Version TB/4

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 28 April 2026 9.30 to 11.10

Module 4C6

ADVANCED LINEAR VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C6 Advanced Linear Vibration data sheet (7 pages).
Supplementary pages: extra copies of Figs. 1a, 1b, and 1c (Question 1)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Experimental modal analysis is used to estimate the modal properties of a test object.

Note that this question makes use of Figs. 1a, 1b, and 1c. An additional copy of each figure is provided on the supplementary pages: hand any annotated copies in with your solution to show your working.

(a) Identify three options for providing input excitation to the object, and three instrumentation options for measuring the structural response. For each option, highlight one key advantage and one key disadvantage. [20%]

(b) A random input force $f(t)$ is used to excite the test object, and the velocity response $y(t)$ is measured at a single point. The signals are divided into frames, and $F(\omega)$ and $Y(\omega)$ denote the Fourier Transforms of a single frame of the input and output respectively. The cross-spectral density $S_{fy}(\omega)$ is estimated as $\mathrm{E}[FY^*]$, where * denotes the complex conjugate and $\mathrm{E}[\cdot]$ denotes the average over frames.

(i) Define two estimates of the transfer function from the input force to the output velocity, in terms of the spectral densities $S_{ff}(\omega)$, $S_{yf}(\omega)$, $S_{fy}(\omega)$, and $S_{yy}(\omega)$. [10%]

(ii) If it is known that measurement noise is present on the output signal only, show which estimate is better. [10%]

(iii) Identify three reasons why the coherence function may be less than 1 at a given frequency. [15%]

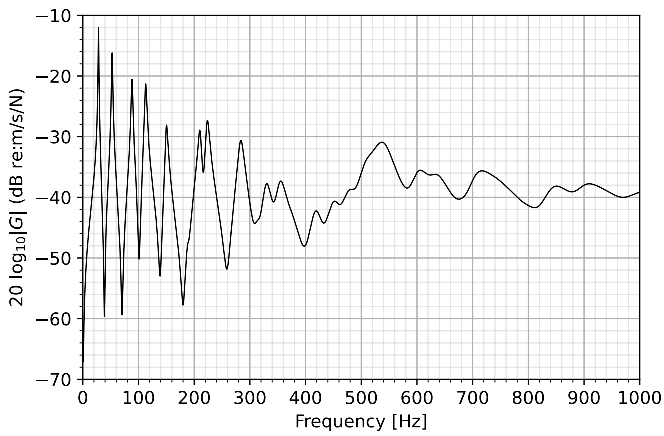
(c) An example Frequency Response Function is shown in Fig. 1a and a labelled Nyquist plot for frequencies near the first mode is shown in Fig. 1b. Note that for this test object, the modal density is approximately constant.

(i) Estimate the modal overlap factors for the frequency ranges $0 - 200$ Hz, $200 - 400$ Hz, and $400 - 600$ Hz. [10%]

(ii) Over what frequency range is modal analysis likely to be most effective? Justify your answer. [10%]

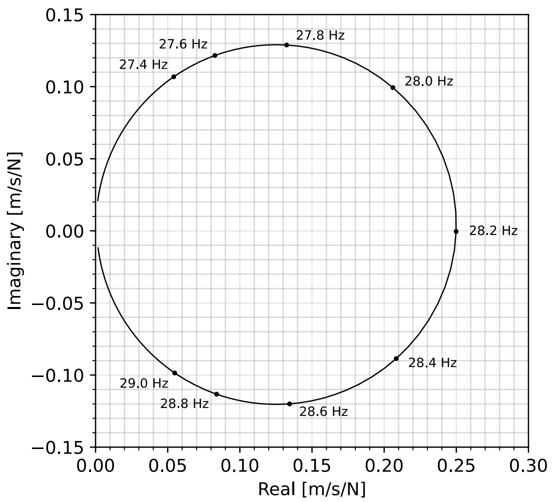
(iii) Use the Nyquist plot in Fig. 1b to estimate the natural frequency, Q-factor, and modal amplitude for the first mode. [15%]

(d) The Schroeder decay curve from a transient response is shown in Fig. 1c for the frequency range $600 - 1000$ Hz. By assuming an exponential decay envelope and a band centre frequency of 800 Hz, estimate the band-averaged Q-factor from the curve. [10%]



