UNIVERSITY OF CAMBRIDGE

Question number **Module 4A15 Solutions 2026** Sheet number 1

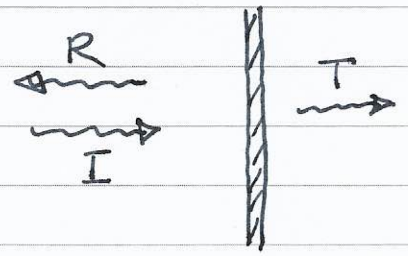
1 a) +ve x-direction plane wave has $p'(x,t) = f(t - x/c)$

(DATASHEET)

$$x = 0: \quad f(t) = I\cos\omega t \Rightarrow p'(x,t) = I\cos\omega(t - x/c)]$$

$$= \mathrm{Re}\left\{\underline{\underline{I e^{i\omega(t - x/c)}}}\right\}$$

b)



Define $p'(x=0^-,t) = p_-$

$p'(x=0^+,t) = p^+$

and likewise for u', particle velocity

$$u^- = \frac{I}{\rho c}e^{i\omega t} - \frac{R}{\rho c}e^{i\omega t}; \quad u^+ = \frac{T}{\rho c}e^{i\omega t}$$

Velocity continuous: $I - R = T$ (1)

Sheet dynamics (Newton II): $p_- - p_+ = \mu . i\omega \frac{T}{\rho c} e^{i\omega t}$

i.e. $$I + R - T = \frac{i\omega\mu}{\rho c} T \qquad (2)$$

Add (1), (2): $$2I = T\left(2 + \frac{i\omega\mu}{\rho c}\right)$$

$$\underline{\underline{T = \frac{I}{1 + \frac{i}{2}\frac{\omega\mu}{\rho c}}}}$$

(i): $$R = I - T = \frac{\frac{i}{2}\frac{\omega\mu}{\rho c}}{1+\frac{i}{2}\frac{\omega\mu}{\rho c}} I$$

$$R = |R|e^{i\phi_R} \quad \text{with} \quad |R| = \frac{\frac{1}{2}\frac{\omega\mu}{\rho c}}{\sqrt{1+\frac{1}{4}\left(\frac{\omega\mu}{\rho c}\right)^2}} I$$

$$\phi_R = \pi/2 - \tan^{-1}\left(\frac{1}{2}\frac{\omega\mu}{\rho c}\right)$$

$$T = |T|e^{i\phi_T} \quad \text{with} \quad |T| = \frac{I}{\sqrt{1+\frac{1}{4}\left(\frac{\omega\mu}{\rho c}\right)^2}} \qquad \phi_T = -\tan^{-1}\left(\frac{1}{2}\frac{\omega\mu}{\rho c}\right)$$

(ii) $\frac{\omega\mu}{\rho c} \ll 1 \Rightarrow R \simeq 0,\ T \simeq I.$

Same behaviour as ~~a rigid~~ original configuration with no sheet; for low enough frequency sheet mass is negligible.

(iii) $\frac{\omega\mu}{\rho c} \gg \Rightarrow R \simeq I,\ T \simeq 0.$

Sheet appears as rigid barrier; inertia too great to permit significant motion at these frequencies.

2 a) DATASHEET: $p' = \dfrac{f(t - r/c_0)}{r}$ (outgoing waves)

t: time

r: distance from source

Harmonic form with specified amplitude: $p' = P\,\mathrm{Re}\left\{ \dfrac{e^{i\omega(t - r/c_0)}}{r} \right\}$

b) Specification gives $r^2 = (X-d)^2 + Y^2 + Z^2$

Define $r_0 = \sqrt{X^2+Y^2+Z^2}$; $r = \sqrt{r_0^2 - 2Xd + d^2}$

$$\simeq r_0\left(1 - \frac{Xd}{r_0^2}\right) = r_0 - \frac{X}{r_0} d$$

$$p' \simeq P\,\mathrm{Re}\left\{ \frac{e^{i\omega(t - r_0/c_0)}}{r_0} \underbrace{e^{+i X/r_0 \, \omega d/c_0}}_{\text{modifying factor}} \right\}$$

