Version XN/3

EGT3
ENGINEERING TRIPOS PART IIB

Thursday 1 May 2025 9.30 to 11.10

Module 4C8

VEHICLE DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C8 Vehicle Dynamics data sheet (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Figure 1 shows a pitch-plane model of a vehicle with two degrees of freedom. The sprung mass is m and the pitch inertia is I. The vertical and pitch displacements at the centre of mass are z and θ. The wheelbase is $L = 2a$. The front and rear suspensions each have stiffness k and damping c. The road vertical displacement input to the front axle is z_{r1} and the input to the rear axle is z_{r2}. The forward speed of the vehicle is U.

(a) Numerical values of the parameters are: m = 1000 kg, I = 1400 kg m^2, a = 1.25 m, k = 80 kN m^{-1}, c = 3.8 kN s m^{-1}, U = 10 m s^{-1}.

(i) If $z_{r1} = z_{r2}$, sketch a graph of the magnitude of the transfer function between road vertical velocity input and sprung mass vertical acceleration response. Use linear axes and a frequency range of 0 Hz to 10 Hz. Mark clearly in your sketch the important values of frequency and magnitude, and show clearly how they were calculated. [30%]

(ii) The vehicle travels along a rough road so that z_{r2} is a delayed version of z_{r1}. Sketch a graph of the magnitude of the transfer function between road vertical velocity input and sprung mass vertical acceleration response. Use linear axes and a frequency range of 0 Hz to 10 Hz. Mark clearly in your sketch the important values of frequency and magnitude, and show clearly how they were calculated. [30%]

(b) It has been suggested that to improve ride comfort, the excitation of the pitch mode of the vehicle can be minimised by varying the suspension stiffness according to vehicle speed. By considering excitation of the undamped vehicle travelling along a rough road show that a suitable design would require $k \propto U^2$ and state the constant of proportionality in terms of the vehicle parameters. Do not use the numerical values of part (a). [20%]

(c) It is further suggested that the excitation of the bounce mode could be minimised at the same time as minimising the pitch excitation. Find the constraints on the vehicle parameters needed for this to happen and comment on whether it is practical to satisfy these constraints. Do not use the numerical values of part (a). [20%]

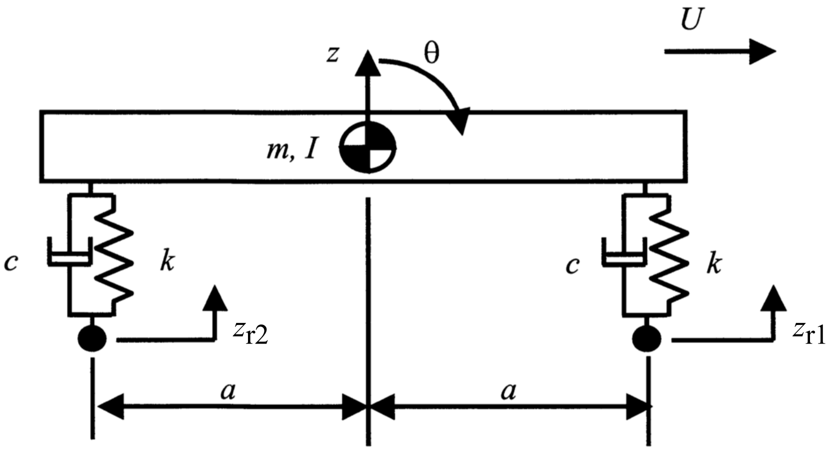

Fig. 1

2 The vertical displacement profiles of the left-hand and right-hand wheel tracks z_L and z_R of a randomly rough road surface can each be considered as the combination of an average vertical displacement z_V and a roll displacement z_ϕ, as shown in Fig. 2. z_L and z_R have the same mean-square spectral density $S_z(n)$. The mean-square spectral densities of z_V and z_ϕ are respectively $S_{z_V}(n)$ and S_{z_ϕ}. They are related by $S_{z_\phi}(n) = |G(n)|^2 S_{z_V}(n)$. Here n is the wavenumber (cycles m^{-1}),

$$|G(n)|^2 = \frac{n^2}{n_c^2 + n^2} ,$$

and n_c is a 'cut-off' wavenumber.

(a) Show that

$$S_{z_V}(n) = S_z(n)\frac{n_c^2 + n^2}{n_c^2 + 2n^2} .$$

[15%]

(b) The displacement spectral density of a road profile is $S_z(n) = \kappa n^{-2}$ where κ is a roughness coefficient. Show that the displacement excitation of a vehicle moving over the road can be given by $S_z(\omega) = 2\pi U\kappa\omega^{-2}$, where U is the vehicle speed and ω is the frequency (rad s^{-1}). [20%]

(c) Derive an expression for the spectral density of vertical displacement $S_{z_V}(\omega)$ as a function of frequency ω, vehicle speed U, roughness κ and cut-off wavenumber n_c. Using logarithmic axes, sketch a graph of $S_{z_V}(\omega = \omega_1)$ against speed U to show how the spectral density at frequency $\omega = \omega_1$ varies with speed. Mark clearly in your sketch the important features. [40%]

