Version AA/6

EGT3
ENGINEERING TRIPOS PART IIB

Friday 8 May 2026 9.30 to 11.10

Module 4A15

ACOUSTICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A15 Aeroacoustics data sheet (5 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A harmonic plane wave propagates in an acoustic medium with density ρ_0 and sound speed c_0. The angular frequency is ω, and the motion is in the positive x direction.

(a) Given that the sound pressure at the origin, $p'(x = 0, t)$, is $I \cos \omega t$, what is $p'(x, t)$? Express your answer as the real part of a complex quantity. [15%]

(b) (i) A thin sheet, of mass per unit area μ, is now placed at $x = 0$. As a result, a reflected wave arises in $x < 0$, and in $x > 0$ the original disturbance is replaced by a transmitted wave. What are the magnitudes and phases of the reflected and transmitted waves? (Give the phases in terms of their values at $x = 0$, relative to that of the incident wave at the same location.) [45%]

(ii) If the frequency is such that $\omega\mu/\rho_0 c_0 \ll 1$, what reflected and transmitted waves arise? Give a physical explanation for your answer. [20%]

(iii) The high-frequency limit is $\omega\mu/\rho_0 c_0 \gg 1$. What are the reflected and transmitted waves in this case? Again, give a physical explanation. [20%]

2 (a) A spherically symmetric point source in a medium with sound speed c_0 oscillates harmonically at angular frequency ω. Its strength is such that the associated pressure fluctuations have amplitude P at a distance of 1 m. Give an expression for the pressure field. You should define any additional variables that you introduce. [15%]

(b) The source is placed at $(x, y, z) = (d, 0, 0)$. The pressure is measured at (X, Y, Z), far away from the source (i.e. $\omega X/c_0 \gg 1$, $\omega Y/c_0 \gg 1$, $\omega Z/c_0 \gg 1$). In contrast, $\omega d/c_0 \sim O(1)$. On this basis, find an approximation to the pressure at the measurement point, consisting of the field due to a source at $(0, 0, 0)$ modified by a complex-exponential multiplying factor. [35%]

(c) A second source, identical to that of part (b) and vibrating in phase with it, is placed at $(-d, 0, 0)$.

(i) What is the pressure at the measurement point now? [10%]

(ii) How does the pressure vary with X on a sphere of radius $\sqrt{X^2 + Y^2 + Z^2} = R$? [10%]

(iii) Explain, with the aid of sketches, the physical origin of the pressure-field behaviour found in part (ii). [15%]

(iv) What is the influence of the parameter $\omega d/c_0$ on the pressure field of part (ii), and why does it arise? [15%]

 (TURN OVER

3 (a) Explain physically why higher-order circumferential modes in a circular duct have higher cut-on frequencies than lower-order modes. [20%]

(b) An axial fan with $B = 7$ blades and radius 200 mm operates in a cylindrical duct of the same diameter. The duct contains a downstream stator row with $S = 5$ vanes.

Table 1 shows values of κ_{mn}, the mth zero of $\mathrm{d}J_n(z)/\mathrm{d}z$, where J_n is the Bessel function of the first kind of order n. For $|n| > 6$, use $\kappa_{1n} \approx |n| + 0.809|n|^{1/3}$.

Assuming sound speed of $343\,\mathrm{ms}^{-1}$ and neglecting mean flow:

(i) Determine the maximum rotational speed $N_{\max}$ (in rpm) such that all *rotor-alone* tones at the blade passing frequency are cut-off. [25%]

(ii) At the design speed of 6000 rpm, determine which rotor-stator interaction modes propagate at the blade passing frequency and at twice the blade passing frequency. Comment on whether the stator vane number is a good choice. [55%]

	$n = 0$	$n = \pm 1$	$n = \pm 2$	$n = \pm 3$	$n = \pm 4$	$n = \pm 5$	$n = \pm 6$
$m = 1$	0.000	1.841	3.054	4.201	5.318	6.416	7.501
$m = 2$	3.832	5.331	6.706	8.015	9.282	10.520	11.735
$m = 3$	7.016	8.536	9.969	11.346	12.682	13.987	15.268

Table 1

4 A Helmholtz resonator with neck cross-sectional area A, effective neck length l, and cavity volume V is mounted as a side-branch on a duct of cross-sectional area S, as shown in Fig. 1. The acoustic medium has density ρ_0 and sound speed c_0. A plane harmonic acoustic wave of amplitude I is incident from the left.

