Q1. (a) Taylor's hypothesis connects temporal gradients with spatial gradients, i.e. ; it is assumed that $\frac{\partial}{\partial t} \approx U \frac{\partial}{\partial x}$, with U the mean velocity. It is valid in mainly one-dimensional mean flow and when the turbulent fluctuations are small compared to U, i.e. $u' \ll U$.

Dissipation (if we assume homogeneous isotropic turbulence) is $\varepsilon = 15\nu\overline{\left(\frac{\partial u'}{\partial x}\right)^2}$. If we have $\frac{\partial}{\partial t}$ from a hot wire measurement, using Taylor's hypothesis we get $\frac{\partial}{\partial x} \Rightarrow \varepsilon$.

(b) Integral timescale $= T' = \int_0^\infty u' u'(\tau)\, d\tau \,/\, \overline{u'^2}$

where τ is the time delay between the measurements, taken at the same point but at two times.

Eddy turnover time is best understood as $T = \frac{L_t}{u}$

L_t: integral lengthscale & u: rms of turb fluctuations (σ or $\sqrt{k}$). If $u \ll U$, then $T' = \frac{U}{u} T$

(c) As the turbulence decays, we expect k to decrease, ε to decrease (both eventually reaching zero), and the integral lengthscale to change slowly as the Re_t decreases & the Kolmogorov length scale increases. The small scales are the first to decay so the eddy range decreases as we go downstream. Data shows that L_t slowly increases with distance; eventually the mathematical definition of the integral lengthscale becomes singular because $\overline{u'^2} \to 0$.

Q1 (a) $L_t \sim x^m$

$$\frac{dk}{dt} = -\varepsilon \,, \quad \varepsilon = \frac{k^{3/2}}{L}$$

$$\frac{dk}{dt} = -k^{3/2} \cdot x^{-m} \Rightarrow \int k^{-3/2}\, dk = -\int x^{-m}\, dx$$

$$\left[\frac{1}{1-\frac{3}{2}} k^{-1/2} \right]_{k_0}^{k} = -\frac{1}{1-m} \left[x^{1-m} \right]_{x_0}^{x}$$

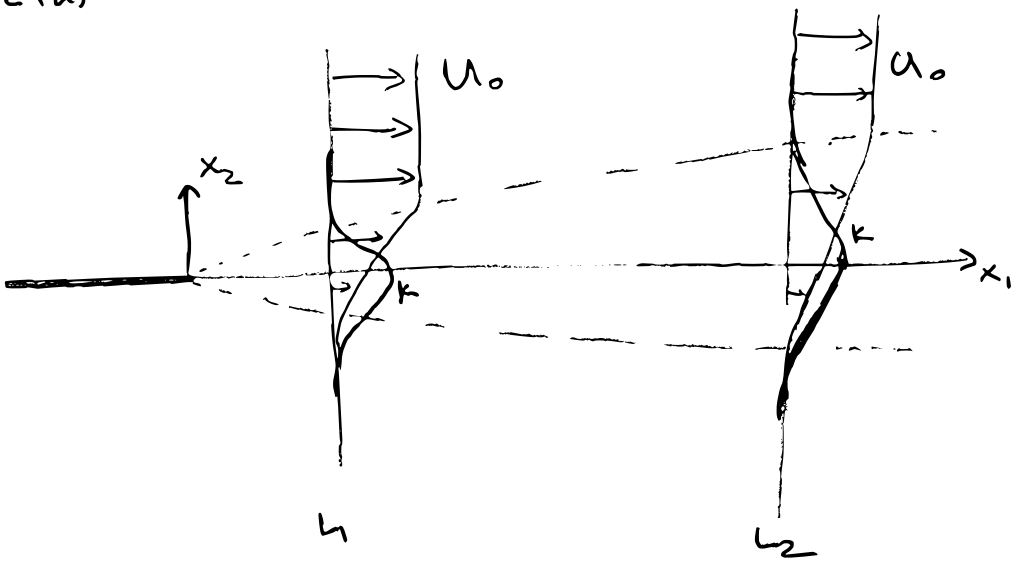
$$(\Rightarrow) 2\left(k_0^{-1/2} - k^{-1/2}\right) = \frac{1}{m-1}\left(x^{1-m} - x_0^{1-m}\right)$$

