + B\frac{\partial^4 h'}{\partial x^4} \qquad \text{where} \quad A = -\frac{\rho g h_0^3}{3\mu}, \quad B = -\frac{\gamma h_0^3}{3\mu}}$$

20%

SECTION 2(c) Finding dispersion relation

With $\hat{h}$ a constant amplitude, seek normal-mode solutions of the form

$$h'(x,t) = \hat{h}\, e^{st+ikx}$$

Compute derivatives:

$$\frac{\partial h'}{\partial t} = s\hat{h}e^{st+ikx}, \quad \frac{\partial^2 h'}{\partial x^2} = -k^2\hat{h}e^{st+ikx}, \quad \frac{\partial^4 h'}{\partial x^4} = k^4\hat{h}e^{st+ikx}.$$

Substitute into linearised equation and cancel the common factor $\hat{h}e^{st+ikx} \neq 0$ to yield:

$$s(k) = -Ak^2 + Bk^4$$

Substituting for A and B, the dispersion relation is:

$$s(k) = \frac{\rho g h_0^3}{3\mu}k^2 - \frac{\gamma h_0^3}{3\mu}k^4, \quad \text{i.e.} \quad \boxed{s(k) = \frac{h_0^3}{3\mu}\left(\rho g k^2 - \gamma k^4\right)}$$

*The term proportional to $\rho g k^2$ is destabilising (gravity);
*The term proportional to γk^4 is stabilising (surface tension);
*Viscosity is a factor in both stabilising and destabilising terms and so does not alter whether this flow is stable or unstable — only the rate of growth/decay

20%

SECTION 2(d) Comparison with observed droplet spacing

Given $s(k)$ is real, instability requires $s(k) > 0$, i.e. $\rho g - \gamma k^2 > 0 \quad \Rightarrow \quad k^2 < \frac{\rho g}{\gamma}$.
The most unstable wavenumber maximises $s(k)$. Differentiating gives:

$$\frac{ds}{dk} = \frac{2h_0^3 k}{3\mu}(\rho g - 2\gamma k^2).$$

Setting $\frac{ds}{dk} = 0$ (with $k \neq 0$) gives:

$$k_m^2 = \frac{\rho g}{2\gamma} \qquad \left(\text{clearly} k_m^2 < \frac{\rho g}{\gamma}, \quad \text{confirming that } s(k_m) > 0\right).$$

Thus, the most unstable wavelength (for comparison with the observed 2-3cm wavelength) is:

$$\lambda_m = \frac{2\pi}{k_m} = 2\pi\sqrt{\frac{2\gamma}{\rho g}}.$$

Given, for water:

$$\rho = 1000 \text{ kg/m}^3, \qquad \gamma = 0.072 \text{ N/m}, \qquad g = 9.81 \text{ m/s}^2.$$

Compute:

$$\lambda_m = 2\pi\sqrt{\frac{0.144}{9810}} \approx 2.4 \times 10^{-2} \text{ m} = 2.4 \text{ cm},$$

$\Longrightarrow$ predicted droplet spacing (of ≈ 2.4 cm) is in good agreement with that observed (2–3 cm).

30%

Q3.

Governing equations are

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0 \qquad \dots\dots (2)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad \dots\dots (3)$$

$$\mu\frac{\partial^2 h}{\partial t^2} + \beta\frac{\partial h}{\partial t} = T\frac{\partial^2 h}{\partial x^2} + p \qquad \dots\dots (4)$$

(a) When $u=U$ and $h=H$, both constants in x and t, equation (2) is satisfied, equation (3) becomes.

$$\frac{\partial p}{\partial x} = 0 \quad \Rightarrow \quad p = \text{constant} = 0 \text{ from boundary condition.}$$

Equation (4) is then also satisfied. So $u=U$ & $h=H$ is a steady state. The pressure $p=0$ in the conduit. 10%

(b) Perturbing (2), (3) and (4) gives

$$\frac{\partial h'}{\partial t} + U\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0$$

$$\mu\frac{\partial^2 h'}{\partial t^2} + \beta\frac{\partial h'}{\partial t} = T\frac{\partial^2 h'}{\partial x^2} + p'.$$

20%

(c) Multiplying the linearized version of (4) by $\sin kx$ $(k=\pi/L)$ and integrating yields.

