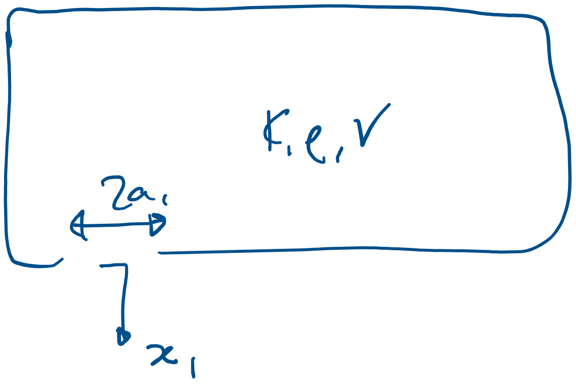
4C6 Advanced Linear Vibrations 2025

Qu1

(a)(i)



Hole area $A_1 = \pi a_1^2$

Apply "F = ma" — "double flange" case

$$pA_1 = m\ddot{x}_1 \quad , \quad m = 1.7 a_1 A_1 \rho = 1.7 \pi a_1^3 \rho$$

$$p = -K\frac{dV}{V}$$

$$= -\frac{K}{V} \cdot x_1 A_1$$

$$-\frac{K}{V} A_1^2 x_1 = m\ddot{x}_1 \Rightarrow 1.7 a_1 \cancel{A_1} \rho \ddot{x}_1 + \frac{K}{V} A_1^{\cancel{2}} x_1 = 0$$

$$\ddot{x}_1 + \frac{K A_1}{\rho \cdot 1.7 a_1 \cdot V} x_1 = 0 .$$

c^2

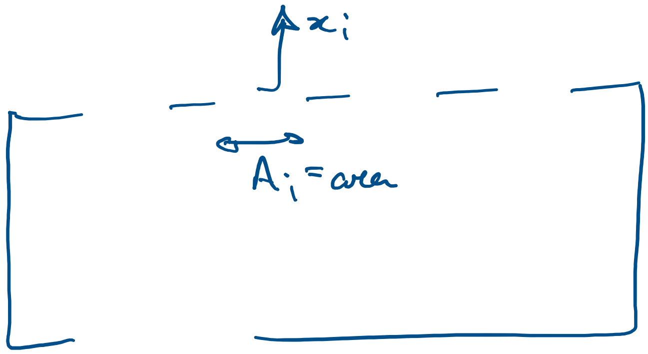
$$\omega_n^2 = c^2 \frac{A_1}{1.7 a_1 V} = c^2 \frac{\pi a_1^{\cancel{2}}}{1.7 \cancel{a_1} V}$$

(Check expect $\omega_n^2 = c^2 \frac{S}{VL}$ ✓)

$$\omega_n = 1.36 . c \sqrt{\frac{a_1}{V}}$$

[20%]

(ii)



$$p A_i = m_i \ddot{x}_i \,, \qquad m_i = 1.7 a_i A_i \rho = 1.7 \pi a_i^3 \rho$$

$$p = -K \frac{dV}{V} = -\frac{K}{V}\left(\sum_j A_j x_j \right)$$

$$A_i \left(-\frac{K}{V} \cdot \sum_j A_j x_j \right) = m_i \ddot{x}_i$$

$$m_i \ddot{x}_i + \frac{K A_i}{V} \cdot \sum_j A_j x_j = 0$$

$$m_i \ddot{x}_i + \frac{K A_i}{V} \left(A_1 x_1 + A_2 x_2 + A_3 x_3 + \dots \right) = 0$$

$$\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \end{bmatrix} + \frac{K}{V} \underbrace{\begin{bmatrix} A_1^2 & A_1 A_2 & A_1 A_3 \cdots \\ A_2 A_1 & A_2^2 & A_2 A_3 \cdots \\ & \ddots & \end{bmatrix}}_{\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} \begin{bmatrix} A_1 & A_2 & A_3 & \cdots \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

gives $\underline{\underline{M}}\,\underline{\ddot{x}} + \underline{k}\,\underline{k}^T \underline{x} = 0$

with $M = \mathrm{diag}(m_i)$

$$\underline{k} = \sqrt{\frac{K}{V}} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \end{bmatrix} = \sqrt{\frac{K}{V}}\,\pi \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ \vdots \end{bmatrix}$$

