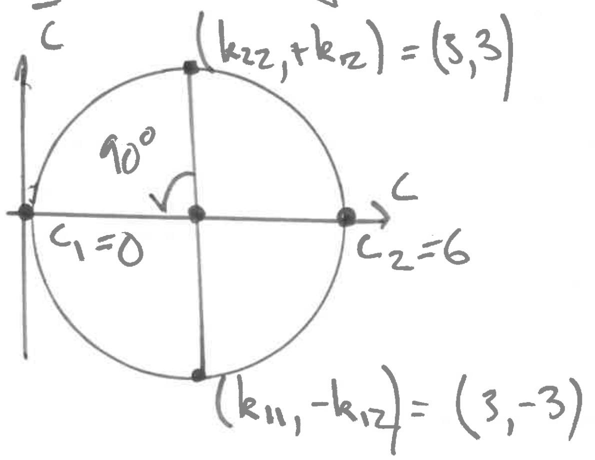
4017 | 1.

1) a) $\bar{z} = \tfrac{1}{2}k_{11}x^2 + k_{12}xy + \tfrac{1}{2}k_{22}y^2$

for convenience : significance ⇒ centre of curvature lies below surface of curving

k_{11}, k_{22} : curvatures w.r.t. 1, 2 axes
k_{12} : twisting w.r.t. 1-2 coordinate system.

b).

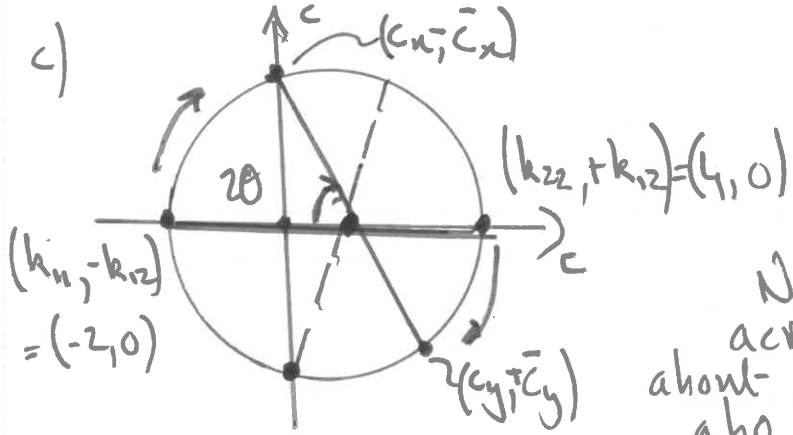


$\bar{c}$ = twist : c = curvature

Since centre at (3,0) and radius = 3 principal curvatures are (0, 6)

⇒ cylindrical, hence developable

c)



Diameter can be rotated until first curvature is zero, as shown. No curving along but across = straight line, about which surface is also twisted: can rotate in opposite dirn by same (dashed) to give second straight line

Centre of circle at (1,0) : radius = 3.

$\bar{c}_x^2 = 3^2 - 1$ (Pythagoras) ⇒ $-\bar{c}_x = \sqrt{8}$: $\bar{c}_y = -\sqrt{8}$, $c_y = 2$

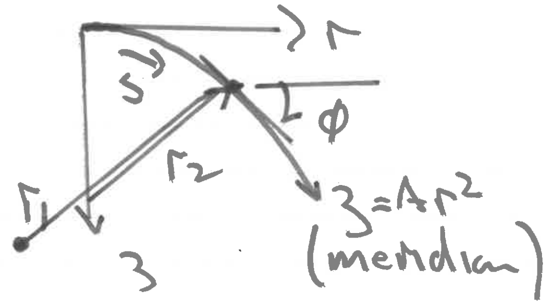
⇒ $(c_x, -\bar{c}_x) = (0, \sqrt{8})$: $(c_y, +\bar{c}_y) = (2, -\sqrt{8})$

∴ twist = $\sqrt{8}$, curving across lines = 2 [units: 1/m]