$$\int_0^L \sin kx \left\{\mu\frac{\partial^2}{\partial t^2}(a(t)\sin kx) + \beta\frac{\partial}{\partial t}(a(t)\sin kx) - T\frac{\partial^2}{\partial x^2}(a(t)\sin kx) - p'(x,t)\right\} dx = 0$$

$$\Rightarrow \left[\mu\frac{d^2a}{dt^2} + \beta\frac{da}{dt} + Tk^2 a\right]\int_0^L \sin^2 kx\, dx - \int_0^L p'(x,t)\sin kx\, dx = 0$$

$$\left[\mu\frac{d^2a}{dt^2} + \beta\frac{da}{dt} + Tk^2 a\right]\frac{L}{2} = \int_0^L p'(x,t)\sin kx\, dx.$$

20%

(d) $h' = a(t)\sin kx$, where $k = \pi/L$. $\left[\text{here } \dot{a} = \frac{da}{dt}\right]$

$$\Rightarrow H\frac{\partial u'}{\partial x} = -\dot{a}\sin kx - Uka\cos kx. \Rightarrow \boxed{u' = \frac{\dot{a}}{kH}\cos kx - \frac{Ua}{H}\sin kx - \frac{\dot{a}}{kH}}$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial p'}{\partial x} = -\frac{\partial u'}{\partial t} - U\frac{\partial u'}{\partial x} = -\left[\frac{\ddot{a}}{kH}\cos kx - \frac{U\dot{a}}{H}\sin kx - \frac{\ddot{a}}{kH}\right] - U\left[-\frac{\dot{a}}{H}\sin kx - \frac{Uka}{H}\cos kx\right]$$

$$= \cos kx\left\{-\frac{\ddot{a}}{kH} + \frac{U^2k}{H}a\right\} + \sin kx\left\{\frac{2U\dot{a}}{H}\right\} + \frac{\ddot{a}}{kH}.$$

$$\Rightarrow \frac{p'}{\rho} = \sin kx\left\{-\frac{\ddot{a}}{k^2H} + \frac{U^2a}{H}\right\} - \cos kx\left\{\frac{2U\dot{a}}{kH}\right\} + \frac{\ddot{a}x}{kH} + \frac{2U\dot{a}}{kH}.$$

$$\Rightarrow \frac{1}{\rho}\int_0^L p'\sin kx\,dx = \left\{-\frac{\ddot{a}}{k^2H} + \frac{U^2a}{H}\right\}\int_0^L \sin^2 kx\,dx - \frac{2U\dot{a}}{kH}\int_0^L \sin kx\cos kx\,dx$$
$$+ \frac{\ddot{a}}{kH}\int_0^L x\sin kx\,dx + \frac{2U\dot{a}}{kH}\int_0^L \sin kx\,dx.$$

$$\boxed{\Rightarrow \frac{1}{\rho}\int_0^L p'\sin kx\,dx = \frac{L}{2}\left\{-\frac{\ddot{a}}{k^2H} + \frac{U^2a}{H}\right\} + \frac{\ddot{a}L}{k^2H} + \frac{4U\dot{a}}{k^2H}}$$

Aside:

$$\int_0^L \sin kx\,dx = -\frac{\cos kx}{k}\Big|_0^L = \frac{2}{k}$$

$$\int_0^L x\sin kx\,dx = -\frac{x\cos kx}{k}\Big|_0^L + \int_0^L \frac{\cos kx}{k}dx$$
$$= -\frac{L\cos\pi}{k} + \cancel{\frac{\sin kx}{k^2}\Big|_0^L}^{\,0}$$
$$= L/k.$$

$$\frac{L}{2}\left[\mu\ddot{a} + \beta\dot{a} + Tk^2a\right] = \frac{\rho L}{2}\left\{-\frac{\ddot{a}}{k^2H} + \frac{U^2a}{H}\right\} + \frac{\rho\ddot{a}L}{k^2H} + \frac{4\rho U\dot{a}}{k^2H}$$

$$\boxed{\Rightarrow \underbrace{\left[\mu - \frac{\rho}{k^2H}\right]}_{\text{equivalent mass}}\ddot{a} + \underbrace{\left[\beta - \frac{8\rho U}{k^2LH}\right]}_{\text{equivalent damping}}\dot{a} + \underbrace{\left[Tk^2 - \frac{\rho U^2}{H}\right]}_{\text{equivalent spring stiffness}}a = 0}$$

20%

e) The ordinary differential equation for $a(t)$ is identical to a damped harmonic oscillator.

$$m'\frac{d^2a}{dt^2} + \beta'\frac{da}{dt} + k'a = 0,$$

where $m' = \mu - \frac{\rho}{k^2H}$

$\beta' = \beta - \frac{8\rho U}{k^2HL}$

$k' = Tk^2 - \frac{\rho U^2}{H}$, $k = \frac{\pi}{L}$.