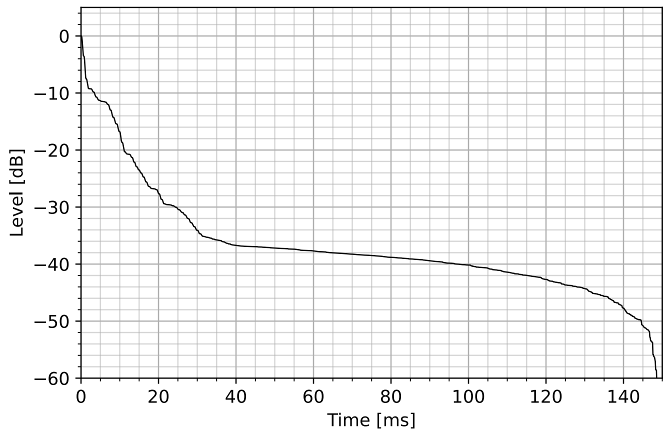
20 log10|G| (dB re:m/s/N)
Frequency [Hz]

(a)



Imaginary [m/s/N]
Real [m/s/N]
27.4 Hz
27.6 Hz
27.8 Hz
28.0 Hz
28.2 Hz
28.4 Hz
28.6 Hz
28.8 Hz
29.0 Hz

(b)



Level [dB]
Time [ms]

(c)

Fig. 1

2 A metamaterial is to be designed to prevent axial wave propagation in a bar for certain frequency ranges. This is to be achieved by adding a periodic array of spring-mass resonators along the length of the bar, as illustrated in Fig. 2a. Each unit cell consists of a uniform bar of length L, cross-sectional area A, Young's modulus E, and density ρ, and a spring-mass resonator attached at one end with mass m and stiffness k.

(a) Find the transfer matrix $\mathbf{T}_b$ for a uniform bar element as shown in Fig. 2b such that:

$$\begin{bmatrix} Y(L) \\ F(L) \end{bmatrix} = \mathbf{T}_b \begin{bmatrix} Y(0) \\ F(0) \end{bmatrix}$$

You may use the result that the frequency-domain axial displacement in the bar can be written $Y(x) = A \sin kx + B \cos kx$, such that $y(x,t) = Y(x)\mathrm{e}^{i\omega t}$, where x is the position along the bar, k is the wavenumber, and the force $F(x) = -EA\frac{\mathrm{d}Y}{\mathrm{d}x}$. [20%]

(b) Find the transfer matrix $\mathbf{T}_r$ across the connection point of a spring-mass resonator, as shown in Fig. 2c, such that:

$$\begin{bmatrix} Y_R \\ F_R \end{bmatrix} = \mathbf{T}_r \begin{bmatrix} Y_L \\ F_L \end{bmatrix}$$

Note that the displacement at the connection point is the same on both sides of the resonator connection point ($Y_R = Y_L$). [20%]

(c) Find the transfer matrix $\mathbf{T}_{br}$ for a single unit cell of the metamaterial, such that:

$$\begin{bmatrix} Y_R \\ F_R \end{bmatrix} = \mathbf{T}_{br} \begin{bmatrix} Y(0) \\ F(0) \end{bmatrix}$$

[10%]

(d) Consider a 2×2 transfer matrix for a passive unit cell of the form:

$$\mathbf{T} = \begin{bmatrix} a(\omega) & 0 \\ b(\omega) & c(\omega) \end{bmatrix}$$

Find an expression for the dispersion relation for an infinite array of unit cells in the form $\cos\gamma = f(\omega)$, where γ is the propagation constant for the periodic structure. [30%]

(e) Fig. 2d shows a plot of $f(\omega)$ for a particular choice of parameters. Identify the stop-bands (the frequency ranges where axial wave propagation is not possible). Briefly explain why axial waves cannot propagate in the stop-bands in terms of the propagation constant. [20%]