[30%]

(iii) Try: $u = M^{-1}k$ as given:

$$kk^T u = \omega^2 M u$$

$$\Rightarrow \quad kk^T M^{-1} k = \omega^2 M M^{-1} k$$

$$k\underbrace{\left(k^T M^{-1} k\right)}_{\text{scalar}} = \omega^2 k$$

so $\omega_N^2 = k^T M^{-1} k$

& $\omega_N = \sqrt{k^T M^{-1} k}$

Need eval's of $M^{-1}K = M^{-1}(kk^T)$

rank of $K = 1$ (as created from 1 vector)

so $\omega_{1,2,\ldots,N-1} = 0$ (all ways for air to flow in one hole & out another)

except one $\omega_N = \sqrt{k^T M^{-1} k}$

[30%]

(iv) Close hole ⇒ identical analysis but $(N-1)\times(N-1)$ matrices

so one non-zero frequency:

$$\omega_{N-1}^{2} = \sqrt{k'^T M'^{-1} k'}$$

where k' is $(N-1)\times 1$ & M' is $(N-1)\times(N-1)$

& elements removed correspond to hole that has been closed.

now $$k^T M^{-1} k = \sum_i \sum_j A_i A_j / M_{ij}$$

all terms +ve, so removing summations by one term makes this smaller, ie frequency reduced.

original frequencies: $0, 0 \ldots 0, \omega_N$

new frequencies: $0, 0 \ldots 0, \omega_{N-1}'$

$$0 < \omega_{N-1}' < \omega_N$$

so interlacing satisfied //

[10%]

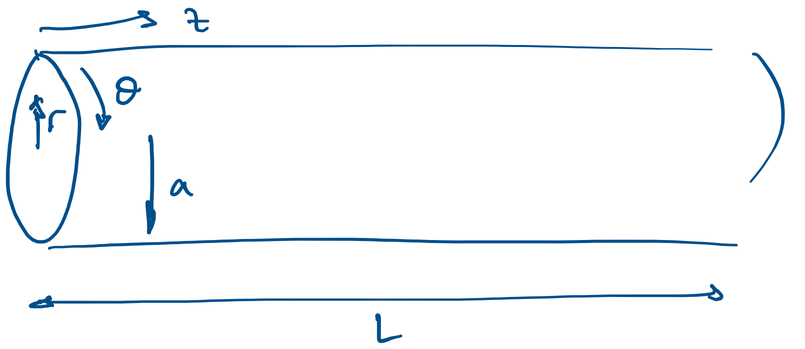
(b) $M = \text{diag}(m_i)$

$$K = \begin{bmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

Banded structure, like chain of masses & springs. But one way for air to pass through, so $(N-1)$ non-zero natural frequencies // [10%]

Qu2

(a)



$$c^2\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}\right) = \frac{\partial^2 p}{\partial t^2}$$

$$p = f(r,\theta)\, e^{i(\omega t - \lambda z)}$$

$$\Rightarrow c^2\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} - \lambda^2 f\right) = -\omega^2 f$$

$$c^2\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right) = \left(c^2\lambda^2 + \omega^2\right) f$$

LHS is equiv to PDE for membrane from datasheet as required

[10%]

(b) by S.o.V let $f = R(r)G(\theta)$

$$\Rightarrow c^2\left(R''G + \frac{1}{r}R'G + \frac{1}{r^2}RG'' - \lambda^2 RG\right) = -\omega^2 RG$$

$$\times \frac{r^2}{c^2}\cdot\frac{1}{RG} \Rightarrow r^2\frac{R''}{R} + \frac{rR'}{R} + \frac{G''}{G} - \lambda^2 r^2 = -\frac{\omega^2 r^2}{c^2}$$

$$\underbrace{r^2\frac{R''}{R} + r\frac{R'}{R} + \left(\frac{\omega^2}{c^2} - \lambda^2\right)r^2}_{R(r)} = \underbrace{-\frac{G''}{G}}_{G(\theta)} = \text{const}$$

circ symmetry so $G = C\sin n\theta + D\cos n\theta$

& const $= n^2$.

$$\Rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} + \left(k^2r^2 - n^2\right) = 0$$

where $k^2 = \frac{\omega^2}{c^2} - \lambda^2$

sol is Bessel function $R = J_n(kr)$

for z: harmonic & $Z(0) = Z(L) = 0$ as open;

so $Z = A\sin\frac{m\pi z}{L}$

ie $\lambda = \frac{m\pi}{L}$, $B = 0$.