4D17/2

2) a) Membrane Hypothesis: when out-of-plane loading is resisted by in-plane forces alone.

b)



$z = Ar^2$: (parabolic) ← constant

$\underbrace{1/r_1 = d\phi/ds \; : \; 1/r_2 = 1/r \, dz/ds}_{\text{meridional + latitudinal curvatures (principal)}}$

s: coordinate (intrinsic) : $\delta s \neq \delta r$.

$$\sin\phi = dz/ds : \cos\phi = dr/ds \Rightarrow \tan\phi = dz/dr.$$

$$\therefore \tan\phi = 2Ar \Rightarrow \underline{r = \tan\phi/2A} : \frac{dr}{d\phi} = \frac{1}{2A}.\sec^2\phi$$

$$1/r_1 = d\phi/ds = d\phi/dr . dr/ds = \frac{2A}{\sec^2\phi}.\cos\phi = \underline{\underline{2A\cos^3\phi}}$$

$$1/r_2 = 1/r \, dz/ds = \frac{2A}{\tan\phi}.\sin\phi = \underline{\underline{2A\cos\phi}}$$

With internal pressure: use eqlm of cap.

$$2\pi r N_\phi \sin\phi = p.\pi r^2 \Rightarrow \underline{\underline{N_\phi = pr/2\sin\phi}}$$

$$\Rightarrow N_\phi = \frac{p}{2\sin\phi}.\frac{\tan\phi}{2A} = \underline{p/4A\cos\phi}$$

Vertical eqlm $\Rightarrow N_\phi/r_1 + N_\theta/r_2 = p$

$$\frac{p}{\cancel{4A\cos\phi}_{2}}\cancel{2A}\cos^{\cancel{3}\,2}\phi + N_\theta.2A\cos\phi = p.$$

$$\Rightarrow \quad p.\cancel{A}\cos^2\phi/2 + N_\theta.2A\cos\phi = p.$$

$$N_\theta = \underline{\underline{p\left[1 - 1/2\cos^2\phi\right]/2A\cos\phi}}$$

4M17/2

b) contd.: $N_\phi = p/_{4A\cos\phi}$; $N_\theta = p[1 - \frac{1}{2}\cos^2\phi]/_{2A\cos\phi}$

c) For small ϕ, $\cos\phi \simeq 1 \Rightarrow N_\phi \sim p/_{4A}$

$N_\theta \simeq p[1 - \frac{1}{2}]/_{2A} \simeq p/_{4A}$

$\Rightarrow N_\phi \simeq N_\theta$ when ϕ small: BIAXIAL STRESSES

d) Relationship $N_\theta = p \, dA/_{ds}$: Zick's Law.

For small ϕ $N_\theta = P/_{4A}$ from above.

$dA/_{ds} = dA/_{d\phi} \cdot d\phi/_{ds}$

Area enclosed $= \frac{1}{2} \overset{r_2}{\downarrow}{r_2}^2 \phi = \frac{1}{2}\left(\frac{1}{2A}\right)^2 \phi \simeq \phi/8A^2$

Arc-length $\sim s = r_2\phi \simeq \phi/2A$.

$\Rightarrow dA/_{d\phi} \simeq \frac{1}{8A^2}$: $d\phi/_{ds} \simeq 2A$

$\Rightarrow dA/_{ds} \simeq 1/_{4A}$:

However $N_\theta/_p = 1/_{4A}$ via eqlm $\therefore$ both equal according to Zick's Law

4D17/3

3) General: $w = \frac{B}{2}\left[x^2 - \nu y^2\right] + Cxy$

a) Shape properties:

Cartesian coordinates $\Rightarrow K_x = -\frac{\partial^2 w}{\partial x^2} : K_y = -\frac{\partial^2 w}{\partial y^2} : K_{xy} = \frac{-\partial^2 w}{\partial x \partial y}$

