

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 6 May 2026 14 to 15.40

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

The NS equations in cylindrical coordinates (1 page)

4A12 Data Card (3 pages)

CUED approved calculator allowed

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the room.

1 A large conference room of dimensions L , W and H giving a volume $V = LHW$, is at ambient conditions. It is ventilated by air input through N small jets from nozzles of diameter d and velocity U_0 , distributed across various walls and injecting air at various orientations. The air flows out of the room through N large openings each of area A_{out} , distributed at all walls and the ceiling. The outlets are large enough for the mean kinetic energy to be negligible, however the turbulence exiting the room should not be neglected.

(a) It is assumed that the above arrangement produces homogeneous isotropic non-decaying turbulence in the room. By considering conservation of mechanical energy of the flow, and assuming a uniform integral length scale, derive an expression for the characteristic turbulent velocity $\sqrt{k}$ in the room, where k is the turbulent kinetic energy. State and discuss your assumptions. [30%]

(b) By reference to the $k - \varepsilon$ turbulence model, discuss what happens to the turbulence length scale and the turbulent kinetic energy if the ventilation is suddenly switched off. [30%]

(c) Consider the flow near one of the jets, and assume it is far enough from the other jets and any exhausts to be treated as a single round jet in stagnant infinite surroundings. Sketch the streamlines of the jet and jet-induced flow. Sketch the distributions of the individual Reynolds stresses in a direction normal to the jet axis, and discuss carefully the mechanisms (such as production, convection and dissipation) explaining the relative magnitudes of these stresses. [40%]

- 2 (a) State the definitions of *integral timescale*, *eddy turnover time*, and *Lagrangian integral timescale* and discuss their physical significance and their connection in the case of homogeneous isotropic turbulence with uni-directional mean flow. [25%]
- (b) Starting from the definitions of the Kolmogorov length and time scales as given in the Data Card, derive expressions relating these to the large-scale turbulent velocity and the integral length scale. [25%]
- (c) Consider homogeneous isotropic turbulence with uni-directional mean flow entering a plane contraction. Considering the sign of the production terms in the individual Reynolds stress equations and not otherwise, discuss how anisotropy develops at the exit of the contraction. [25%]
- (d) Discuss qualitatively how your answer in part (c) changes if the fluid is heated so that the dynamic viscosity increases by a factor of two. Explain your reasoning. [25%]

- 3 (a) Derive the vorticity equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega$$

from the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$$

You may use the vector identities in the Data Card.

[20%]

- (b) Consider an infinite flat disc located at $z = 0$ in Cartesian coordinates. The disc is translating with a constant velocity $(U, 0, 0)$, or rotating with a constant angular velocity $(0, 0, \Omega)$, in an otherwise initially quiescent liquid. Here U and Ω are constant. Discuss whether a steady-state solution can be achieved in each case.

[20%]

- (c) For the rotating disc in part (b), a steady axisymmetrical boundary layer develops on the disc as a result of viscous coupling with the surrounding fluid. Assume that the velocity field in cylindrical coordinates is (u_r, u_θ, u_z) .

- (i) Determine the order of magnitude of the boundary layer thickness δ . [10%]

- (ii) What are the orders of magnitude of the radial velocity component u_r and the azimuthal velocity component u_θ ? Express u_r and u_θ in self-similar forms involving the boundary layer thickness δ . [15%]

- (iii) Consider the continuity equation. Deduce the order of magnitude of the axial velocity component u_z , and hence express u_z in a self-similar form. [15%]

- (iv) Using the above self-similar forms for u_r , u_θ , and u_z , derive an ordinary differential equation from the radial momentum equation in the NS equations in the cylindrical coordinates. What are the appropriate boundary conditions? [20%]

4 A vortex sheet with time varying thickness δ takes the form

$$\boldsymbol{\omega} = (0, 0, \Omega), \quad \Omega = \frac{u_0}{\delta} \exp \left[-\frac{x^2}{\delta^2} \right], \quad \delta = \delta(t)$$

where x , y , and z are Cartesian coordinates, and u_0 is a constant. The velocity field associated with the vortex sheet is $\mathbf{u} = (0, V(x, t), 0)$.

(a) Sketch the velocity and vorticity fields. [10%]

(b) Starting from the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}$$

derive an equation governing Ω , and hence show that

$$\delta(t) \delta'(t) = 2\nu$$

where δ' denotes the time derivative of δ . [30%]

(c) Determine $\delta(t)$ and describe physically what is happening to the vortex sheet. Evaluate $\int_{-\infty}^{+\infty} \Omega(x, t) dx$. What is the significance of the requirement that u_0 is constant? [20%]

(d) The vortex sheet is now subjected to the irrotational straining flow

$$u = -\alpha x, \quad w = \alpha z$$

so that the velocity field is

$$\mathbf{u} = (-\alpha x, V(x, t), \alpha z)$$

where α is a positive constant.