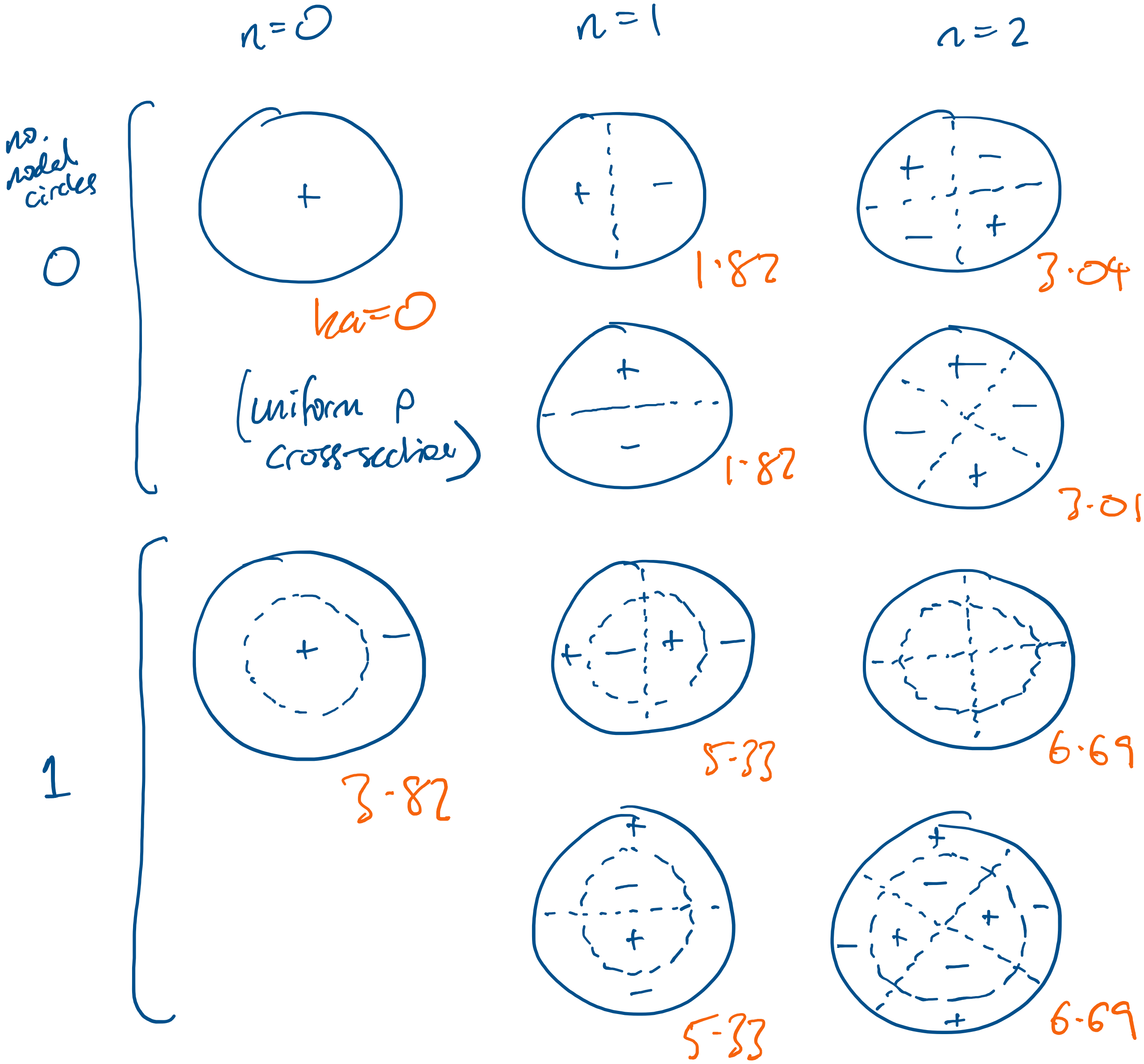
Gives: $p = J_n(kr) \cdot \cos n\theta \cdot \sin\frac{m\pi z}{L}$