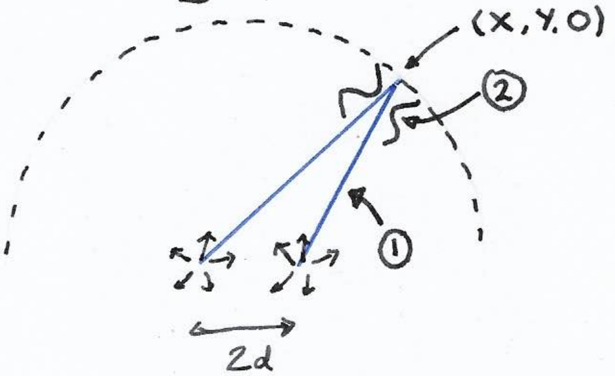
c) (i) Source fields superpose:

$$p' = P\,\mathrm{Re}\left\{ \frac{e^{i\omega(t - r_0/c_0)}}{r_0} \left(e^{i X/r_0 \, \omega d/c_0} + e^{-i X/r_0 \, \omega d/c_0} \right) \right\}$$

$$= 2P \cos\left(\frac{X}{r_0} \frac{\omega d}{c_0} \right) \mathrm{Re}\left\{ \frac{e^{i\omega(t - r_0/c_0)}}{r_0} \right\}$$

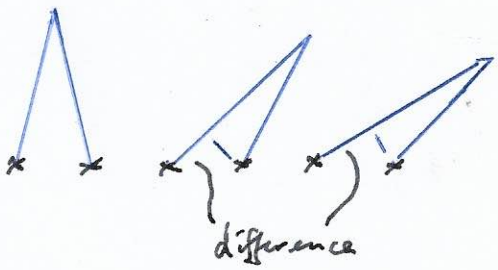
(ii) As r_0 $\left(= \sqrt{X^2+Y^2+Z^2} = R\right)$ is fixed on the sphere, the only X variation arises from the term $\cos\left(\frac{X}{r_0}\frac{\omega d}{c_0}\right)$

(iii) Consider x-y plane:



- Path lengths from sources to measurement point differ (1)
- Hence phases of source contributions also differ (2)
- The resulting interference varies with the path-length difference, which increases continuously with X:

N.B. Interference isn't monotonic, but oscillates between constructive and destructive as difference increases.



(iv) $\omega d/c_0$ determines how quickly the interference pattern varies with x. It represents the source spacing on a path-length plot normalised on the wavelength. When it is larger, the path difference at a given angle is greater (in wavelengths). Hence the interference pattern oscillates between its extremes more rapidly.

3 a) Higher-order modes require more acoustic wavelengths to fit around the duct circumference.

$p' \sim e^{in\theta}$ & n is high.

Total wavenumber $k = \sqrt{k_x^2 + k_r^2} = \omega / c_0$

$k_r = \kappa_{mn} / a$. For propagation k_x must be real

$$\therefore \left(\frac{\omega}{c_0}\right)^2 > \left(\frac{\kappa_{mn}}{a}\right)^2$$

higher n implies higher κ_{mn} & hence a higher cut-on frequency.

b) (i) Rotor alone modes have $n = B = 7$

For cut-on freq. for $n=7$:

$$\kappa_{1,7} \sim 7 + 0.809 \, 7^{1/3} = 8.55$$

$$\omega_{cutoff} = \kappa_{1,7} \cdot \frac{c_0}{a} = 8.55 \times \frac{343}{0.2}$$

$$= 14663.3 \text{ rad/s}$$

$$f_{cutoff} = \frac{14663.3}{2\pi} = 2333.7 \text{ Hz}$$

For cut-off,

$$f_{BPF} < f_{cut\,off}$$

$$f_{BPF} = \frac{NB}{60} = \frac{7N}{60} \quad (N \text{ in rpm})$$

$\therefore \quad \frac{7N}{60} < 2333.7$

$N_{max} \sim 20{,}000$

b)(ii) Operating conditions:

$f_{BPF} = 700\,Hz$

$2 \times f_{BPF} = 1400\,Hz$

Rotor-stator interaction modes:

$n = MB \pm NS$

At BPF, M=1, $n = B \pm NS = 7 \pm 5N$

N	n	$k_{1,\lvert n\rvert}$	f_{cutoff} (Hz)	Cut-off?
2	-3	4.2	1146	✓
1	2	3.054	833	✓
0	7	8.55	2333	✓
1	12	13.9	> 2333	✓
⋮				

