Q1 (a) input

- Instrumented hammer: + no added mass
  - hard to control bandwidth / get clear pulse
- EM Shaker: + can choose input signal
  - added mass
- Counter rotating masses: + can deliver large force
  - sinusoidal / swept input only.

response

- Accelerometer: + good linearity & cost effective
  - added mass
- laser vibrometer: + non-contact, high linearity, wide bandwidth
  - expensive, bulky.
- laser triangulation: + non-contact
  - limited bandwidth

20%

(b) (i) $H_1 = \dfrac{S_{yf}}{S_{ff}}$, $H_2 = \dfrac{S_{yy}}{S_{fy}}$

10%

(ii) with output noise:

$$H_1 = \frac{S_{yf}}{S_{ff}} = \frac{E[(Y+N)F^*]}{S_{ff}}$$

$$= \frac{E[YF^*] + \cancel{E[NF^*]}^{\,0}}{S_{ff}}$$

$$= \frac{S_{yf}}{S_{ff}} = H_{true}$$

So H_1 is best in this case.

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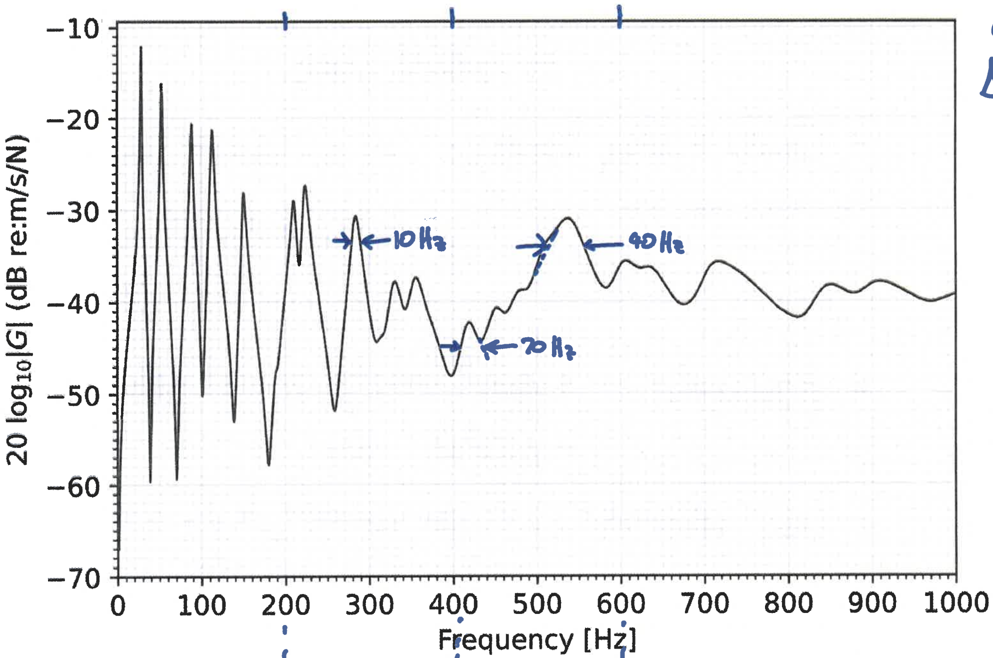
(iii) • Uncorrelated noise on input/output signals

• Nonlinearity in the system

• Too short frame lengths reducing coherence @ peaks. 15%

(c) no. modes: 5 5 5 = 15 modes in 0–600



i.e. $\Delta f_n \simeq \frac{600}{15}$

$= \underline{40\,Hz}$.

Bandwidth (approx average): 0.8Hz | 10Hz | 30Hz

↓ from Nyquist of 1st mode! OR could measure

Modal overlap: $\frac{0.8}{40}$ | $\frac{10}{40}$ | $\frac{30}{40}$

$M = \frac{f_{3dB}}{\Delta f_n}$ = 0.02 | 0.25 | 0.75 10%

(ii) Modal analysis best when $M \ll 1$, so modal analysis most effective for $f < 400\,Hz$, but may be ok to 600Hz. Answers in the range 400–600Hz accepted.