& $p = J_n(kr)\sin n\theta \cdot \sin\frac{m\pi z}{L}$

& $k^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{L}\right)^2$

[30%]

(c) Note: BC's require zero gradient, ie stationary points of Bessel functions in order to find right "ka" values



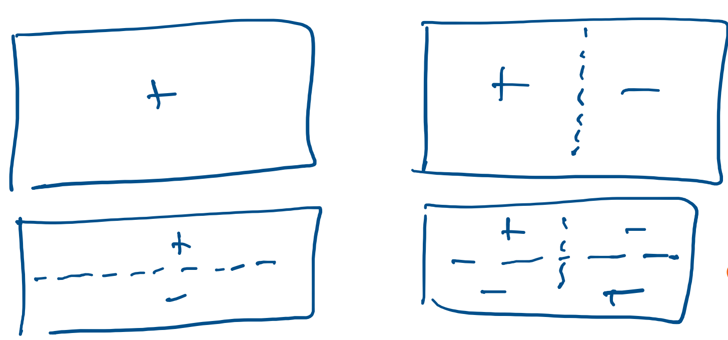
[30%]

(d) (i) $p = \sin\frac{n\pi x}{L_x} \cdot \sin\frac{m\pi y}{L_y} \cdot \sin\frac{p\pi z}{L_z}$

for room dimensions $L_x \times L_y \times L_z$.

ie stretched string modes in each direction.

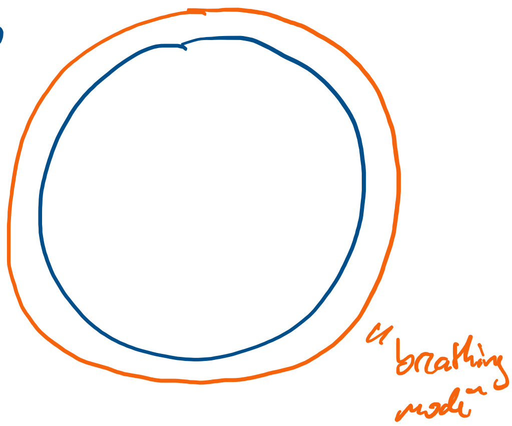
2D plan view



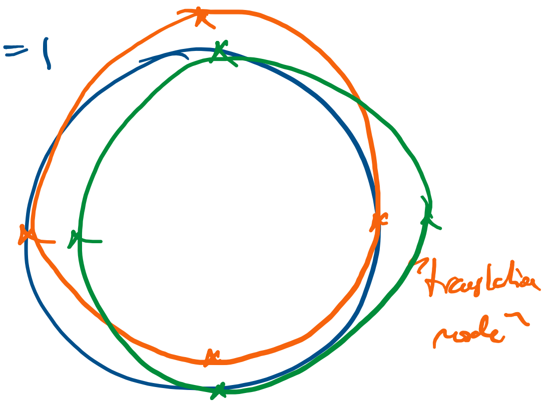
[10%]

(ii) Circ symmetry: $f(\theta) = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$