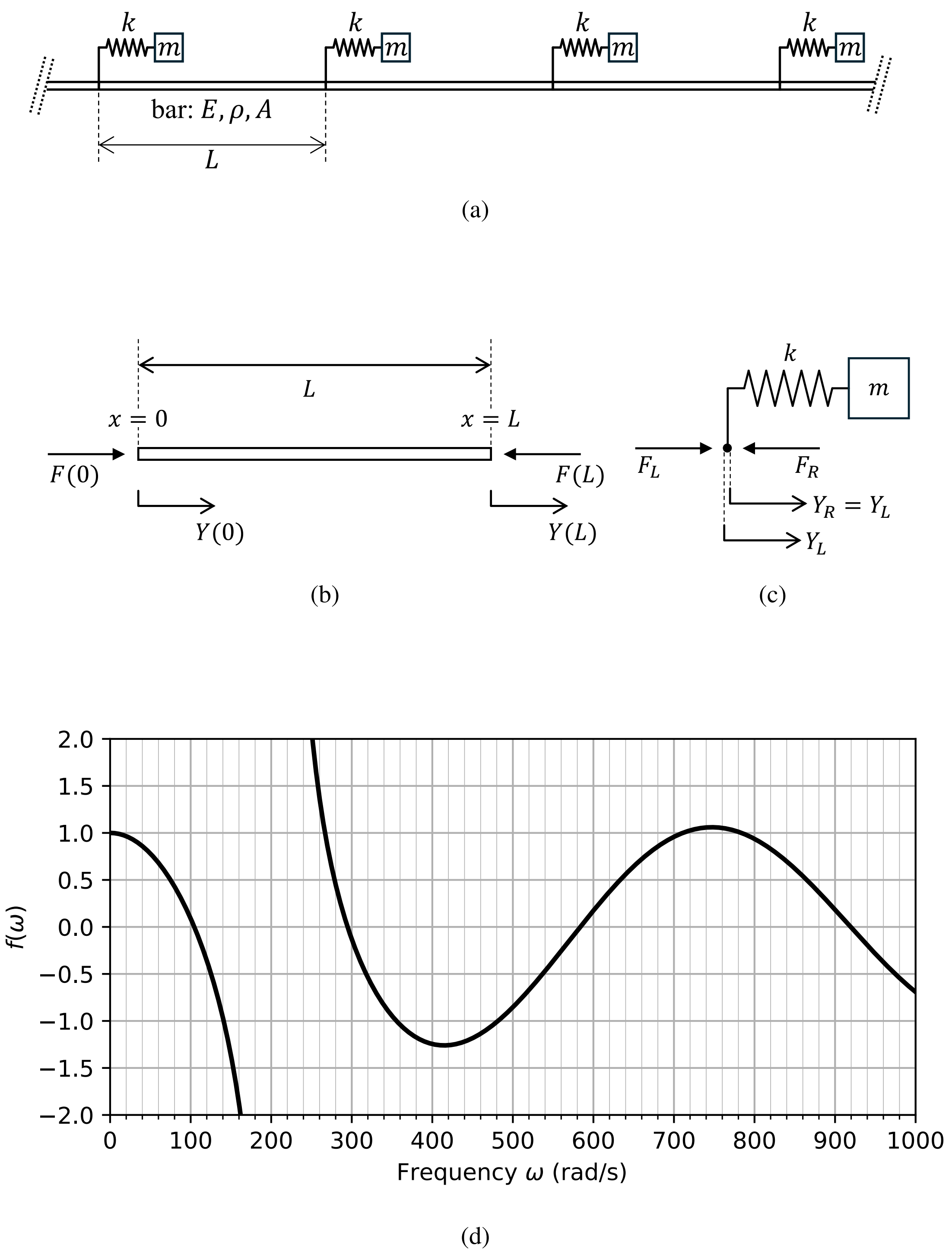


Fig. 2

(TURN OVER

3 The partial differential equation governing free 1D acoustic wave propagation in a duct with non-uniform cross-sectional area is given by:

$$\frac{\partial^2 p}{\partial x^2} + \frac{1}{A(x)}\frac{dA}{dx}\frac{\partial p}{\partial x} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

where $p(x,t)$ is the acoustic pressure, $A(x)$ is the cross-sectional area which can vary with distance along the duct x, and c is the speed of sound.

(a) Find the mode shapes $P(x)$ and natural frequencies ω_n for a duct of length L that is open at both ends and has constant cross-sectional area $A(x) = A_0$. [10%]

(b) Consider a duct where the cross-sectional area can be written as $A(x) = A_0 x^n$ for $0 \leq x \leq L$, where A_0 and n are positive constants. The duct is open at $x = L$.

(i) Using separation of variables $p(x,t) = P(x)\mathrm{e}^{i\omega t}$, find the ordinary differential equation satisfied by $P(x)$. [10%]

(ii) Using the transformation $P(x) = x^{-\nu}\phi(x)$, where $\nu = (n-1)/2$, show that the ordinary differential equation governing $\phi(x)$ can be written in the form of Bessel's differential equation:

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \left(\alpha^2 x^2 - \beta^2\right)\phi = 0$$

and find expressions for the constants α and β. [40%]