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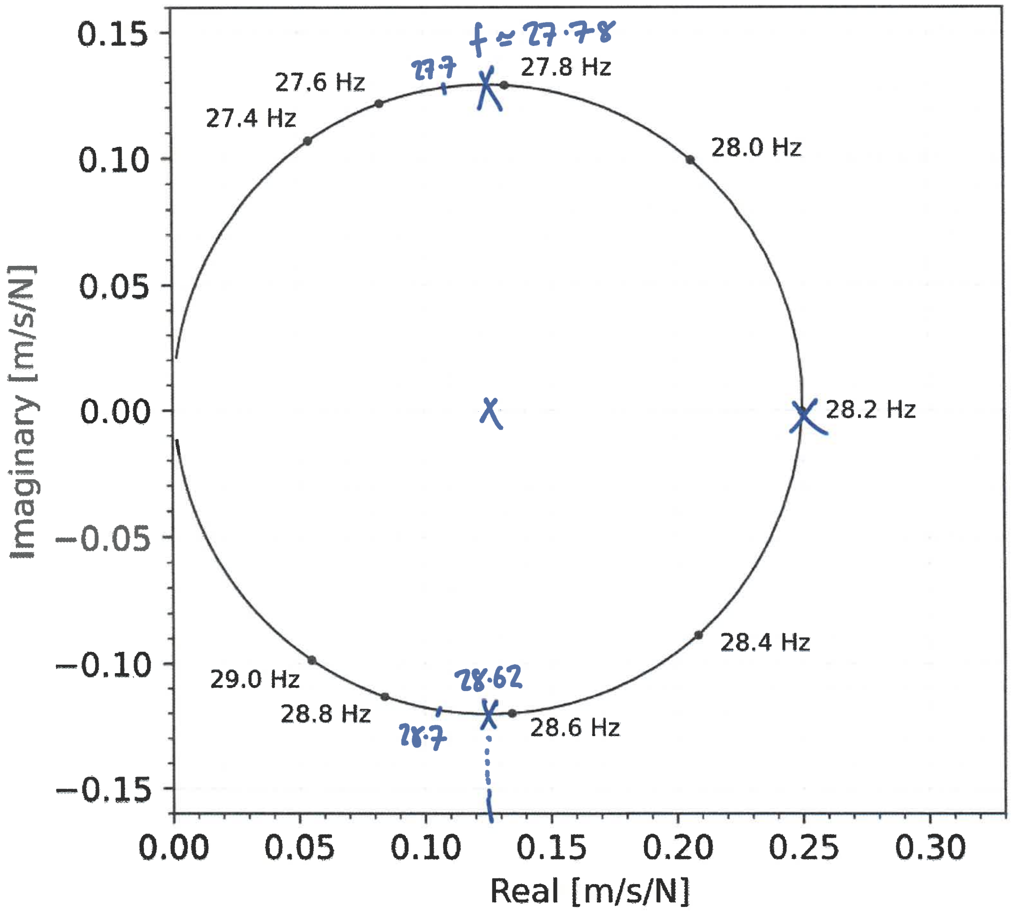
(iii)



