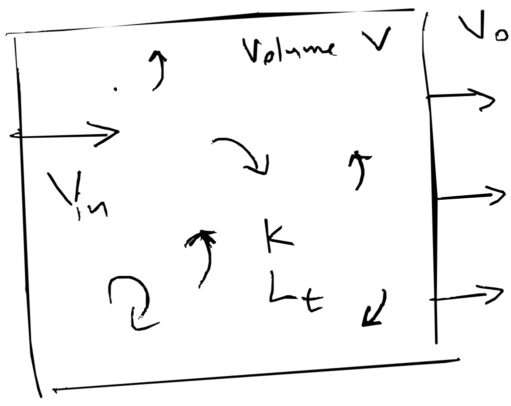
2026 4A12 Cribs

Q1 (a)



$V_{out} << V_{in} \left(\frac{A_{out}}{A_{in}} >> 1\right)$

$V = LWH$

$\rho = const$

steady-state; consider mean kinetic energy eqtn + k-eqtn, & integrate over space. There are fluxes of E_{in} & E_{out} & k_{out}; k is dissipated throughout the volume by ε (per unit mass); no body forces; neglect viscous terms. Hence:

$$\dot m_{in} E_{in} + \underbrace{\dot m_{in} k_{in}}_{=0} = \underbrace{\dot m_{out} E_{out}}_{\substack{\approx 0 \text{ following} \\ \text{the assumption} \\ V_{out} << V_{in}}} + \dot m_{out} k + V\rho\varepsilon$$

$$\Rightarrow \dot m \tfrac{1}{2} V_{in}^2 = \dot m k + V\rho\varepsilon$$
$$= \dot m k + V\rho \frac{k^{3/2}}{L_t} = \dot m k + V\rho \frac{k}{L_t/\sqrt{k}}$$

Using $\dot m = \rho \dot Q$ (volume flow rate) & $\tau_{res} = \frac{V}{\dot Q}$ (residence time), we have $\frac{1}{2} V_{in}^2 = k + \frac{\tau_{res}}{\tau_{turb}} k$

$$\Rightarrow \boxed{k = \tfrac{1}{2} V_{in}^2 \left(1 + \frac{\tau_{res}}{\tau_{turb}}\right)^{-1}}$$

(b) Homogeneous decaying turbulence:

$$\frac{dk}{dt} = -\varepsilon = -\frac{k^{3/2}}{L_t} \quad (1)$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2} \frac{\varepsilon^2}{k} = -C_{\varepsilon 2} \frac{k^2}{L_t^2} \quad (2)$$

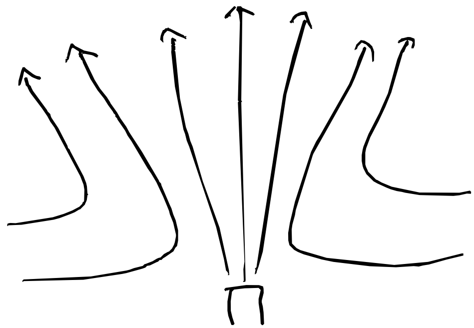
$$\left(\varepsilon = \frac{k^{3/2}}{L_t} \Leftrightarrow \varepsilon^2 = \frac{k^3}{L_t^2}\right)$$

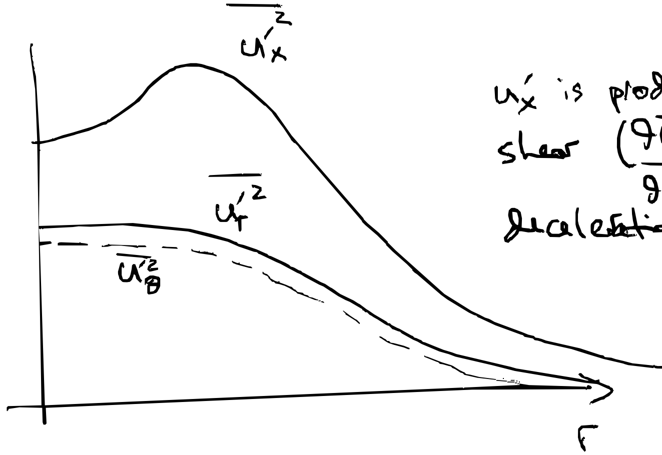
If ventilation switched off, no production mechanism. Homogeneous conditions imply no convection or turbulent diffusion terms $\Rightarrow$ k-ε model is simplified to Eqtns 1 & 2 above.

From (1), k will decay with time. Without simultaneous solution of 1 & 2 we cannot tell what happens to L_t; if $L_t \sim t^m$ $(m>0)$, the solution of 1 & 2 will tell us what m is. (Actually, experiment gives m & we get $C_{\varepsilon 2}$ from solution of Eqtn 1-2).

L_t increases mildly during the decay.