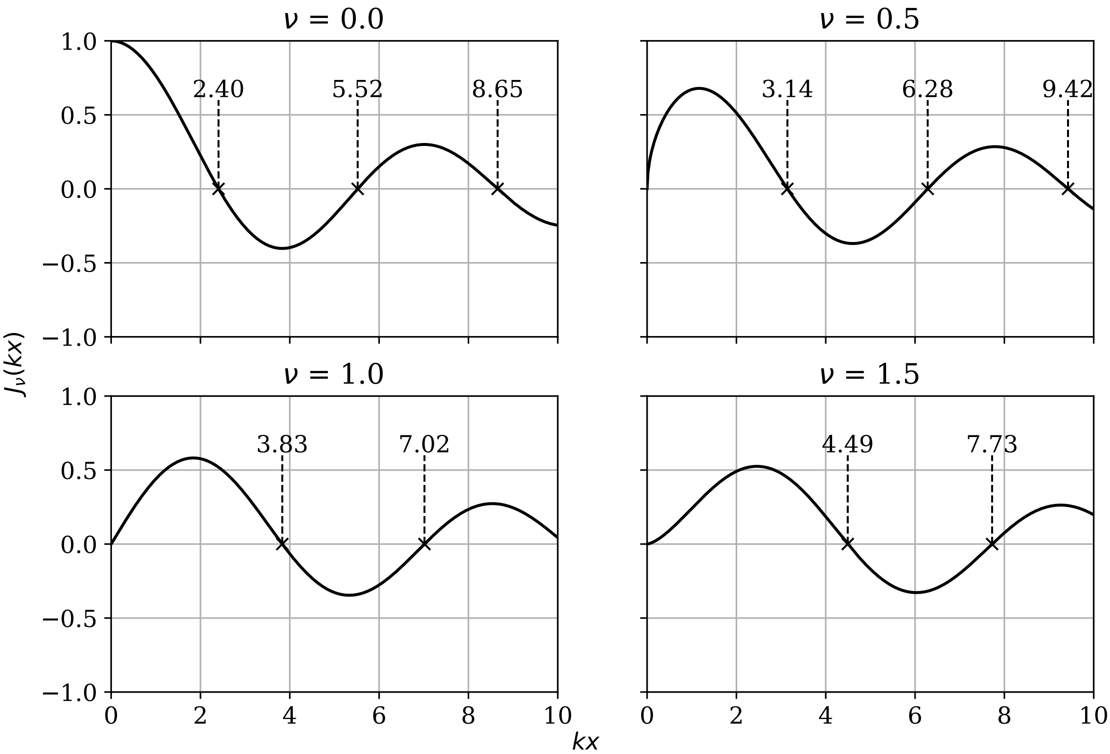
(iii) The spatial part of the solution $P(x) = x^{-\nu}\phi(x)$ can be written in terms of Bessel functions:

$$P(x) = x^{-\nu}\left[C_1 J_\nu(kx) + C_2 Y_\nu(kx)\right]$$

where $k = \omega/c$ is the wavenumber, C_1 and C_2 are constants, and J_ν and Y_ν are Bessel functions of the first and second kind, respectively. Without detailed analysis, explain why $C_2 = 0$ for this problem. [10%]

(iv) Using the relevant labelled plots of the Bessel functions $J_\nu(kx)$ in Fig. 3, find the first two natural frequencies of the duct for the cases $n = 1, 2, 3$. [30%]

 (cont.



$\nu = 0.0$
$\nu = 0.5$
$\nu = 1.0$
$\nu = 1.5$
2.40
5.52
8.65
3.14
6.28
9.42
3.83
7.02
4.49
7.73
1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
$J_\nu(kx)$
kx

Fig. 3

4 A thin rectangular plate of length a, width b, thickness h, density ρ, Young's modulus E, and Poisson ratio ν is simply supported on all edges (displacement and bending moment are zero at the edges). The partial differential equation (PDE) governing free transverse vibration of the plate is given by:

$$\rho h \frac{\partial^2 u}{\partial t^2} + D\left(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) = 0$$

where u is the transverse displacement, and $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity of the plate. The mode shapes for the simply supported plate are given by:

$$\phi_{mn}(x, y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

where m, n are positive integers.