$\Delta f_{3dB} \simeq 28.62 - 27.78 = 0.84$

$f_n \simeq 28.2\ Hz$

$\zeta_n \simeq \dfrac{0.84}{2 \times 28.2} = 0.015$, $Q = 34$

$\text{radius} = \dfrac{a_n}{4\zeta_n \omega_n} = 0.125$

so $a_n = 0.125 \cdot 4 \cdot 0.015 \cdot 2\pi \cdot 28.2$

$= 1.33\ kg^{-1}$

15%

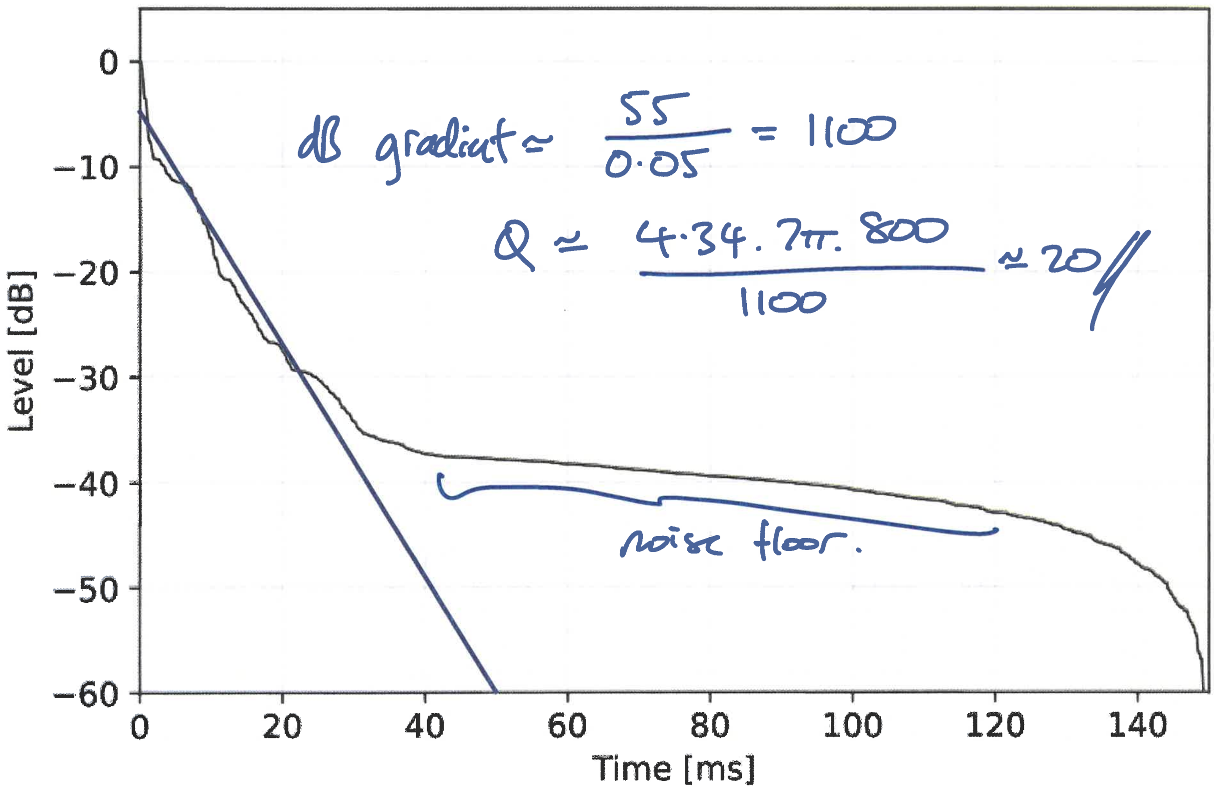
(d) $E = \int_t^\infty g^2\, dt$ (Schroeder integral).

assuming envelope squared follows: $E = E_0 e^{-2\zeta_n \omega_n t}$

$\Rightarrow 10 \log_{10} E = C - 2\zeta \omega_n t \cdot 10 \log_{10} e$

$$= C - \frac{\omega_n}{Q} 10 \log_{10} e \, . \, t$$

$$\text{dB gradient} = -4.34 \frac{\omega_n}{Q}$$



$$\text{dB gradient} \simeq \frac{55}{0.05} = 1100$$

$$Q \simeq \frac{4.34 . 7\pi . 800}{1100} \simeq 20$$

$Q \simeq 20$ (approx 15–25 accepted if method correct)

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use α, β to distinguish from Area A

Q2 (a) $Y = \alpha \sin kx + \beta \cos kx \Rightarrow \beta \Rightarrow \alpha \sin kL + \beta \cos kL$

$$F = -EAY' = -EA\alpha k \cos kx + EA\beta k \sin kx \Rightarrow \underbrace{-EA\alpha k}_{x=0} \Rightarrow \underbrace{-EA\alpha k \cos kL + EA\beta k \sin kL}_{x=L}$$