(i) Starting from the vorticity equation, derive an equation governing Ω . [20%]

(ii) Determine the value of α in terms of ν and the initial sheet thickness δ_0 such that the vortex sheet remains steady in time. [20%]

END OF PAPER

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Navier-Stokes Equations in the Cylindrical Coordinates

$$\text{Continuity: } \frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial(u_z)}{\partial z} = 0 ,$$

$$r : \quad \rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] ,$$

$$\begin{aligned} \theta : \quad \rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \right. \\ &\quad \left. + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right] , \\ \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right] &. \end{aligned}$$

$$z : \quad \rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right] .$$

The Biot-Savart Law

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}') d\mathbf{V}'}{|\mathbf{x} - \mathbf{x}'|^3}$$

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Integral Theorems

$$\text{Gauss : } \int (\nabla \cdot \mathbf{A}) dV = \oint \mathbf{A} \cdot d\mathbf{S}$$

$$\text{Stokes : } \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{l}$$

Vector Identities

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

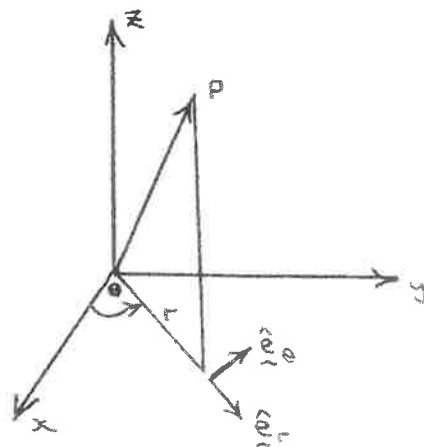
Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r}(rA_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r} \frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right)$$



Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \frac{\partial \overline{u'_i u'_j}}{\partial x_j} + \bar{g}_i$$

Mean scalar:

$$\frac{\partial \bar{\phi}}{\partial t} + \bar{u}_i \frac{\partial \bar{\phi}}{\partial x_i} = D \frac{\partial^2 \bar{\phi}}{\partial x_i^2} - \frac{\partial \overline{u'_i \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u'_i u'_i}/2$):

$$\begin{aligned} \frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = & -\frac{1}{\rho} \frac{\partial \overline{u'_j p'}}{\partial x_j} - \frac{1}{2} \frac{\partial \overline{u'_j u'_i u'_i}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ & - \overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u'_i}{\partial x_j} \right)^2} + \overline{g'_i u'_i} \end{aligned}$$

The $k - \varepsilon$ model:

$$\begin{aligned} \frac{\partial k}{\partial t} + \bar{u}_i \frac{\partial k}{\partial x_i} &= \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_i} \right) + P_k - \varepsilon \\ \frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} &= \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k} \\ \nu_t &= C_\mu \frac{k^2}{\varepsilon} \\ P_k &= \frac{1}{2} \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)^2 \\ C_\mu &= 0.09, \quad c_{\varepsilon 1} = 1.44, \quad c_{\varepsilon 2} = 1.92, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3 \end{aligned}$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u'_i}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \bar{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - \overline{2\phi' u'_j \frac{\partial \phi'}{\partial x_j}} - 2 \overline{\phi' u'_j} \frac{\partial \bar{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \bar{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i} \left((D + D_{turb}) \frac{\partial \sigma^2}{\partial x_i} \right) + 2D_{turb} \left(\frac{\partial \bar{\phi}}{\partial x_i} \right)^2 - 2\bar{N}$$

Scalar dissipation:

$$2\bar{N} = 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2 \frac{\varepsilon}{k} \sigma^2 = 2 \frac{u}{L_{turb}} \sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1 :

$$\frac{u}{L_{turb}} \sim \frac{\partial \bar{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= (\nu^3/\varepsilon)^{1/4} \\ \tau_K &= (\nu/\varepsilon)^{1/2} \\ v_K &= (\nu\varepsilon)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu \frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u'_i u'_j} &= -\nu_{turb} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) + \frac{2}{3} k \delta_{ij} \\ \overline{u'_j \phi'} &= -D_{turb} \frac{\partial \bar{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u'_1 u'_2} = -\nu_{turb} \frac{\partial \bar{u}_1}{\partial x_2}$$