(a) Verify that the solution $u = A_{mn}\phi_{mn}(x, y)\mathrm{e}^{i\omega t}$, where A_{mn} is the amplitude, satisfies the PDE and find an expression for the natural frequencies ω_{mn}. [20%]

(b) Suggest three passive methods that could be used to attenuate vibration of the plate, and briefly describe the principle of each method. [10%]

(c) The potential energy of the plate vibrating in the (m, n) mode is given by:

$$V = \frac{1}{2}K_{mn}A_{mn}^2, \qquad \text{where} \quad K_{mn} = \frac{Dab\pi^4}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2$$

The loss factor associated with the Young's modulus E is η_E. Use the correspondence principle to find an expression for the Q-factor of the plate modes. [20%]

(d) A spring of stiffness k and loss factor η_k is attached between the ground and the plate at position (x_0, y_0). With the same material damping as in part (c), find an estimate for the Q-factor of the (m, n) mode of the plate, assuming that the mode shapes remain unchanged. Your answer may be left in terms of K_{mn} and $\phi_0 \equiv \phi_{mn}(x_0, y_0)$. [40%]

(e) What can be said about the natural frequencies of the undamped plate if the spring stiffness $k \to \infty$? Justify your answer. [10%]

END OF PAPER

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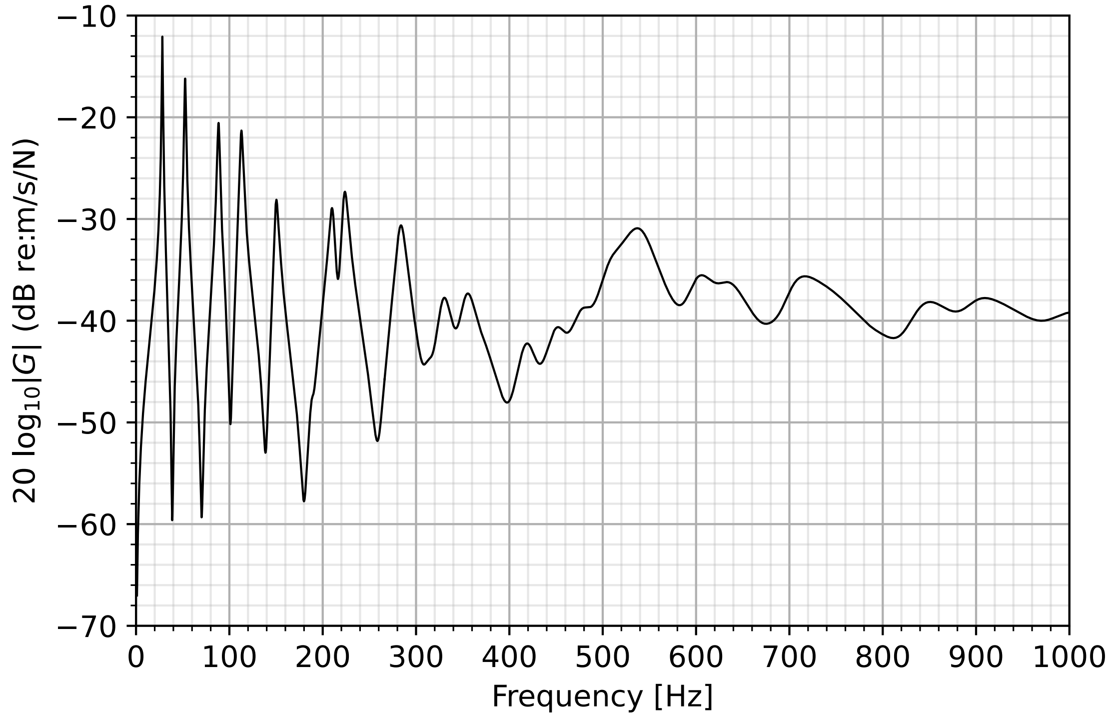

Extra copy of Fig. 1a: Frequency Response Function for Question 1.