we take our starting point a distance x_0 where the turbulent kinetic energy is k_0

$$\Rightarrow k_0^{-1/2} - k^{-1/2} = \frac{1}{2(m-1)}\left(x^{1-m} - x_0^{1-m}\right) . \; //$$

Q1-2

Q2 (a)



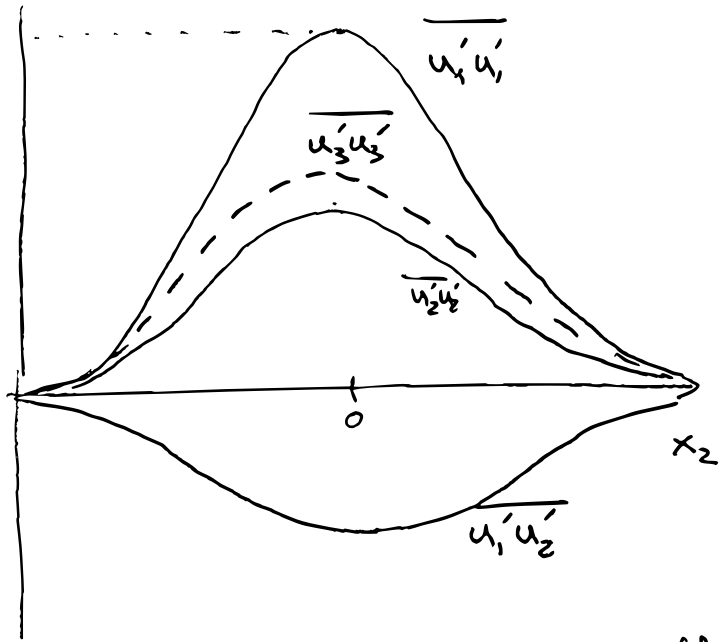
$\frac{K \text{ at the centreline } (x_2=0)}{U_0}$ = constant

This is the key observation in self-preserving thin shear flows. The mixing layer grows with distance, but the characteristic velocity does not change (unlike in jets, for example). So K stays constant. The mean velocity is very well approximated by an error function profile. The K profile is like a Gaussian.

If the cross-stream distance is normalised by the shear layer thickness, at all x_1 the mean velocity profile is identical.

Q2 (b)

$\frac{\sqrt{u_1'u_1'}}{U_0} \sim 10\%$



$\overline{u_1'u_1'}$ is higher than the other components because it is generated by the shear. $\overline{u_2'u_2'}$ & $\overline{u_3'u_3'}$ receive their energy through the pressure interaction terms. The slight expansion of the flow in the cross-stream direction causes the $\overline{u_2'u_2'}$ to be slightly reduced compared to the $\overline{u_3'u_3'}$ (similar to what happens in plane expansion). All 3 peak at $x_2=0$.

The sign of $\overline{u_1'u_2'}$ is opposite to $\frac{\partial U}{\partial x_2}$ so it is negative.

Q2-(c)

$u' = u_1'$, $U = \overline{u_1}$
$v' = u_2'$ $V = \overline{u_2}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{continuity}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y} \qquad \text{y-mom}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} \qquad \text{x-mom}$$

From orders of magnitude in continuity eqtn

$V = O\left(U_0 \frac{\delta}{L}\right)$ (U_0 = scale of streamwise velocity)