(d) With reference to your sketch in part (c), discuss the influence of vehicle speed on the vertical excitation of the sprung mass modes (natural frequency ω_1) and unsprung mass modes (natural frequency ω_2) of a vehicle using values $n_c = 0.1$ cycles m^{-1}, $\omega_1 = 5$ rad s^{-1} and $\omega_2 = 50$ rad s^{-1}. [25%]

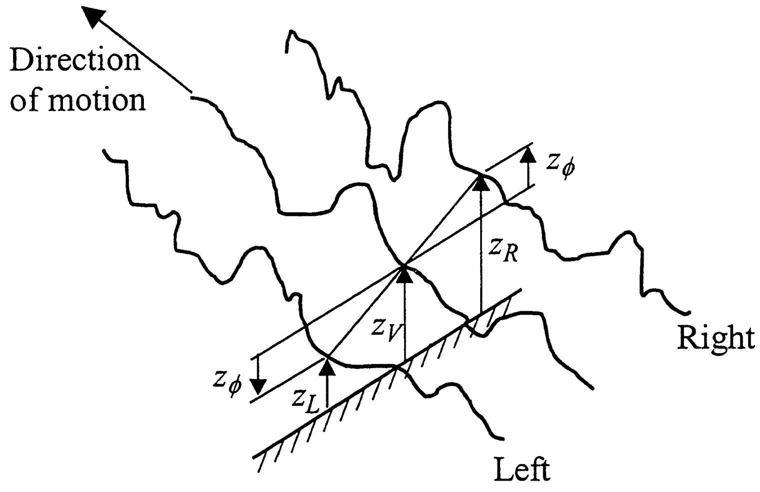

Fig. 2

3 Figure 3 shows the 'bicycle' model of a car, with freedom to sideslip with velocity v and yaw at rate Ω. The car has mass m and yaw moment of inertia I. It moves at a constant forward velocity u on a horizontal surface. The lateral creep coefficients at the front and rear axles of the bicycle model are C_f and C_r. The lengths a and b, and the front wheel steer angle δ, are defined in the figure.

(a) Show that, under suitable assumptions, the equations of motion for the bicycle model in a coordinate frame rotating with the car are given by

$$m(\dot{v} + u\Omega) + (C_f + C_r)\frac{v}{u} + (aC_f - bC_r)\frac{\Omega}{u} = C_f\delta$$

$$I\dot{\Omega} + (aC_f - bC_r)\frac{v}{u} + (a^2C_f + b^2C_r)\frac{\Omega}{u} = aC_f\delta$$

State the assumptions needed to derive these equations of motion. [30%]

(b) Derive the transfer function between front wheel steer angle input and car yaw rate response, assuming the car exhibits neutral steer. Sketch the yaw rate as a function of time in response to a step input in front wheel steer angle. Mark clearly in your sketch the important features. [30%]

(c) A constant front wheel steer angle δ_0 is applied to the car. After some time, the car follows a circular path of radius R steadily. Using the equations of motion of the car, derive an expression for R. Sketch the variation of R with forward velocity u for three cases: understeer, oversteer, and neutral steer. Comment on the variation of R with u for a real car. [40%]

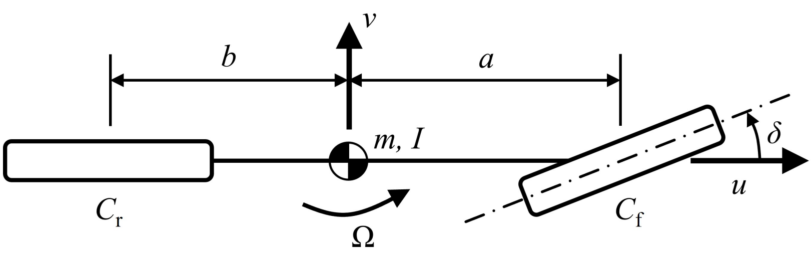

Fig. 3

4 A ‘coned’ railway wheelset with effective wheel conicity ϵ, average wheel rolling radius r (the wheel rolling radius when the wheelset is in central position with respect to the track), and track gauge $2d$ is moving on a horizontal track at a steady speed u. The wheelset has a small lateral tracking error y and small yaw angle θ. The coefficients of both lateral and longitudinal creep of the wheels are C.

(a) The wheelset is travelling along a straight track. Show that, under suitable assumptions, the lateral tracking of the wheelset may be modelled as a simple harmonic motion, with the angular natural frequency given by

$$u\sqrt{\frac{\varepsilon}{rd}}$$

State the assumptions needed to derive this angular natural frequency. [30%]

(b) The same wheelset is negotiating a curve of constant radius. The radius of curvature of the track centre-line is R. The creep velocities result in a total longitudinal force X, a total lateral force Y, and a total yaw moment N. Show that

$$X = 0,$$
$$Y = 2C\left(\frac{\dot{y}}{u} + \theta\right),$$
$$N = 2dC\left(\frac{\epsilon y}{r} - \frac{d\dot{\theta}}{u} - \frac{d}{R}\right)$$

Indicate the directions of the creep velocities and creep forces at each wheel using a sketch of the wheelset. Indicate the directions of Y and N on the same sketch. State the assumptions and compare them with the assumptions made in part (a). [35%]

(c) A rigid bogie comprises two wheelsets at a spacing of $2a$, joined by a rigid frame. The bogie supports a wagon of mass M.