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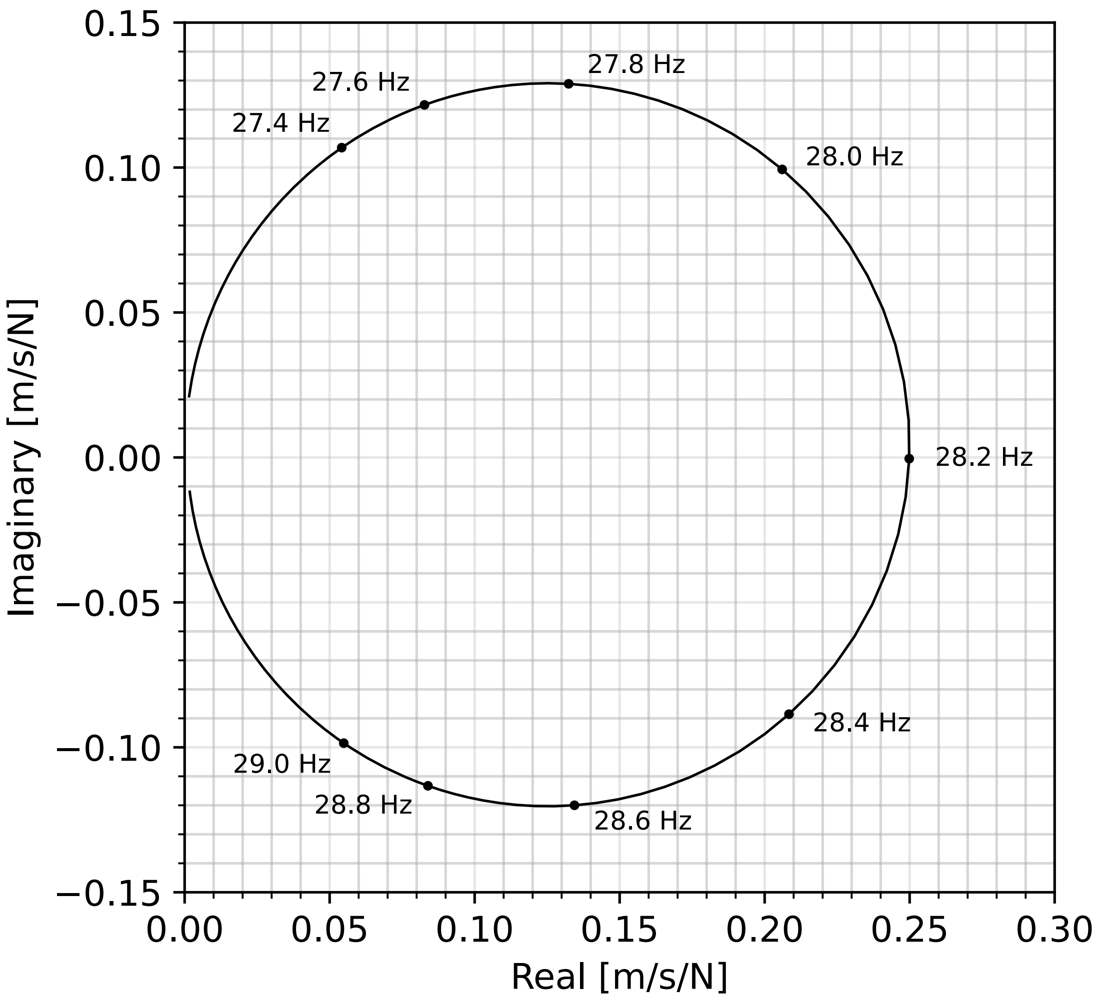

Extra copy of Fig. 1b: Nyquist plot of Frequency Response Function near Mode 1 for Question 1.

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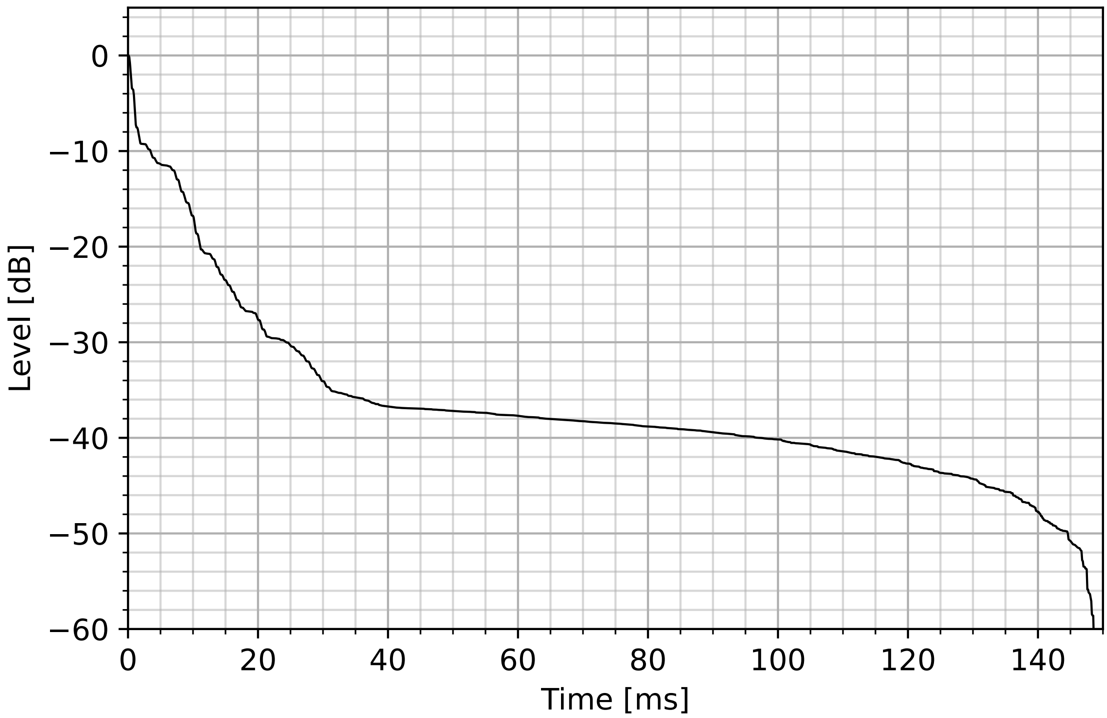
ENGINEERING TRIPOS PART IIB
Tuesday 28 April 2026, Module 4C6, Question 1.