(a) By considering mass conservation in the cavity and momentum conservation in the neck, derive an expression relating the pressure perturbation at the neck entrance p_1' to the mass flow rate into the resonator m' for harmonic disturbances of frequency ω. State clearly the assumptions required. [30%]

(b) The result of part (a) can be written in the form:

$$p_1' = Z'(\omega)\, m'$$

where $Z'(\omega)$ is a function of ω, l, A, V, and c_0. Let R, and T denote the strengths of the reflected, and transmitted pressure waves in the duct. By applying continuity of pressure and conservation of mass flow rate at the junction, derive an expression for the reflection coefficient $|R/I|$ in terms of S, c_0, and Z'. [50%]

(c) Using your expressions from parts (a) and (b), or otherwise, find the frequency at which $|R/I|$ is maximum and find its maximum value. Comment on whether this prediction is physically reasonable, and state what additional physical effect must be included in the model to correct this. [20%]

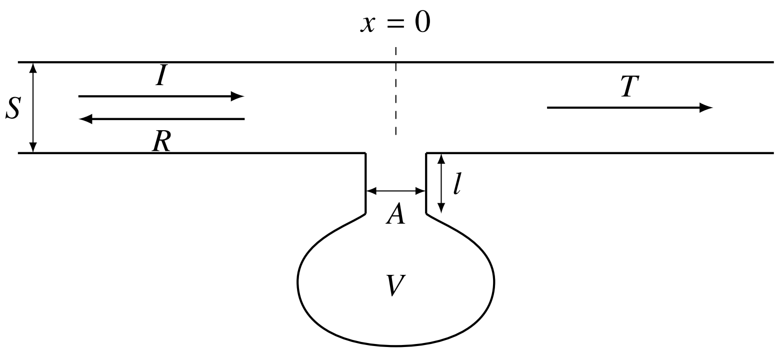

Fig. 1

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Module 4A15 Aeroacoustics Data Sheet

USEFUL DATA AND DEFINITIONS

Physical Properties

Speed of sound in an ideal gas $\sqrt{\gamma RT}$, where T is temperature in Kelvins

Units of sound measurement

$$\begin{aligned}
\text{SPL (sound pressure level)} &= 20\log_{10}\left(\frac{p'_{rms}}{2\times 10^{-5}\text{Nm}^{-2}}\right)\text{dB}\\
\text{IL (intensity level)} &= 10\log_{10}\left(\frac{\text{intensity}}{10^{-12}\text{watts m}^{-2}}\right)\text{dB}\\
\text{PWL (power level)} &= 10\log_{10}\left(\frac{\text{sound power output}}{10^{-12}\text{watts}}\right)\text{dB}
\end{aligned}$$

Definitions

Surface impedance Z_s, relates the pressure perturbation applied to a surface, p', to its normal velocity v'; $p' = Z_s v'$

Characteristic impedance of a fluid $\rho_0 c_0$

Specific impedance of a surface $Z_s/(\rho_0 c_0)$

Wavenumber $k = \omega/c_0 = 2\pi/\lambda$, where λ is the wavelength

Helmholtz number (or compactness ratio) = kD, where D is a typical dimension of the source.

Strouhal number $= \omega D/(2\pi U)$ for sound of frequency ω (in rad/s), produced in a flow with speed U, length scale D.

Basic equations for linear acoustics

Conservation of mass

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot\mathbf{v}' = 0$$

Conservation of momentum

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0$$

Isentropic

$$c_0^2 = \left. \frac{dp}{d\rho} \right|_S$$

Wave equation

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

Energy density

$$e = \frac{1}{2} \rho_0 v'^2 + \frac{1}{2\rho_0 c_0^2} p'^2$$

Intensity $\mathbf{I} = p'\mathbf{v}'$

Velocity potential ϕ' satisfies the wave equation and $p' = -\rho_0 \frac{\partial \phi'}{\partial t}$, $\mathbf{v}' = \nabla \phi'$.