$\Rightarrow$ $\underline{\underline{K_x = -B : K_y = +\nu B : K_{xy} = C(-1)}}$

b) $M_x = D\left[K_x + \nu K_y\right] = D\left[-B + \nu^2 B\right] = -DB\underline{\underline{\left[1-\nu^2\right]}}$ (M_x)

bending moment; flexural rigidity

$M_y = D\left[K_y + \nu K_x\right] = D\left[\nu B + \nu . -B\right] = \underline{\underline{0}}$ (M_y)

(twist) $M_{xy} = D[1-\nu]K_{xy} = -D\underline{\underline{[1-\nu]C}}$

$\Rightarrow M_x, M_{xy}$ constant and nonzero: M_y = zero thro'out

c) Now must consider plate eqlbrm equations (in x,y coordinates).

$Q_x = \partial M_x/\partial x + \partial M_{xy}/\partial y = 0$: so is $Q_y \left[= \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right]$

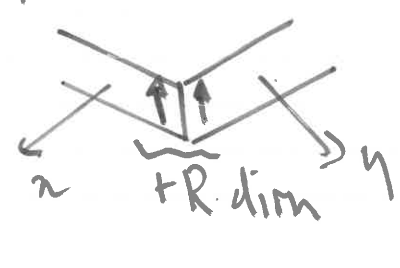
And since $\partial Q_x/\partial x + \partial Q_y/\partial y = -p$ from transverse eqlm,

Then $p = 0$: no external pressure applied

Note also that Kelvin's Edge Resultant in both directions

$V_x = Q_x + \partial M_{xy}/\partial y = 0$ (as is V_y)

However corner forces do persist $R = 2M_{xy} \neq 0 = -2D(1-\nu)C$



4D17/3

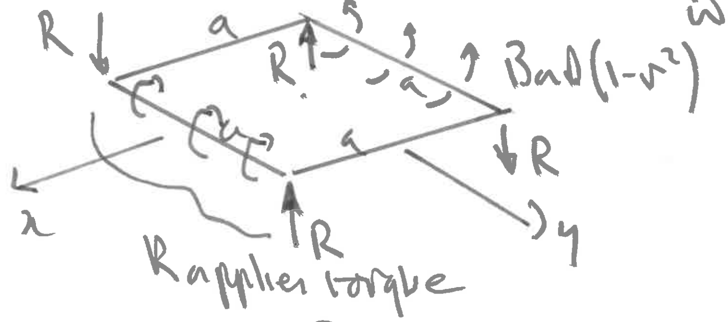
c) contd.

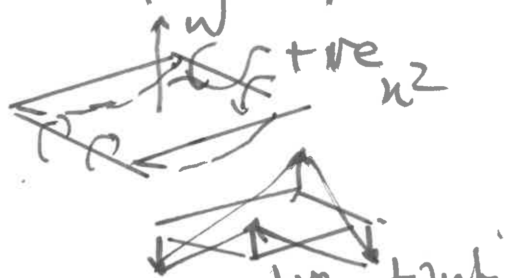
There is also a moment $M = aM_x$ applied along x dirn to cause bending

$\uparrow$ ×a, side-length because M_x = constant.

$\therefore M = -BaD[1-\nu^2]$

in opposite directn to defn of M_x (and consistent with deflectn).





Given $D = Et^3/12(1-\nu^2) \Rightarrow -M = Ba.\frac{Et^3}{12\cancel{(1-\nu^2)}}\cancel{(1-\nu^2)}$