Extra copy of Fig. 1c: Schroeder decay curve for Question 1.

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Part IIB Data Sheet

Module 4C6 Advanced Linear Vibration

1 Vibration Modes and Response

Discrete Systems

1. Equation of motion

The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

2. Kinetic Energy

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

3. Potential Energy

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

Continuous Systems

1. Equation of motion

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 2 for examples.

2. Kinetic Energy

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

See Section 2 for examples.

4. Natural frequencies and mode shapes

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 2) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_{j} q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1, x_2, t) = \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1, x_2, t) \approx \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos \omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos \omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\Big]$$

for $t \geq 0$ (with small damping).

1.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\widetilde{T}} = \frac{\mathbf{y}^T\mathbf{K}\mathbf{y}}{\mathbf{y}^T\mathbf{M}\mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 2.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\widetilde{T}}$ is *stationary* near each mode.

2 Governing equations for continuous systems

2.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

2.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

2.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

2.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$
\begin{aligned}
\cos\alpha\cosh\alpha + 1 = 0, &\quad \alpha_1 = 1.8751 \\
\cos\alpha\cosh\alpha - 1 = 0, &\quad \alpha_1 = 4.7300 \\
\tan\alpha - \tanh\alpha = 0, &\quad \alpha_1 = 3.9266
\end{aligned}
$$

Some devices for vibration excitation and measurement

Moving coil electro-magnetic shaker

LDS V101: Peak sine force 10N, internal armature resonance 12kHz. Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, stroke 2.5mm, shaker body mass 0.9kg

LDS V650: Peak sine force 1kN, internal armature resonance 4kHz. Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg

LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz. Frequency range 5 – 1.7kHz, armature suspension stiffness 72kN/m, armature mass 250kg, stroke 50mm, shaker body mass 13000kg

Piezo stack actuator

FACE PAC-122C
Size 2×2×3mm, mass 0.1g, peak force 12N, stroke 1μm, u
nloaded resonance 400kHz

Impulse hammer

IH101
Head mass 0.1kg, hammer tip stiffness 1500kN/m, force transducer sensitivity 4pC/N, internal resonance 50kHz