For the y-momentum, the orders of magnitudes are:

$$\left(U_0 \frac{U_0 \delta/L}{L}\right) \quad \left(U_0 \frac{\delta}{L} \frac{U_0 \delta/L}{\delta}\right) \quad (?) \left(\frac{u^2}{L}\right) \quad \left(\frac{u^2}{\delta}\right)$$

$$\left(\frac{U_0^2}{u^2}\frac{\delta^2}{L^2}\right)\frac{u^2}{\delta} \quad \left(\frac{U_0^2}{u^2}\frac{\delta^2}{L^2}\right)\frac{u^2}{\delta} \quad (?) \quad \left(\frac{\delta}{L}\right)\frac{u^2}{\delta} \quad \frac{u^2}{\delta}$$

It is known that $\frac{u}{U_0} > \frac{\delta}{L}$ in thin shear flows $\Rightarrow$ all terms vanish as $\frac{\delta}{L} \to 0$

so that the pressure term must be of order $\frac{u^2}{\delta}$. $\Rightarrow$ y-mom becomes $\frac{\partial \overline{v'v'}}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$

$\Rightarrow p + \rho\overline{v'v'} = p_\infty$

$\Rightarrow \frac{\partial p}{\partial x} + \rho\frac{\partial \overline{v'v'}}{\partial x} = 0$ To be used later for x-mom

Q2-(a) cont'd

Therefore, x-momentum becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y}$$

$$\left(\frac{U_0^2}{L}\right)\ \left(\frac{U_0\delta}{L}\ \frac{U_0}{\delta}\right)\ \left(\frac{u^2}{L}\right)\ \left(\frac{u^2}{L}\right)\ \left(\frac{u^2}{\delta}\right)$$

$$\left(\frac{U_0^2}{u^2}\frac{\delta}{L}\right)\frac{u^2}{\delta}\ \left(\frac{U_0^2}{u^2}\frac{\delta}{L}\right)\frac{u^2}{\delta}\ \ \frac{\delta}{L}\frac{u^2}{\delta}\ \ \frac{\delta}{L}\frac{u^2}{\delta}\ \ \frac{u^2}{\delta}$$

3rd & 4th terms are negligible as $\frac{\delta}{L} \to 0$

so for the eqtn to balance,

$\frac{U_0^2}{u^2}\frac{\delta}{L}$ must be $O(1)$. Hence, $\frac{u}{U_0} \sim \left(\frac{\delta}{L}\right)^{1/2}$

Since $\frac{u}{U_0}$ is constant (self-preservation),

$\frac{\delta}{L}$ = constant, so $\delta \sim x$.

Simplified x-momentum eqtn is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = -\frac{\partial \overline{u'v'}}{\partial y}.$$

Q3 - 1

(a) There may be a similarity solution because there is no time & length scales in this problem

(b) $u_\theta = \frac{\Gamma}{2\pi r}$

$$\frac{\partial u_\theta}{\partial r} = \frac{1}{2\pi}\frac{\partial}{\partial r}\left(\frac{\Gamma}{r}\right) = \frac{1}{2\pi}\left(\frac{1}{r}\frac{\partial \Gamma}{\partial r} - \frac{\Gamma}{r^2}\right)$$

$$\frac{\partial^2 u_\theta}{\partial r} = \frac{1}{2\pi}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}\right) = \frac{1}{2\pi}\left(\frac{1}{r}\frac{\partial^2 \Gamma}{\partial r^2} - \frac{2}{r^2}\frac{\partial \Gamma}{\partial r} + \frac{2\Gamma}{r^3}\right)$$

Substition in Eqn. for u_θ

$$\frac{1}{2\pi r}\frac{\partial \Gamma}{\partial t} = \frac{\nu}{2\pi}\left(\frac{1}{r}\frac{\partial^2 \Gamma}{\partial r^2} - \frac{2}{r^2}\frac{\partial \Gamma}{\partial r} + \frac{2\Gamma}{r^3} + \frac{1}{r^2}\frac{\partial \Gamma}{\partial r} - \frac{\Gamma}{r^3} - \frac{\Gamma}{r^3}\right)$$

$$\boxed{\frac{\partial \Gamma}{\partial t} = \nu\left(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r}\right)}$$