The solution may be found as a linear combination of e^{st}, where s satisfies

$$m's^2 + \beta's + k' = 0$$

$$s_{1,2} = \frac{-\beta' \pm \sqrt{\beta'^2 - 4m'k'}}{2m'}. \quad \Rightarrow a(t) = Ae^{s_1t} + Be^{s_2t}.$$

The real part of $s_{1,2}$ determines whether $a(t)$ grows or decays, and therefore the instability.

There are two modes of instability

(i) Growing oscillations if $Tk^2 - \frac{\rho U^2}{H} > 0$ $\left[\text{i.e.} \quad U < \left(\frac{THk^2}{\rho}\right)^{1/2} = \left(\frac{TH\pi^2}{\rho L^2}\right)^{1/2} \right]$

and $\frac{8\rho U}{k^2 LH} > \beta$ i.e negative damping and $\boxed{U > \frac{\beta k^2 LH}{8\rho} = \frac{\beta \pi^2 H}{8\rho L}}$

This requires $\frac{\beta \pi^2 H}{8\rho L} < \left(\frac{TH}{\rho}\right)^{1/2} \frac{\pi}{L} \Rightarrow \boxed{\beta < \frac{8}{\pi}\left(\frac{T\rho}{H}\right)^{1/2}}$

(ii) Steady deformation if the equivalent spring constant becomes negative while the equivalent damping is positive.

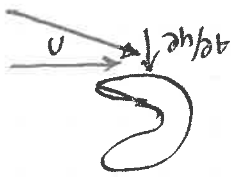
$\boxed{\Rightarrow U > \left(\frac{TH}{\rho}\right)^{1/2} \frac{\pi}{L}}$ with $\boxed{\beta > \frac{8}{\pi}\left(\frac{T\rho}{H}\right)^{1/2}}$

30%

Q4.

a) $$\lambda \frac{\partial^2 h}{\partial t^2} + \beta \frac{\partial h}{\partial t} - T\frac{\partial^2 h}{\partial x^2} = f(x,t).$$ 10%

b) By virtue of the motion of the wire (local vertical speed $\frac{\partial h}{\partial t}$), the wind appears to make an angle of attack $\alpha = -\frac{\partial h/\partial t}{U}$ and speed $V = \sqrt{U^2 + \left(\frac{\partial h}{\partial t}\right)^2} \approx U$ (to linear order).



The lift force is $-\frac{1}{2}\rho U^2 c m \alpha \cong +\frac{1}{2}\rho U^2 c m \frac{\partial h/\partial t}{U} = +\frac{1}{2}\rho U c m \frac{\partial h}{\partial t}$.

The vertical component of this lift force is $\frac{1}{2}\rho U c m \frac{\partial h}{\partial t}\cos\alpha$, but because $\alpha \ll 1$, $\cos\alpha \approx 1$. Therefore, $f(x,t) = \frac{1}{2}\rho U c m \frac{\partial h}{\partial t}$.

With this aerodynamic force,

$$\lambda \frac{\partial^2 h}{\partial t^2} + \beta \frac{\partial h}{\partial t} - T\frac{\partial^2 h}{\partial x^2} = \frac{1}{2}\rho U c m \frac{\partial h}{\partial t}.$$ 20%

c) Let $\beta' = \beta - \frac{1}{2}\rho U c m$ be the effective damping. Substitute normal modes, $h(x,t) = e^{i\omega_n t}\sin k_n x$, where $k_n = \frac{n\pi}{L}$ to satisfy boundary conditions at $x=0$ and $x=L$.

The frequency is given by
$$\lambda\omega_n^2 - \beta' i\omega_n = T k_n^2 \Rightarrow \omega_n = \frac{i\beta \pm \sqrt{-\beta^2 + 4\lambda T k_n^2}}{2\lambda}$$ 50%

d) Instability occurs when $\beta' < 0$ i.e. $\boxed{U > \frac{2\beta}{\rho c m}}$.

All modes become unstable simultaneously. 20%