All cut-off. No modes propagate at BPF

*

$\rightarrow$ 1400 Hz

At 2BPF, M = 2, $n = 14 \pm 5N$

N	n	$k_{1,\lvert n \rvert}$	$f_{cut\,off}$	cut off?
3	1	1.841	502	No.
2	4	5.318	1411	Yes ✓
1	9	10.7	> 1400 Hz	Yes ✓
0	14		> 1400 Hz	Yes ✓

$n = \pm 1$ mode propagates at 2BPF

S = 5 is a good choice, because.

a. B & S are co-primes, so plane waves are cut off

b. All BPF modes — dominant tonal noise — are cut-off.

c. At 2BPF only $n = \pm 1$ mode propagates. This has a low cut-on freq (502 Hz) and is thus difficult to cut off.

4

(a)

Mass conservation:

$$\dot{m}(t) = -V \frac{\partial \rho_2'}{\partial t} = \text{mass flow rate out of the cavity.}$$

For a time-harmonic response:

$$\frac{\partial}{\partial t} \to i\omega$$

$$\Rightarrow \dot{m} = -i\omega V \rho_2'$$

$$p_2' = c_0^2 \rho_2' = -\frac{c_0^2 \dot{m}}{V i\omega}$$

Conservation of momentum in the neck:

$$\frac{p_2' - p_1'}{l} = \rho_0 \frac{\partial u'}{\partial t}$$

$$u' = \frac{\dot{m}}{\rho_0 A}.$$

$$\therefore\ p_2' - p_1' = \frac{i\omega l m'}{A}$$

$$p_1' = -\left(\frac{c_0^2}{V i\omega} + \frac{i\omega l}{A}\right)\dot{m} = \left(\omega^2 - \frac{c_0^2 A}{V l}\right)\frac{l \dot{m}}{i\omega A}$$

$$\dot{m} = m'$$

(b) $$Z'(\omega) = \left(\omega^2 - \frac{c_0^2 A}{V l}\right)\frac{l}{i\omega A}$$

Plane waves $\Rightarrow$

$$u' = \pm \frac{p'}{\rho_0 c_0} \qquad \begin{array}{ll} + & \longrightarrow \\ - & \longleftarrow \end{array} \text{ wave}$$

At $x = 0$.

Pressure continuity:

$$I + R = T = p_1' \qquad -(1)$$

Conservation of mass $\Rightarrow$ Vol. velocity continuity

$$\therefore \frac{S}{\rho_0 c_0}(I - R) = \frac{S}{\rho_0 c_0} T + \frac{m'}{\rho_0} \qquad -(2)$$

From (a), with $p_1' = T$:

$$\cancel{m' = \frac{T}{\frac{c_0^2}{i\omega}}} \qquad m' = \frac{T}{Z'}$$

From (1), $R = T - I$

$$\cancel{\therefore \frac{S_0}{\rho_0 c_0}(I - T + I) = \frac{S}{\rho_0 c_0} T + \frac{T}{\rho_0 Z'}}$$

From (1), $T = I + R$

$$\therefore \frac{S}{\rho_0 c_0}(I - R) = \left(\frac{S}{\rho_0 c_0} + \frac{1}{\rho_0 Z'}\right)(I + R)$$

$$\cancel{I} - R = \cancel{I} + R + \frac{c_0 (I + R)}{S Z'}$$

$$-2R - \frac{c_0 R}{S Z'} = \frac{c_0 I}{S Z'}$$

$$\frac{R}{I} = \frac{-c_0}{2 S Z' + c_0}$$

$\therefore$ the reflection coefficient,

$$\left|\frac{R}{I}\right| = \frac{c_0}{|2 S Z' + c_0|}$$

(c) $$Z' = i\left(\frac{\omega L}{A} - \frac{c_0^2}{V \omega}\right)$$

It's purely imaginary

$\therefore \left|\frac{R}{I}\right|$ is a maximum when

$$Z' = 0$$

$$\Rightarrow \frac{\omega l}{A} - \frac{c_0^2}{V \omega} = 0$$

or $$\boxed{\omega = c_0 \sqrt{\frac{A}{V l}}}$$ at the resonance freq of the Helmholtz resonator

$$\left|\frac{R}{I}\right|_{max} = 1$$

$\therefore$

$\therefore$ there is no transmission of acoustic waves through the duct.

In reality there would be some damping in the neck, which would give Z' a real part, so some energy would be absorbed and the reflection would be incomplete.