Q1(c)





u_x' is produced mainly by shear $\left(\frac{\partial \bar{u}_x}{\partial r}\right)$ & by the deceleration $\left(\frac{\partial \bar{u}_x}{\partial x} < 0\right)$

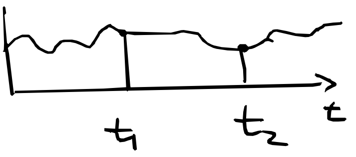
u_r' & u_θ' mainly fed by the streamwise turbulence through the pressure terms & minor production by $\frac{\partial \bar{u}_r}{\partial x}$ term. u_θ' from pressure terms.

Convection: +ve contribution ($\overline{u_x'^2}$ higher upstream)

Turb diffusion: diffuses turbulence from the region of production (off-axis) to the outer regions & towards the axis

Dissipation: important everywhere.

Q2 (a) If $u(t)$

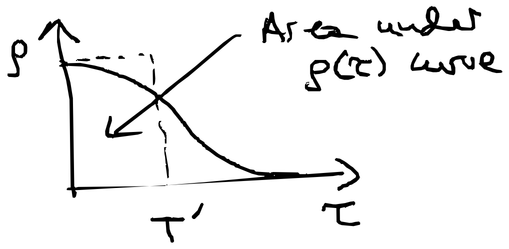


$t_2 - t_1 = \tau$

$\overline{u' \cdot u'(\tau)}$ = Autocorrelation

$\dfrac{\overline{u' u'(\tau)}}{\overline{u'^2}}$ = Autocorrelation coefficient ρ

Integral timescale T' :



$$T' = \int_0^\infty \rho(\tau)\, d\tau$$

Eddy turnover time $T_t = \dfrac{L_t}{\sqrt{k}}$, where

L_t = integral lengthscale & k is turbulent kinetic energy

Lagrangian integral timescale : $T_L = \displaystyle\int_0^\infty R(\tau) d\tau$

where $R_L(\tau)$ is equal to $\dfrac{\overline{V'V'(\tau)}}{\overline{V'^2}}$ where

V' is the velocity fluctuation following a fluid particle.

In homogeneous isotropic non-decaying turbulence with mean flow U,

$T' = \dfrac{L_t}{U}$, $T_t = \dfrac{L_t}{\sqrt{k}}$, $\overline{V'^2} = k$ & we don't know about $R_L(\tau)$.

(b) If $\eta_K = (\nu^3/\varepsilon)^{1/4}$ & $\tau_K = (\nu/\varepsilon)^{1/2}$ (Data Card)

& if $Re_t = \dfrac{\sqrt{k} \cdot L_t}{\nu}$, $T_t = \dfrac{L_t}{\sqrt{k}}$, $\varepsilon = \dfrac{k^{3/2}}{L_t}$

substitute to the definitions to show that

$\eta_K = L_t \, Re_t^{-3/4}$ & $\tau_K = T_t \, Re_t^{-1/2}$

(c) Production of Reynolds stress $\overline{u_i' u_j'}$ is

$P_{ij} = -\overline{u_i' u_k'} \dfrac{\partial \bar{u}_j}{\partial x_k}$ (repeated summation over index k)

$$\Rightarrow P_{11} = -\left(\underbrace{\overline{u_1' u_1'} \frac{\partial \bar{u}_1}{\partial x_1}}_{>0 \text{ for contraction}} + \underbrace{\overline{u_1' u_2'} \frac{\partial \bar{u}_1}{\partial x_2}}_{0 \text{ no } x_2 \text{ gradient}} + \underbrace{\overline{u_1' u_3'} \frac{\partial \bar{u}_1}{\partial x_3}}_{0 \text{ no } x_3 \text{ gradient}} \right)$$

$\therefore$ P_{11} is negative $\Rightarrow \overline{u_1'^2} \downarrow$

$$P_{22} = -\overline{u_2' u_2'} \underbrace{\frac{\partial \bar{u}_2}{\partial x_2}}_{<0 \text{ for contraction}} \Rightarrow P_{22} \text{ is positive} \Rightarrow \overline{u_2'^2} \uparrow$$

Entry : isotropic turbulence, exit : anisotropic

(a) If fluid is heated, its kinematic & dynamic viscosity changes. This changes the <u>scale</u> at which dissipation occurs (e.g. η_K), but does not change the production mechanism. Therefore anisotropy development at end of contraction is irrelevant to heating.