(c) $\Gamma(r,t) = f(\eta)$, $\eta = r/(\nu t)^{1/2}$

$$\frac{\partial \eta}{\partial t} = \frac{-r}{2(\nu t)^{1/2}}\frac{1}{t} = -\frac{\eta}{2t}, \quad \frac{\partial \eta}{\partial r} = \frac{1}{(\nu t)^{1/2}}$$

$$-\frac{\eta}{2t}\frac{df}{d\eta} = \nu\left(\frac{1}{\nu t}\frac{d^2 f}{d\eta^2} - \frac{1}{r(\nu t)^{1/2}}\cdot\frac{df}{d\eta}\right)$$

$$-\frac{\eta}{2}\frac{df}{d\eta} = \frac{d^2 f}{d\eta^2} - \frac{(\nu t)^{1/2}}{r}\frac{df}{d\eta}$$

$$\boxed{\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} - \frac{1}{\eta}\right)\frac{df}{d\eta} = 0}$$

(d) $\left(\ln f'\right)' = \frac{f''}{f'} = \frac{1}{\eta} - \frac{\eta}{2}$

$$\ln f' = \ln \eta - \frac{\eta^2}{4} + C$$

$$f' = C\eta e^{-\eta^2/4} = B\left(e^{-\eta^2/4}\right)'$$

$$f = A + B e^{-\eta^2/4}$$

At $\eta = 0$, $f(\eta) = 0 \Rightarrow B = -A$, $f = A(1 - e^{\eta^2/4})$

When $\eta \to +\infty$, $f(\eta) = \Gamma_0$, $\boxed{f = \Gamma_0(1 - e^{\eta^2/4})}$

Hence $u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$

(e) (i) when $r \gg (4\nu t)^{1/2}$, $u_\theta \approx \frac{\Gamma_0}{2\pi r}$, $\left| e^{-\frac{r^2}{4\nu t}} \right| \ll 1$

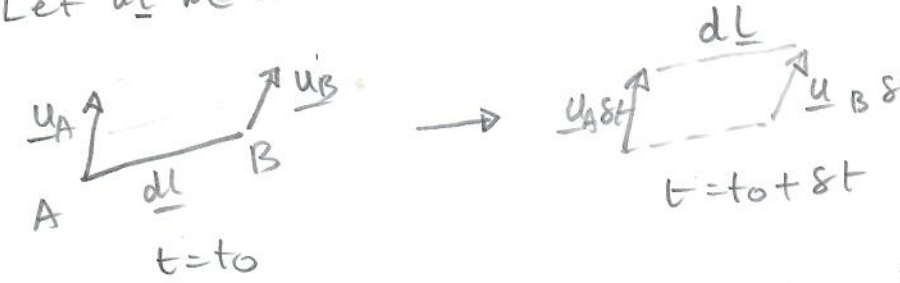
(ii) when $r \ll (4\nu t)^{1/2}$

$$u_\theta = \frac{\Gamma_0}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right) \approx \frac{\Gamma_0}{2\pi r}\left[1 - \left(1 - \frac{r^2}{4\nu t}\right)\right] = \frac{\Gamma_0 r}{8\pi\nu t}$$

corresponds to almost uniform (solid) rotation with angular velocity $\Gamma_0 / 8\pi\nu t$.

(a) Let $d\underline{l}$ be a material line element between A & B

Q 4-1



The change of $d\underline{l}$ in time δt is: $\delta(d\underline{l}) = (\underline{u}_B - \underline{u}_A)\delta t$

so the rate of change $d\underline{l}$ is

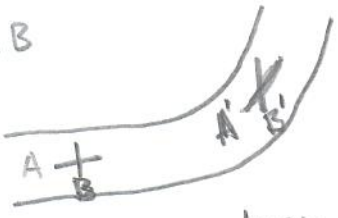
$$\boxed{\frac{D}{Dt}(d\underline{l}) = \underline{u}_B - \underline{u}_A = d\underline{l} \cdot \nabla \underline{u}}$$

Compare this with the vorticity eqn.

$$\boxed{\frac{D}{Dt}(\underline{\omega}) = \underline{\omega} \cdot \nabla \underline{u}}$$

$\underline{\omega}$ & $d\underline{l}$ obey the same equation, so if at $t=0$, we have $\underline{\omega} = \lambda d\underline{l}$, then $\underline{\omega} = \lambda d\underline{l}$ for all subsequent time.