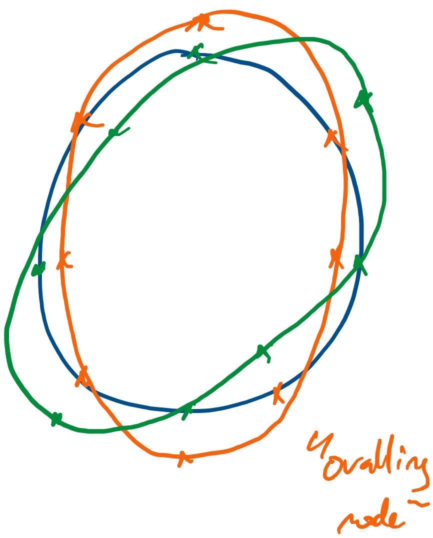
$n=0$



$n=1$



$n=2$

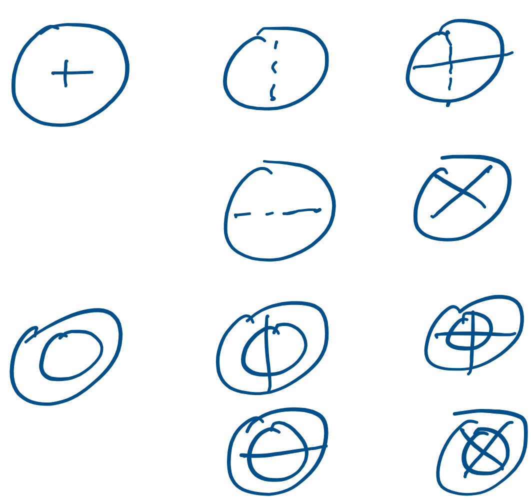


. . .

[10%]

(iii) Circular plate : expect similar to membrane,

& $g(\theta) = \sin n\theta, \cos n\theta$



[10%]

Qu 3

(a)

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad K = \begin{bmatrix} k+s & -s \\ -s & k+s \end{bmatrix}$$

$$\omega_1^2 = k/m \qquad \omega_2^2 = \frac{k+2s}{m}$$

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(By symmetry & inspection)

[10%]

(b) $M\ddot{x} + C\dot{x} + Kx = 0.$

so $(K + i\omega C - \omega^2 M)x = 0$ for unforced damped vibration.

Rayleigh damping:

$$[K + i\omega(\alpha K + \beta M) - \omega^2 M]u = 0.$$

$$[(1+i\omega\alpha)K + (i\omega\beta - \omega^2)M]u = 0$$

$$\left[K + \frac{i\omega\beta - \omega^2}{1+i\omega\alpha} M\right]u = 0$$

same form as $[K - \omega^2 M]u = 0$

so u unchanged but "ω" is changed

Result does not depend on specific "M".

[20%]

(c)(i) $C = \alpha K + \beta M = 0.1K + 0.1M$

$$= 0.1\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} + 0.1\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 & \\ & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.1 \\ -0.1 & 0.6 \end{bmatrix}$$

$$z = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \qquad \dot{\underline{z}} = \underline{\underline{A}}\,\underline{z}$$

$$\underline{\underline{A}} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0.5 & -0.3 & 0.05 \\ 0.5 & -2 & 0.05 & -0.3 \end{bmatrix}$$

[20%]

(ii) • Mode shapes unchanged, so for $u_1 = \begin{bmatrix} 1 \\ 1 \\ \lambda \\ \lambda \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0.5 & -0.3 & 0.05 \\ 0.5 & -2 & 0.05 & -0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \lambda \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ \lambda \\ \lambda \end{bmatrix}$$

r3: $\quad -2 + 0.5 - 0.3\lambda + 0.05\lambda = \lambda^2$

$$\lambda^2 + 0.25\lambda + 1.5 = 0.$$

$$\lambda = -0.125 \pm 1.22j \,//$$

$\omega_n = 1.22$, $\quad \zeta\omega_n = 0.125$

$$\zeta = \frac{0.125}{1.22} = 0.1$$

$$Q = 5 \,/ \quad \text{for mode } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

• Next mode: $u_2 = \begin{bmatrix} 1 \\ -1 \\ \lambda \\ -\lambda \end{bmatrix}$

so
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0.5 & -0.3 & 0.05 \\ 0.5 & -2 & 0.05 & -0.3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ \lambda \\ -\lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \\ \lambda \\ -\lambda \end{bmatrix}$$

r3 :
$$-2 - 0.5 - 0.3\lambda - 0.05\lambda = \lambda^2$$

$$\lambda^2 + 0.35\lambda + 2.5 = 0$$

$$\lambda = -0.175 \pm 1.57j$$

so $\zeta = \dfrac{0.175}{1.57} = 0.11$, $Q = 4.5$ // for mode $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

[30%]

(d) $C = \alpha K^2$, $M = \gamma I$

$$[K + i\omega\alpha K^2 - \omega^2\gamma I] u = 0$$

note $Ku = \omega^2 \gamma u$ for undamped modes

$\times K \Rightarrow$
$$K^2 u = \omega^2 \gamma K u$$
$$= \omega^2 \gamma [\omega^2 \gamma u]$$
$$= \omega^4 \gamma^2 u$$
ie u is also evec of K^2.