(a) [20%] $\nabla\times(\nabla\times A)=\cdots$, $\nabla\times(\nabla\times \underline{u})=-\nabla^2\underline{u}$

$\nabla(\underline{A}\cdot\underline{B})=\cdots \Rightarrow \underline{u}\nabla\underline{u}=\frac{1}{2}\nabla|\underline{u}|^2-\underline{u}\times\underline{\omega}$

$$\Rightarrow \frac{\partial \underline{u}}{\partial t}-\underline{u}\times\underline{\omega}=-\nabla\left(\frac{1}{2}|\underline{u}|^2+\frac{p}{\rho}\right)-\nu\nabla\times(\overbrace{\nabla\times\underline{u}}^{\underline{\omega}})$$

Take the curl, $\nabla\times(\underline{A}\times\underline{B})=\cdots$

$$\frac{\partial \underline{\omega}}{\partial t}-\underline{\omega}\cdot\nabla\underline{u}+\underline{u}\nabla\underline{\omega}=-\nu\nabla\times(\nabla\times\underline{\omega})=\nu\nabla^2\underline{\omega}$$

$$\Rightarrow \boxed{\frac{\partial \underline{\omega}}{\partial t}+\underbrace{\underline{u}\nabla\underline{\omega}}_{\text{advection}}=\underbrace{\underline{\omega}\cdot\nabla\underline{u}}_{\text{stretching}}+\underbrace{\nu\nabla^2\underline{\omega}}_{\text{diffusion}}}$$

(b) [20%] (i) when disc moves with $(U,0,0)$, the flow is unidirectional $(u(z),0,0)$, and the vorticity equation is $\frac{\partial \omega}{\partial t}=\nu\frac{\partial^2\omega}{\partial z^2}$, no advection, no stretching only diffusion, hence unsteady

(ii) when disc rotates, the centrifugal force expels the fluid out along the disc, to conserve the mass there is an inward flow toward the disc, hence against vorticity diffusion. Hence steady solution is possible.

(c) (i) [10%] Determine the length scale from Ω & ν.

$$\delta=\sqrt{\nu/\Omega}$$

(ii) [15%] On the disc, the θ-component is of order $r\Omega$, so is the r-component. Hence similarity form $u_r=\Omega r F(\eta)$, $u_\theta=\Omega r G(\eta)$

$$\eta=z/\delta$$

(iii) [15%] In cylindrical coordinates Q3-2

continuity Eqn: $\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$

$$O\left[\frac{1}{r}\frac{1}{r}(r \times r\Omega)\right] \qquad \underset{0}{\parallel} \qquad \frac{u_z}{\delta}$$

hence $\Omega \sim \frac{u_z}{\delta}$, $u_z \sim \delta\Omega = \sqrt{\nu\Omega}$

Hence similarity form $u_z = \sqrt{\nu\Omega}\, H(\eta)$

(iv) [20%] The terms in the radial momentum Eqn.

$$u_r \frac{\partial u_r}{\partial r} = \Omega^2 r F^2$$

$$u_z \frac{\partial u_r}{\partial z} = \sqrt{\nu z}\, H r\Omega F'/\delta = \Omega^2 r F' H$$

$$-\frac{u_\theta^2}{r} = -\Omega^2 r G^2$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_r}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(r\Omega F) = \cancel{\frac{\Omega F}{r}}$$

$$-\frac{u_r}{r^2} = -\frac{\Omega r F}{r^2} = \cancel{-\frac{\Omega F}{r}}$$

$$\frac{\partial^2 u_r}{\partial z^2} = \frac{\Omega r F''}{\delta^2} = \frac{\Omega^2 r F''}{\nu}$$

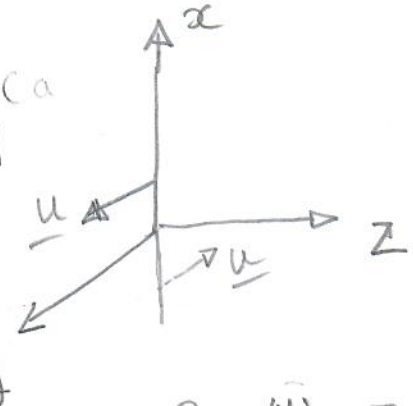
Hence the ODE is $F^2 + F'H - G^2 = F''$

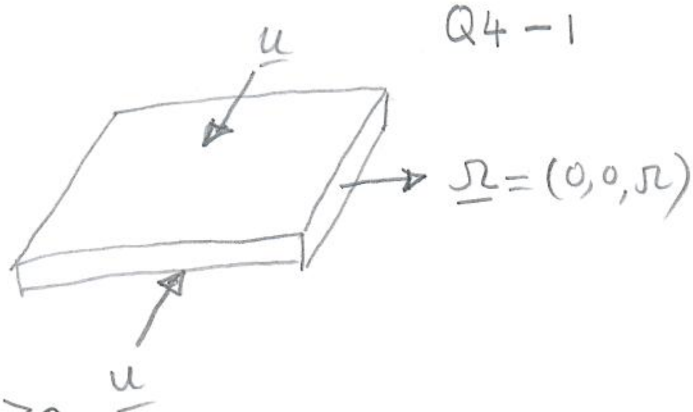
The BCs are

at $z=0$, $F=0$, $G=1$, $G=0$

at $z=+\infty$, $F=0$, $G=0$, no BC for H.