Piezo accelerometer

B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale range +/-5000g

DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale range +/-2000g

B&K4370 Mass 10g sensitivity 100pC/g, 1-4.8kHz, full-scale range +/-2000g

MEMS accelerometer

ADKL202E
265mV/g
Full scale range +/- 2g
DC-6kHz

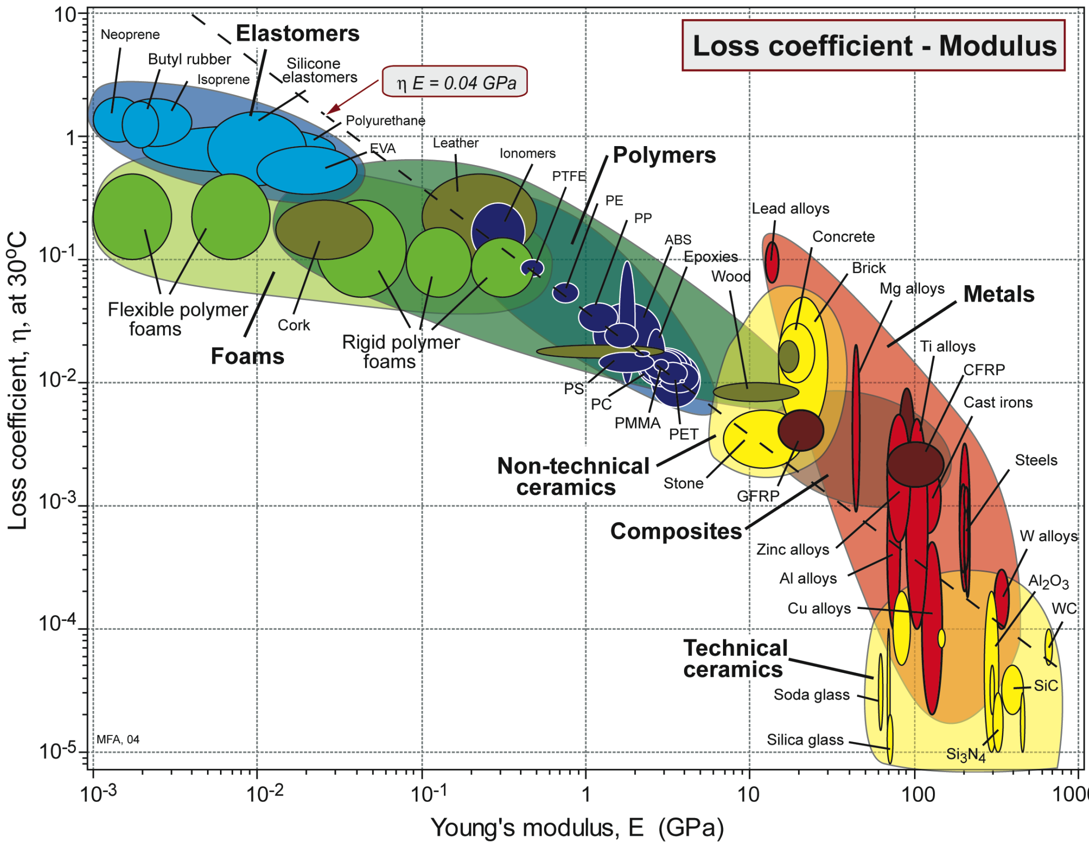
Laser Doppler Vibrometer

Polytec PSV-400 Scanning Vibrometer
Velocity ranges 2/10/50/100/1000 [mm/s/V]

VIBRATION DAMPING

Correspondence principle

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \to E(1+i\eta)$. Typical values of E and η for engineering materials are shown below:



For a complex natural frequency ω:

$$\omega \simeq \omega_n\left(1+i\zeta_n\right) \simeq \omega_n\left(1+i\eta_n/2\right) \simeq \omega_n\left(1+i/2Q_n\right)$$

and

$$\omega^2 \simeq \omega_n^2\left(1+i\eta_n\right) \simeq \omega_n^2\left(1+i/Q_n\right)$$

Free and constrained layers

For a 2-layer beam: if layer j has Young's modulus E_j, second moment of area I_j and thickness h_j, the effective bending rigidity EI is given by:

$$EI = E_1 I_1 \left[1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh} \right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1}.$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12} \left[\frac{h_{31} - d}{1+g} \right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[\frac{E_2 h_2}{2} (h_{21} - d) + E_3 h_3 (h_{31} - d) \right] \left[\frac{h_{31} - d}{1+g} \right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

G_2 is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

Viscous damping, the dissipation function and the first-order method

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2} \underline{\dot{y}}^t \, C \, \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and C is the (symmetric) dissipation matrix.

If the system has mass matrix M and stiffness matrix K, free motion is governed by

$$M \underline{\ddot{y}} + C \underline{\dot{y}} + K \underline{y} = 0.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix}$. If $\underline{z} = \underline{u} e^{\lambda t}$ then $\underline{u}, \lambda$ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

where 0 is the zero matrix and I is the unit matrix.

THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume V with a neck of effective length L and cross-sectional area S has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where c is the speed of sound in air.

The end correction for an unflanged circular neck of radius a is $0.6a$.

The end correction for a flanged circular neck of radius a is $0.85a$.

VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension T and mass per unit area m undergoes small transverse free vibration with displacement w, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates x, y or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates r, θ.

For a circular membrane of radius a the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta\, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where J_n is the Bessel function of order n and k is determined by the condition that $J_n(ka) = 0$. The first few zeros of J_n's are as follows:

	$n=0$	$n=1$	$n=2$	$n=3$
$ka=$	2.404	3.832	5.135	6.379
$ka=$	5.520	7.016	8.417	9.760
$ka=$	8.654	10.173		

For a given k the corresponding natural frequency ω satisfies

$$k = \omega\sqrt{m/T}\ .$$