Dr A McRobie

2014

PART 2B : 4D6

1. 305 x 165 x 54 UB

Data Book $\Rightarrow$ $m = 54.0\,kg/m$; $I = 11{,}700\ cm^4$; $E = 210\,GPa$.

Additional mass loading $\Rightarrow$ $m_{tot} = 54 + 20 = 74\ kg/m$

a). Considering first and second bending modes (beam inclination has no effect on vibration about equilibrium position):

$$\bar{u}_j = 1 - \cos\frac{j\pi x}{2L}$$

$$M_{eq} = \int_0^L m\bar{u}^2\,dx = m\int_0^L \left(1 - \cos\frac{j\pi x}{2L}\right)^2 dx$$

$$= m\int_0^L 1 - 2\cos\frac{j\pi x}{2L} + \frac{1}{2}\left(1 + \cos\frac{j\pi x}{L}\right) dx$$

$$= m\left[x - \frac{4L}{j\pi}\sin\frac{j\pi x}{2L} + \frac{x}{2} + \frac{L}{2j\pi}\sin\frac{j\pi x}{L}\right]_0^L$$

$$= m\left(\frac{3L}{2} - \frac{4L}{j\pi}\sin\frac{j\pi}{2}\right) = \left(\frac{3}{2} - \frac{4}{j\pi}\sin\frac{j\pi}{2}\right)\times 74\times 6 = \underline{101\ kg}\ (j=1)$$

$$= \underline{\mathbf{666\ kg}}\ (j=2)\quad [3]$$

$$K_{eq} = \int_0^L EI\left(\frac{d^2\bar{u}}{dx^2}\right)^2 dx = EI\int_0^L \left(\frac{(j\pi)^2}{4L^2}\cos\frac{j\pi x}{2L}\right)^2 dx$$

$$= \frac{(j\pi)^4 EI}{16L^4}\int_0^L \frac{1}{2}\left(1 + \cos\frac{j\pi x}{L}\right) dx = \frac{(j\pi)^4 EI}{32L^4}\left[x + \frac{L}{j\pi}\sin\frac{j\pi x}{L}\right]_0^L$$

$$= \frac{(j\pi)^4 EI}{32L^3} = \frac{(j\pi)^4}{32}\times\frac{210\times10^9\times 11{,}700\times10^{-8}}{6^3} = \underline{346\ kN/m}\ (j=1)$$

$$= \underline{5540\ kN/m}\ (j=2)\quad [3]$$

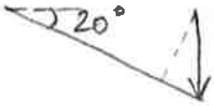
$$\therefore\ \omega_1 = \sqrt{\frac{346\times10^3}{101}} = 58.5\ rad/s$$

$$= \underline{9.3\ Hz}\quad [1]$$

$$\omega_2 = \sqrt{\frac{5540\times10^3}{666}} = 91.2\ rad/s$$

$$= \underline{\mathbf{14.5}\ Hz}.\quad [1]$$

① Continued



$F_N = 2000 \times \cos 20° = 1879N$

b). Require Equivalent modal forces:

$$F_{eq} = \int_0^L f \bar{u}\, dx = F\bar{u}_j(L) = F\left(1-\cos\frac{j\pi}{2}\right) = 1879 \times \left(1-\cos\frac{j\pi}{2}\right) = \underline{1879N}\ (j=1)$$

$$= \underline{3758N}\ (j=2) \quad [2]$$

Using DAF plot on Data Sheet:

For Mode 1, $f_1 = 9.3\,Hz \Rightarrow T_1 = \frac{1}{9.3} = 0.11s$

$\therefore \frac{t_d}{T_1} = \frac{0.1}{0.11} = 0.91 \Rightarrow \underline{DAF \simeq 1.52}$

[1]

$\therefore u_1)_{max} = 1.52 \times \frac{1879}{346 \times 10^3} = 0.0083 = \underline{8.3\,mm}$ [2]

(5.4 mm)

Both occur at tip

For Mode 2, $f_2 = 14.5\,Hz \Rightarrow T_2 = \frac{1}{19.1} = 0.069s$

$\therefore \frac{t_d}{T_2} = \frac{0.1}{0.069} = 1.45 \Rightarrow \underline{DAF \simeq 1.3}$