Autocorrelation $F(\xi)$, the autocorrelation of $f(y)$ is given by

$$F(\xi) = \overline{f(y) f(y+\xi)}$$

$$F(0) = \overline{f^2}$$

Integral length scale, l

$$l\overline{f^2} = \int_{-\infty}^{\infty} F(\xi) d\xi$$

Sound power

Sound power from a source is defined as

$$P = \int_S \bar{\mathbf{I}} \cdot \mathbf{dS} = \int_{S_\infty} \frac{\overline{p'^2}}{\rho_0 c_0} \mathbf{dS}$$

for a statistically stationary source. For an outward propagating spherically symmetrical sound field $P = \frac{\overline{p'^2}}{\rho_0 c_0} 4\pi r^2$, where p' is the acosutic pressure at radius r.

For a sound field, which is a function of spherical polar coordinates r, θ only, and is independent of the azimuthal angle,

$$P = 2\pi r^2 \int_0^\pi \frac{\overline{p'^2}}{\rho_0 c_0} \sin\theta d\theta$$

Simple wave fields

1D or plane wave

The general solution of the 1D wave equation is $p'(x,t) = f(t - x/c_0) + g(t + x/c_0)$, where f and g are arbitrary functions. In a plane wave propagating to the right $p' = \rho_0 c_0 u'$; in a plane wave propagating to the left $p' = -\rho_0 c_0 u'$, u' being the particle velocity.

Spherically symmetric sound fields

The general spherically symmetric solution of the 3D wave equation is

$$\phi'(r,t) = \frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r},$$

where r is the distance from the source; f and g are arbitrary functions.

$\cos\theta$ dependence

The general solution of the 3D wave equation with $\cos\theta$ dependence is

$$p'(\mathbf{x},t) = \frac{\partial}{\partial x}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right] = \cos\theta \frac{\partial}{\partial r}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right]$$

Useful mathematical formulae

Spherical polar coordinates (r, θ, ψ)

Gradient

$$\nabla p' = \left(\frac{\partial p'}{\partial r}, \frac{1}{r}\frac{\partial p'}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p'}{\partial \psi}\right)$$

Divergence

$$\nabla \cdot \mathbf{v}' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v'_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta v'_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v'_\phi}{\partial \psi}$$

Laplacian

$$\nabla^2 p' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p'}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial p'}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p'}{\partial \psi^2}$$

Delta functions

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1D δ-function $\delta(x) = 0$ for $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(ax-b) f(x) dx = f(b/a)/|a|$

3D δ-function $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$

Convolution algebra

Convolution of $f(\mathbf{x})$ and $g(\mathbf{x})$

$$(f \star g)(\mathbf{x}) = \int_{-\infty}^{\infty} f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d\mathbf{y}$$

Commutative properties

$$f \star g = g \star f$$

$$\frac{\partial}{\partial x_i}(f \star g)(\mathbf{x}) = f \star \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \star g$$

Green's function

3D Green's function for wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right) g(\mathbf{x}, t | \mathbf{y}, \tau) = \delta(t-\tau)\delta(\mathbf{x}-\mathbf{y})$$

$$g(\mathbf{x}, t | \mathbf{y}, \tau) = \frac{\delta\left\{|\mathbf{x}-\mathbf{y}| - c_0(t-\tau)\right\}}{4\pi c_0 |\mathbf{x}-\mathbf{y}|}$$

Lighthill's Acoustic Analogy

Lighthill's equation

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right) \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}.$$

For cold, isentropic, low Mach-number jets, T_{ij} can be approximated as:

$$T_{ij} = \rho_0 u_i u_j$$

Lighthill eight power law Acoustic power,

$$P_a \sim \frac{\rho_o d_j^2}{c_0^5} u_j^8,$$

where d_j and u_j are the jet exit diameter and velocity, respectively.

In a cylindrical duct of radius a

The pressure field is given by

$$p'(\mathbf{x},t) = e^{i(\omega t + n\theta)} J_n(z_{mn} r/a)(Ae^{-ikx_3} + Be^{ikx_3}),$$

where z_{mn} is the mth zero of $\dot{J}_n(z)$ and $k = (k_0^2 - z_{mn}^2/a^2)^{1/2}$.

For large azimuthal wavenumber, n

$$z_{1n} \approx n + 1.85 n^{1/3}$$

In a duct of varying area $A(x)$

Webster horn equation

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{A}\frac{\partial}{\partial x}\left(A\frac{\partial p'}{\partial x}\right) = 0$$

Modified Webster horn equation $\psi(x) = \hat{p}(x) A^{1/2}$, $A = \pi a^2$

$$\frac{d^2\psi}{dx^2} + \left(k^2 - \frac{1}{a}\frac{d^2 a}{dx^2}\right)\psi = 0$$