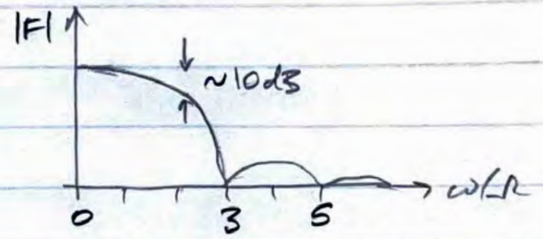
so $[K + i\omega\alpha K^2] u \propto u$ as well, hence mode shapes unchanged //.

Result is specific to $M \propto I$ //

[20%]

(4) a). $M = 0.2\,kg$; $k = 0.5$, 2 or 10 MN/m ; $f_{interest} = 10 - 1000$ Hz.

i). Impulse spectrum takes the form

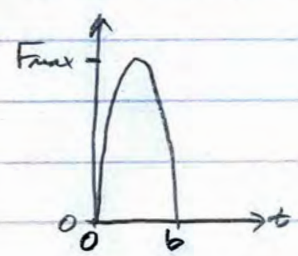


$\Rightarrow$ 'working range' $\approx 2\Omega$

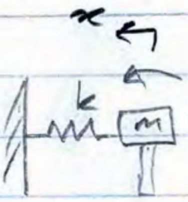
$\therefore$ require $2\Omega = 1000$ Hz $= 2\pi \times 1000$ rad/s $\Rightarrow \Omega = 1000\pi$ rad/s

$\therefore k = \Omega^2 M = (1000\pi)^2 \times 0.2 = 2.0 \times 10^6 = 2$ MN/m 15%

ii). Max. force = 1.3 kN



$b = \frac{\pi}{\Omega} = \frac{\pi}{1000\pi} = 1\,ms$



$x = X \sin \Omega t$

$\dot{x} = \Omega X \cos \Omega t$

$= V \cos \Omega t$

$\therefore V = \Omega X = \Omega \frac{F_{max}}{k} = 1000\pi \times \frac{1300}{2 \times 10^6}$

$= \frac{1.3\pi}{2} = 2\,m/s$ 15%

iii). Since the maximum frequency of interest is 1 kHz, the filter could act as an anti-aliasing filter, removing unwanted signal above 1 kHz and therefore enabling the use of a relatively low sampling frequency (and hence minimising the data storage requirement). However, any filter would introduce signal distortion in the form of magnitude ripples and phase distortion. 10%

iv) For a 1kHz LP filter, we could choose $f_s = 3$ kHz, to satisfy the Nyquist criterion with some margin. 10%

PTO.

4 cont.

b). $H(j,k,\omega)=\sum_{1}^{N}\frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2+2i\omega\omega_n\zeta_n-\omega^2}$, $\zeta_n=\frac{1}{2Q}$

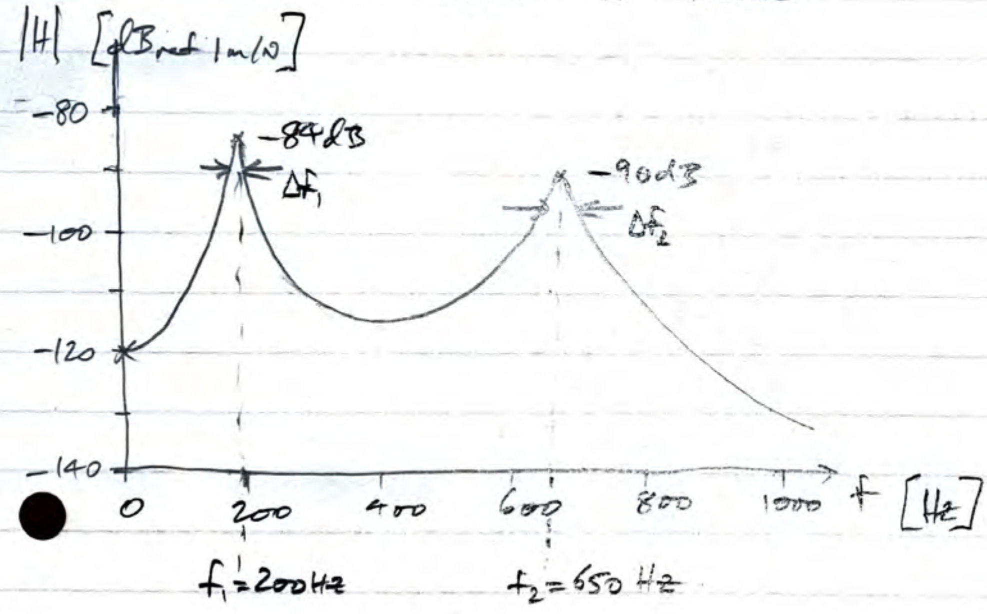
At resonance , $|H|^{(n)}\simeq\left|\frac{u_j^{(n)}u_k^{(n)}}{2i\omega_n^2\zeta_n}\right|=Q\frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2}$