(b) (i) $\omega_z = 0$, a cross AB



enters the bend

leg A (streamline) turns anticlockwise,

To keep $\omega_z = 0$, leg B must turn clockwise,

Hence the velocity at the inner side is larger that of the outer side. the flow speed decreases with r

(ii) Within the bend $\underline{u} = u_r(r)\,\underline{e}_r + u_\theta(r)\,\underline{e}_\theta$

continuity eqn: $\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta(r)) = 0$

u_θ is independent of θ, so $\frac{\partial}{\partial \theta}(u_\theta(r)) = 0$

Hence $r u_r = \text{constant}$

but at the wall $u_r = 0$ hence $\boxed{u_r = 0}$

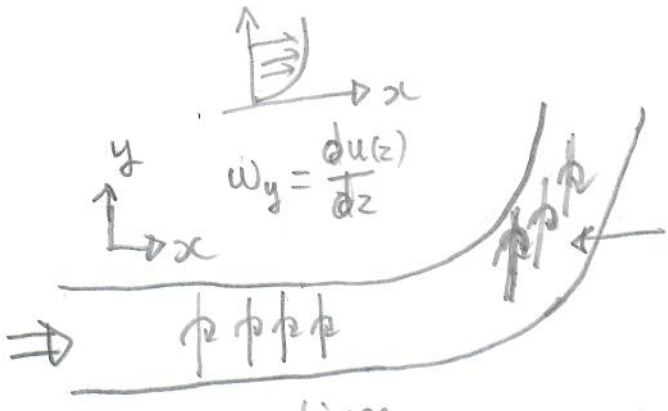
Hence the streamlines are circular

(iii) The bulk flow is irrotational, $\nabla \times \underline{u} = 0$

$$\frac{1}{r}\left(\frac{d}{dr}(r u_\theta) - \frac{du_r}{d\theta}\right) = 0, \quad \frac{d}{dr}(r u_\theta) = 0$$

$$\Rightarrow r u_\theta = \text{constant}, \text{ or } u_\theta = \frac{\text{constant}}{r}$$

(c) (i)

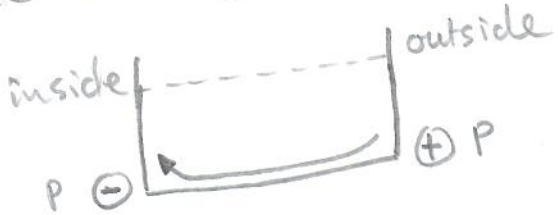


Vortex lines corresponding to wall boundary layer

① In the straight portion of the channel, vorticity are horizontal in y-direction, perpendicular to streamline & vortex lines turn clockwise.

② In the bend, the streamlines turn anti-clockwise

Thus a streamwise component of vorticity - secondary flow (iii) the bounday layer



→ bulk streamlines

--→ - secondary flow in the boundary

The weakness of this argument is that Helmholtz' theorem applies to an inviscid flow. The flow is not inviscid within the boundary layer.

(ii) In the bulk flow $\frac{\partial p}{\partial r} = \frac{1}{\rho}\frac{u_\theta^2}{r}$.

The pressure in the boundary layer is dictated by that of the bulk flow, hence the same $\frac{\partial p}{\partial r}$. Due to the no-slip condition, u_θ is smaller than that of the bulk, i cannot balance $\frac{\partial p}{\partial r}$, hence the pressure gradient push the fluid inward (the secondary flow).