(i) Explain the purpose of incorporating a ’cant angle’ in a rail track. [10%]

(ii) The bogie and wagon are moving along a straight track. Derive an expression for the wavelength of the lateral tracking motion of the bogie. Compare this with the kinematic wavelength of a single wheelset. [25%]

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Engineering Tripos Part IIB

Data sheet for Module 4C8: Vehicle Dynamics

1 Creep Forces in Rolling Contact

1.1 Surface Tractions

Longitudinal force:

$$X = \iint_A \sigma_x \, dA$$

Lateral force:

$$Y = \iint_A \sigma_y \, dA$$

Realigning moment:

$$N = \iint_A (x\sigma_y - y\sigma_x) \, dA$$

where

$\sigma_x, \ \sigma_y$ = longitudinal, lateral surface tractions

$x, \ y$ = coordinates along, across contact patch

A = area of contact patch

1.2 Brush Model

$$\sigma_x = K_x q_x, \quad \sigma_y = K_y q_y \quad \text{for} \quad \sqrt{\sigma_x^2 + \sigma_y^2} \leq \mu p$$

where

$q_x, \ q_y$ = longitudinal, lateral displacements of 'bristles' relative to wheel rim

$K_x, \ K_y$ = longitudinal, lateral stiffness per unit area

μ = coefficient of friction

p = local contact pressure

1.3 Linear Creep Equations

$$\begin{aligned} X &= -C_{11}\xi \\ Y &= -C_{22}\alpha + C_{23}\psi \\ N &= C_{32}\alpha + C_{33}\psi \end{aligned}$$

where X, Y, N are defined as in 1.1 above, and

C_{ij} = coefficients of linear creep

ξ = longitudinal creep ratio = longitudinal creep speed / forward speed

α = lateral creep ratio = (lateral speed / forward speed) - steer angle

ψ = spin creep ratio = spin angular velocity / forward speed

2 Plane Motion in a Moving Coordinate Frame

$$\ddot{\mathbf{R}}_{O_1} = (\dot{u} - v\Omega)\mathbf{i} + (\dot{v} + u\Omega)\mathbf{j}$$

with $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ axis system fixed to body at point O_1

where

u = speed of point O_1 in $\mathbf{i}$ direction

v = speed of point O_1 in $\mathbf{j}$ direction

Ω = angular rate of body in $\mathbf{k}$ direction

3 Routh-Hurwitz Stability Criteria

$$(a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0)\ y = x(t) \qquad \text{is stable if all } a_i > 0$$

$$(a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0)\ y = x(t) \qquad \text{is stable if (i) all } a_i > 0 \text{ and (ii) } a_1a_2 > a_0a_3$$

$$(a_4\frac{d^4}{dt^4} + a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0)\ y = x(t) \qquad \text{is stable if (i) all } a_i > 0 \text{ and (ii) } a_1a_2a_3 > a_0a_3^2 + a_4a_1^2$$

4 Vehicle Vibrations

4.1 Random Vibration

$$E[x^2(t)] = \frac{1}{T}\int_{t=0}^{t=T} x^2(t)\,dt = \int_{\omega=-\infty}^{\omega=\infty} S_x(\omega)\,d\omega \qquad \text{(or } \int_{\omega=0}^{\omega=\infty} S_x(\omega)\,d\omega \;\text{ if } S_x(\omega) \text{ is single sided)}$$

$$S_{\dot{x}}(\omega) = \omega^2 S_x(\omega) \qquad \text{(Spectrum of time derivative of } x \text{ is } \omega^2 \text{ times spectrum of } x\text{)}$$

4.2 Single Input – Single Output System

$$S_y(\omega) = |H_{yx}(\omega)|^2 S_x(\omega)$$
$$y(\omega) = H_{yx}(\omega)\; x(\omega)$$

4.3 Two Input – Two Output System

$$\begin{bmatrix} y_1(\omega) \\ y_2(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} \begin{bmatrix} x_1(\omega) \\ x_2(\omega) \end{bmatrix}$$

$$\begin{bmatrix} S^y_{11}(\omega) & S^y_{12}(\omega) \\ S^y_{21}(\omega) & S^y_{22}(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^* \begin{bmatrix} S^x_{11}(\omega) & S^x_{12}(\omega) \\ S^x_{21}(\omega) & S^x_{22}(\omega) \end{bmatrix} \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^{\mathbf{T}}$$

where * means complex conjugate and T means transpose.

If x_1 and x_2 are uncorrelated, the following equations hold:

$$S_{(x_1+x_2)}(\omega) = S_{x_1}(\omega) + S_{x_2}(\omega)$$

$$S^x_{12}(\omega) = S^x_{21}(\omega) = 0$$
$$E[(x_1(t) + x_2(t))^2] = E[x_1^2(t)] + E[x_2^2(t)]$$