$$\underline{\underline{B = -12M/Eat^3}}$$

Torque from $R = aR \Rightarrow T = aR = a\,\underline{2D(1-\nu)c}$

+ve because we have anticipated directn

$$\Rightarrow T = 2aD(1-\nu)c = \frac{2aEt^3}{12(1-\nu^2)}(1-\nu)c$$

$$\Rightarrow \quad \underline{\underline{c = 6T(1+\nu)/Eat^3}}$$

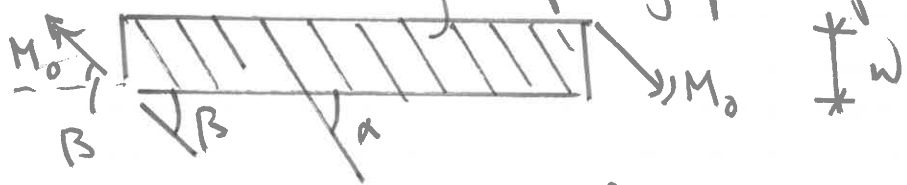
d) Gaussian curvature $= K_x K_y - K_{xy}^2 = -B^2\nu - (-c)^2$

$$= -\nu B^2 - c^2 \neq 0$$

Non-zero GC $\Rightarrow$ in-plane strain from Gauss's Theorem.

4D17/4

4)



a)

$=$ Torque $M_0 \cos\beta$ + Bending moment $M_0 \sin\beta$

BM / Torque = $\underline{\underline{\tan\beta}}$

b) (i) Generators are viable if very thin, displacements moderate and boundary conditions permit.

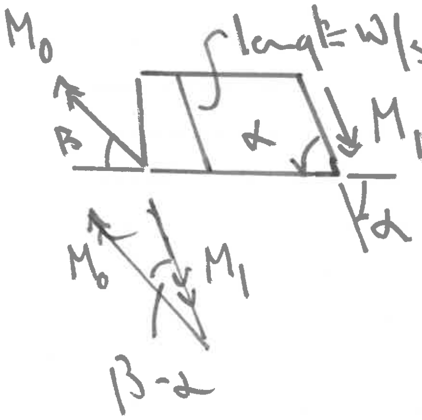
b) (ii) Parallel + uniform because torque and moments applied are equal + opposite.

b) (iii) If $L \gg w$, then end effects, where the pattern is not uniform are minimised.

b) (iv). Mansfield's Inext. Theory deals with non-parallel generators

b) (v) if $\beta = 90° \Rightarrow$ pure bending and $\alpha = 90°$ i.e. transverse generators $\equiv$ cylindrical bending along strip.

c)



M_1 = total bending moment across generator.

$= \underline{M_0 \cos(\beta - \alpha)}$

B.M. / unit length of generator

$M = M_1 / (w/\sin\alpha)$ [constant]

$= M_1 \sin\alpha / w$

Curvature across $= M/D = M_1 \sin\alpha / Dw = \underline{\underline{\dfrac{M_0 \cos(\beta - \alpha)\sin\alpha}{Dw}}}$

4M17/4

d) Strain energy = $U = \int_0^L \int_0^w \frac{1}{2} D \kappa^2 \, dx\, dy$ constant w.r.t. x and y ⇒ × Lw area of strip.

$$\Rightarrow U = \frac{L M_0^2}{2Dw}\left[\cos(\beta-\alpha)\sin\alpha\right]^2$$

$$\partial U/\partial\alpha = (\uparrow)\cdot\left[2\cos(\beta-\alpha)\sin\alpha\right]\times\underbrace{\left[\cos\alpha\cos(\beta-\alpha) + \sin(\beta-\alpha)\cdot\sin\alpha\right]}_{\cos(2\alpha-\beta)}$$

$$\Rightarrow \frac{dU}{d\alpha} = (\;)\cdot\left[\cos(\beta-\alpha)\cdot\sin\alpha\cdot\cos(2\alpha-\beta)\right] = 0$$

when $\cos(\beta-\alpha) = 0 \Rightarrow U = 0$;
$\sin\alpha = 0 \Rightarrow U = 0$ } discount.

$$\Rightarrow \cos(2\alpha-\beta) = 0 \Rightarrow 2\alpha-\beta = \pm\pi/2$$

$$\Rightarrow \underline{\underline{\alpha = \pm\pi/4 + \beta/2}}$$

e) For pure torsion, $\beta = 0 \Rightarrow \alpha = \pm\pi/4$
(twisted with respect to either sense).