$\therefore |H|^{(1)}\simeq\frac{50\times2}{(2\pi\times200)}=6.3\times10^{-5}$ m/N , $|H|^{(2)}\simeq\frac{100\times5}{(2\pi\times650)^2}=3.0\times10^{-5}$ m/N

$20\log_{10}(6.3\times10^{-5})=-84$ dB ; $20\log_{10}(3.0\times10^{-5})=-90$ dB.

Statically, $|H(0)|=\left|\frac{u_j^{(1)}u_k^{(1)}}{\omega_1^2}+\frac{u_j^{(2)}u_k^{(2)}}{\omega_2^2}\right|=\left|\frac{2}{(2\pi\times200)^2}-\frac{5}{(2\pi\times650)^2}\right|$

$=9.7\times10^{-7}\equiv-120$ dB



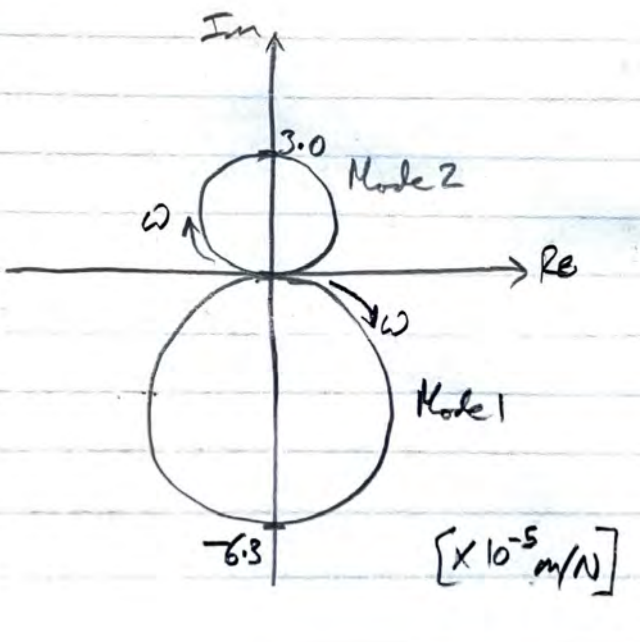
Half-power bandwidths:

$\frac{\Delta f}{f}=\frac{1}{Q}$

$\therefore \Delta f_1=\frac{200}{50}=4$ Hz.

$\Delta f_2=\frac{650}{100}=6.5$ Hz.

25%

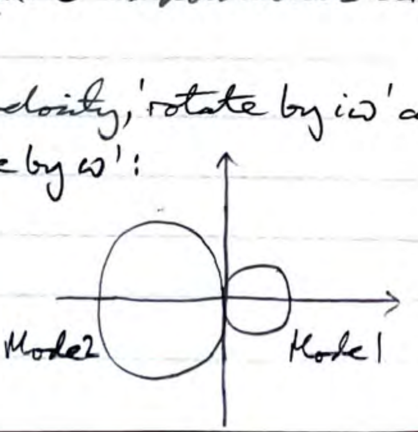


15%

$H=\frac{u_j^{(n)}u_k^{(n)}}{2i\omega_n^2\zeta_n}=\frac{Qa_n}{i\omega_n^2}=-i\frac{Qa_n}{\omega_n^2}$

$\therefore a_n>0\Rightarrow$ 'downwards' circle.

For velocity, 'rotate by $i\omega$' and 'scale by ω':



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