So
$$\begin{bmatrix} Y(0) \\ F(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -EAk & 0 \end{bmatrix}}_{\text{inv} = \frac{1}{EAk}\begin{bmatrix} 0 & -1 \\ EAk & 0 \end{bmatrix}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \begin{bmatrix} Y(L) \\ F(L) \end{bmatrix} = \begin{bmatrix} \sin kL & \cos kL \\ -EAk\cos kL & EAk \sin kL \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

$$\begin{bmatrix} Y(L) \\ F(L) \end{bmatrix} = \begin{bmatrix} \sin kL & \cos kL \\ -EAk\cos kL & EAk \sin kL \end{bmatrix} \begin{bmatrix} 0 & -1/EAk \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y(0) \\ F(0) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos kL & -\frac{1}{EAk} \sin kL \\ EAk \sin kL & \cos kL \end{bmatrix}}_{T_b} \begin{bmatrix} Y(0) \\ F(0) \end{bmatrix}$$

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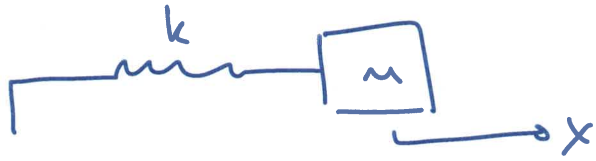
(b)



$\to Y_R = Y_L$

$F_L \rightarrow \quad \leftarrow F_R$

$$F_L - F_R = -m\omega^2 X \quad \text{—①}$$
$$F_L - F_R = k(Y_L - X) \quad \text{—②}$$

need to eliminate X

(2) $\Rightarrow$ $$X = Y_L - \frac{F_L - F_R}{k}$$

$\to$ (1) $\Rightarrow$ $$F_L - F_R = -M\omega^2 \left[Y_L - \frac{F_L - F_R}{k} \right]$$

$$F_R = F_L + M\omega^2 \left[Y_L - \frac{F_L - F_R}{k} \right]$$

$$F_R\left(1 - \frac{M\omega^2}{k}\right) = F_L\left(1 - \frac{M\omega^2}{k}\right) + M\omega^2 Y_L.$$

$$F_R = F_L + \frac{M\omega^2 k}{k - M\omega^2} Y_L$$

so: $$\begin{bmatrix} Y_R \\ F_R \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}}_{T_F} \begin{bmatrix} Y_L \\ F_L \end{bmatrix}, \quad z = \frac{M\omega^2 k}{k - M\omega^2}$$

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(c) $$\begin{bmatrix} Y_R \\ F_R \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \begin{bmatrix} \cos kL & -\frac{1}{EAk}\sin kL \\ EAk\sin kL & \cos kL \end{bmatrix} \begin{bmatrix} Y(0) \\ F(0) \end{bmatrix}$$

$$\begin{bmatrix} Y_R \\ F_R \end{bmatrix} = \underbrace{\begin{bmatrix} \cos kL & -\frac{1}{EAk}\sin kL \\ z\cos kL + EAk\sin kL & -\frac{z}{EAk}\sin kL + \cos kL \end{bmatrix}}_{T_{br}} \begin{bmatrix} Y(0) \\ F(0) \end{bmatrix}$$

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(d) $T = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$

Passive, so $\det(T) = 1$

characteristic eq: $\lambda^2 - tr(T)\lambda + \underset{\substack{\| \\ 1}}{\det(T)} = 0.$