(a) [10%]





$$\Omega = \omega_z = \frac{\partial v}{\partial x} > 0$$

(b) [30%]
$$\frac{\partial \underline{\omega}}{\partial t} + \underline{u}\, \nabla \underline{\omega} = \underline{\omega}\, \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$$

undirection flow, the stretching term = 0
only ω_z component is not zero, the advection term is also zero. Hence

$$\frac{\partial \Omega}{\partial t} = \nu \frac{\partial^2 \Omega}{\partial x^2}, \quad \Omega = \frac{u_0}{\delta} \exp\left[-\frac{x^2}{\delta^2}\right]$$

$$\frac{\partial \Omega}{\partial t} = u_0 \delta' \left(-\frac{1}{\delta^2} + \frac{2x^2}{\delta^4}\right) \exp\left[-\frac{x^2}{\delta^2}\right]$$

$$\frac{\partial \Omega}{\partial x} = \frac{u_0}{\delta} \times \left(-\frac{2x}{\delta^2}\right) \exp\left(-\frac{x^2}{\delta^2}\right) = -\frac{2u_0 x}{\delta^2} \exp\left(-\frac{x^2}{\delta^2}\right)$$

$$\frac{\partial^2 \Omega}{\partial x^2} = -\frac{2u_0}{\delta^3} \exp\left(-\frac{x^2}{\delta^2}\right)\left[1 - \frac{2x^2}{\delta^2}\right] = \frac{2u_0}{\delta}\left[-\frac{1}{\delta^2} + \frac{2x^2}{\delta^4}\right] \exp\left(-\frac{x^2}{\delta^2}\right)$$

Hence $\delta' = \frac{2\nu}{\delta}$, $\quad \delta \delta' = 2\nu$

(c) [20%] $\frac{d\delta^2}{dt} = 4\nu \Rightarrow \delta^2 = \sqrt{4\nu t + \delta_0^2} \Rightarrow \delta = \sqrt{\delta_0^2 + 4\nu t}$

The sheet is diffusing with diffusing length $\sim \sqrt{4\nu t}$

The net flux of vorticity is $\Phi = \int_{-\infty}^{+\infty} \Omega \, dx = \frac{u_0}{\delta} \int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{\delta^2}\right)} dx$

$$= u_0 \int_{-\infty}^{+\infty} e^{-\eta^2} d\eta, \quad \eta = x/\delta$$

The Φ = constant $\times\, u_0$. The flux cannot be changed by diffusion, hence u_0 is constant.

(d) (i) [20%]

Vorticity only has z component $\omega_z = \Omega(x)$

$$\underline{\omega} \cdot \nabla \underline{u} = (0, 0, \Omega) \begin{pmatrix} -\alpha & V' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix} = (0, 0, \alpha\Omega)$$

$\underline{u}\, \nabla \omega$ only has z component.

From vorticity Equation

$$\frac{\partial \Omega}{\partial t} - \alpha x \frac{\partial \Omega}{\partial x} + V(x) \cancel{\frac{\partial \Omega}{\partial y}} = \alpha \Omega + \nu \frac{\partial^2 \Omega}{\partial x^2}$$

(ii) [20%] for steady flow, $\frac{\partial \Omega}{\partial t} = 0$

$$\boxed{-\alpha x \frac{d\Omega}{dx} = \alpha \Omega + \nu \frac{d^2 \Omega}{dx^2}}$$

$$\frac{d\Omega}{dx} = -\frac{2u_0 x}{\delta_0^3} \exp\left(-\frac{x^2}{\delta_0}\right)$$

$$\frac{d^2\Omega}{dx^2} = \frac{2u_0}{\delta_0}\left[-\frac{1}{\delta^2} + \frac{2x^2}{\delta^4}\right] \exp\left(-\frac{x^2}{\delta_0^2}\right)$$

$$\alpha x \frac{2u_0 x}{\delta_0^3} \exp\left(-\frac{x^2}{\delta_0^2}\right) = \frac{\alpha u_0}{\delta_0} \exp\left[-\frac{x^2}{\delta^2}\right] + \frac{2\nu u_0}{\delta_0}\left[-\frac{1}{\delta^2} + \frac{2x^2}{\delta^4}\right] \exp\left(-\frac{x^2}{\delta_0^2}\right)$$

$$\alpha \frac{u_0}{\delta_0}\left(\frac{2x^2}{\delta_0^2} - 1\right) = \frac{2\nu u_0}{\delta_0^3}\left(\frac{2x^2}{\delta_0^2} - 1\right)$$

$$\Rightarrow \boxed{\alpha = \frac{2\nu}{\delta_0^2}}$$