let $\lambda = e^{i\gamma}$

$$e^{2i\gamma} - tr(T)e^{i\gamma} + 1 = 0.$$

$$e^{i\gamma} - tr(T) + e^{-i\gamma} = 0.$$

$$2\cos\gamma = tr(T).$$

$$\cos\gamma = \frac{1}{2} tr(T) = \frac{1}{2}[a(\omega) + c(\omega)]$$

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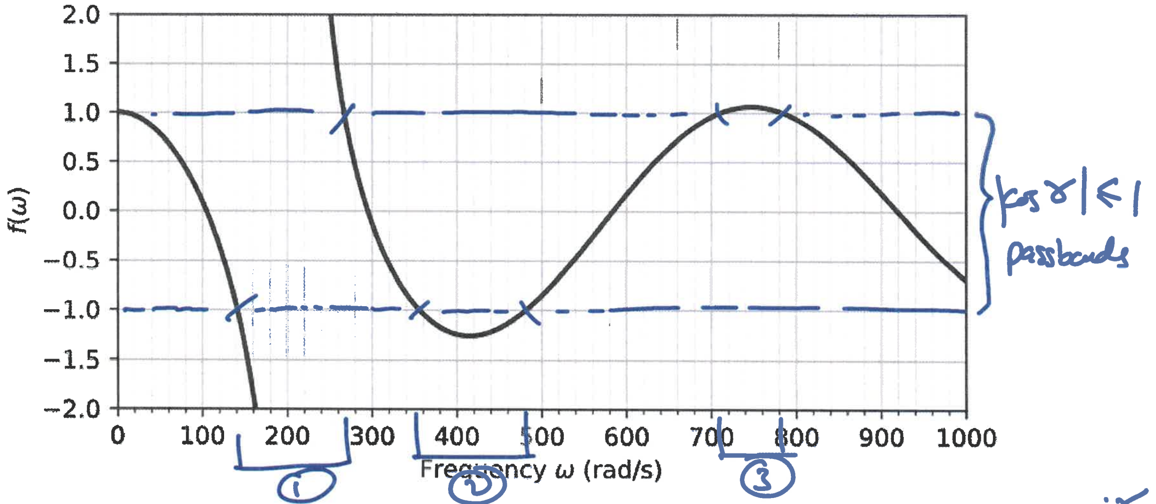
(e)



stopbands where $|f(\omega)| > 1$

stopbands:
1. 140 — 270 Hz
2. 350 — 480 Hz
3. 710 — 790 Hz

Waves propagate as $e^{i\gamma}$ from cell to cell. This happens without attenuation if γ is real.
In stop bands, $\cos\gamma > 1$, so γ is complex, & waves will attenuate from cell to cell.

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Q3 // (a) uniform: $A(x) = A_0$

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$ → wave equation
⇒ sinusoidal modes.

BC's: @ $x = 0, L$, $p = 0$.

$$u_n = \sin\frac{n\pi x}{L} \;//\qquad \omega_n = kc = \frac{n\pi c}{L} \;//$$

10%

(b) $A(x) = A_0 x^n$

(i) $p = P e^{i\omega t}$

→ PDE ⇒ $P'' + \frac{A'}{A}P' + \frac{\omega^2}{c^2}P = 0$ // ODE for P.

10%

(ii) $P = x^{-\nu}\phi(x)$

$$P' = -\nu x^{-(\nu+1)}\phi + x^{-\nu}\phi'$$

$$P'' = \nu(\nu+1)x^{-(\nu+2)}\phi \underbrace{- \nu x^{-(\nu+1)}\phi' - \nu x^{-(\nu+1)}\phi'}_{(2\nu x^{-(\nu+1)}\phi')} + x^{-\nu}\phi''$$

now $A = A_0 x^n$

$A' = A_0 n x^{n-1}$

so $A'/A = \frac{A_0 n x^{n-1}}{A_0 x^n} = n x^{-1}$ //

ODE:
$$\nu(\nu+1)x^{-(\nu+2)}\phi - 2\nu x^{-(\nu+1)}\phi' + x^{-\nu}\phi''$$
$$+ n x^{-1}\left[-\nu x^{-(\nu+1)}\phi + x^{-\nu}\phi'\right] + \frac{\omega^2}{c^2}x^{-\nu}\phi = 0$$

$$\Rightarrow \phi\left[\frac{\omega^2}{c^2}x^{-\nu} - n\nu x^{-1}.x^{-(\nu+1)} + \nu(\nu+1)x^{-(\nu+2)}\right]$$
$$+ \phi'\left[n x^{-1}.x^{-\nu} - 2\nu x^{-(\nu+1)}\right] + x^{-\nu}\phi'' = 0.$$

$$\phi\left[\frac{\omega^2}{c^2}x^{-\nu}+\left(\nu(\nu+1)-n\nu\right)x^{-(\nu+2)}\right]$$
$$+\phi'\left(n-2\nu\right)x^{-(\nu+1)}+x^{-\nu}\phi''=0.$$

terms: $\nu(\nu+1)-n\nu = \frac{n-1}{2}\cdot\frac{n+1}{2}-n\frac{n-1}{2}$
$$=\frac{n^2-1}{4}-\frac{2n^2-2n}{4}=\frac{-n^2+2n-1}{4}=-\left[\frac{(n-1)}{2}\right]^2$$
$$=-\nu^2$$

& $n-2\nu = n-(n-1)=1$

$$\Rightarrow \quad x^{-\nu}\phi''+x^{-(\nu+1)}\phi'+\left[\frac{\omega^2}{c^2}x^{-\nu}-\nu^2x^{-(\nu+2)}\right]\phi=0$$

$\times x^{\nu+2}$: $\quad x^2\phi''+x\phi'+\left(\frac{\omega^2}{c^2}x^2-\nu^2\right)\phi=0$ // as req.

$$\alpha=\frac{\omega^2}{c^2}=k^2,\quad \beta=\nu \;//$$

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(iii) No singularity @ $x=0$ so no contribution of Bessel function of second kind Y_ν, so $C_2=0$.

[Note: $x^{-\nu}J_\nu\to$ finite as $x\to0$ though this is not required for this question]

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(iv)

n	ν	k_1L	k_2L	$\omega_1 = ck$	ω_2
1	0	2.40	5.52	$2.4c/L$	$5.52c/L$
2	0.5	3.14	6.28	$3.14c/L$	$6.28c/L$
3	1	3.83	7.02	$3.83c/L$	$7.02c/L$.

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Q4 (a) $u = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{i\omega t}$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u$$

$$\frac{\partial^4 u}{\partial x^4} = \left(\frac{m\pi}{a}\right)^4 u, \quad \frac{\partial^4 \phi}{\partial y^4} = \left(\frac{n\pi}{b}\right)^4 u$$

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 u$$

$$\Rightarrow \quad -\omega^2 \rho h u + D \cdot \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 u = 0.$$

satisfies PDE for some ω, m, n

non-zero u: $\omega_{m,n} = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \sqrt{D/\rho h}$

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(b)
- Free/constrained damping layer: exploits loss from material in added layer
- Tuned mass damper: add a mass-on-spring tuned to a specific resonance. Splits mode & large amp motion of TMD can be damped locally.
- Add a particle damper: multiple local impacts cause energy to scatter to high frequencies where damped by other mechanisms, in addition to friction damping between particles.

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(c) $D \to D(1+i\eta_E)$

so $V \to V(1+i\eta_E)$

$$\omega_n'^2 = \omega_n^2(1+i\eta_E)$$

$$\omega_n' \simeq \omega_n\left(1+\tfrac{1}{2}i\eta_E\right)$$

$\Rightarrow$ transient decay: $e^{i\omega_n(1+\frac{1}{2}i\eta_E)t}$

$$= e^{i\omega_n t} \cdot e^{-\frac{1}{2}\omega_n \eta_E t}$$

cf $e^{-\zeta\omega_n t} = e^{-\frac{1}{2Q_n}\omega_n t}$

$$\Rightarrow Q_n = \frac{1}{\eta_E} \; //.$$

20%

(d) $V = \frac{1}{2}K_{mn}(1+i\eta_E) + \frac{1}{2}k(1+\eta_k)\underbrace{\phi^2(x_0,y_0)}_{=\phi_0^2}$ for amplitude $A=1$.

$$= \underbrace{\tfrac{1}{2}K_{mn}}_{=V_0} + \underbrace{\tfrac{1}{2}k\phi_0^2 + \tfrac{1}{2}i\left[K_{mn}\eta_E + k\phi_0^2\eta_k\right]}_{=V'}$$

$$\omega_n'^2 = \frac{V_0+V'}{T_0} = \frac{V_0}{T_0}\left(1+V'/V_0\right) = \omega_n^2\left(1+V'/V_0\right)$$

& we need $\eta_{eff} = \dfrac{\mathrm{im}\{\omega_n'^2\}}{\mathrm{re}\{\omega_n'^2\}}$

$$= \frac{K_{mn}\eta_E + k\phi_0^2\eta_k}{K_{mn} + k\phi_0^2} \; //$$

So $Q = \dfrac{K_{mn} + k\phi_0^2}{K_{mn}\eta_E + k\phi_0^2\eta_k} \; //$

40%

(e) $K \to \infty \Rightarrow$ constraint, so new frequencies higher & must satisfy interlacing. //.

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