[1]

$\therefore u_2)_{max} = 1.3 \times \frac{3758}{554 \times 10^3} = \underline{0.88\,mm}$

(0.68 mm)

$\sqrt{8.3^2 + 0.88^2} = 8.3\,mm.$ [2]

Hence peak displacement (at tip) $= \underline{8.3\,mm}$

c). $356 \times 171 \times 57$ UB $\Rightarrow I \times \frac{16040}{11700} = 1.371 \Rightarrow k_{eq_1} = 474\,kN/m$

$\Rightarrow T_1 \times \sqrt{\frac{1.041}{1.371}} = 0.871$

$\underline{\underline{m_{tot}}} \times \frac{57+20}{54+20} = 1.041 \Rightarrow M_{eq_1} = 105\,kg$ [2]

Hence $t_d/T_1 \times 1.15 = 0.91 \times 1.15 = 1.05 \Rightarrow DAF \simeq 1.5$

$\therefore u_1)_{max} = 1.5 \times \frac{1879}{346 \times 10^3 \times 1.371} = 0.0059 = 5.9\,mm.$

Alternatives include propping the cantilever at the tip, or installing a tuned mass damper, tuned to the first natural frequency. [2]

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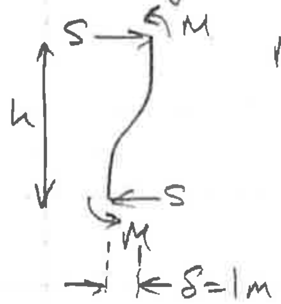
(2)

a). Using Data Book, calculate column stiffnesses:



$M = \frac{6EI}{h^2} \Rightarrow S = \frac{12EI}{h^3} \equiv K$, $EI = 10 MNm^2$

Hence 3m column stiffness, $k_3 = \frac{12 \times 10 \times 10^6}{3^3}$

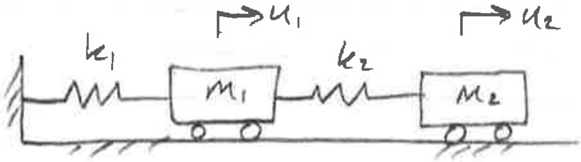
$= 4.44$ MN/m

5m " " , $k_5 = 0.96$ MN/m

$\Rightarrow$ total lower column stiffness, $k_1 = 2k_3 + k_5 = \underline{9.84 \text{ MN/m}}$

" upper " " $k_2 = 2k_3 = \underline{8.88 \text{ MN/m}}$ [3]

Hence equivalent 2DOF model:



$m_1 = 2000$ kg

$m_2 = 1000$ kg.

$k_2(u_2 - u_1) - k_1 u_1 = m_1 \ddot{u}_1$, $m_1 \ddot{u}_1 + (k_1 + k_2)u_1 - k_2 u_2 = 0$

$k_2(u_1 - u_2) = m_2 \ddot{u}_2$, $m_2 \ddot{u}_2 - k_2 u_1 + k_2 u_2 = 0$

$\therefore \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$ or $M\ddot{\underline{u}} + K\underline{u} = 0$ [3]

$K\bar{\underline{u}} = \omega^2 M \bar{\underline{u}}$

$M^{-1}K\bar{\underline{u}} = \omega^2 \bar{\underline{u}}$

Free vibration $\therefore \underline{u} = \bar{\underline{u}} \cos\omega t \Rightarrow [K - \omega^2 M]\bar{\underline{u}} = 0$

$\Rightarrow |K - \omega^2 M| = 0$

$k_1 + k_2 = 9.84 + 8.88 = \underline{18.72 \text{ MN/m}}$

$\therefore$ (dividing by 1000) $\begin{vmatrix} 18720 - 2\omega^2 & -8880 \\ -8880 & 8880 - \omega^2 \end{vmatrix} = 0$

$(18720 - 2\omega^2)(8880 - \omega^2) - 8880^2 = 0$

$2\omega^4 - 36480\omega^2 + 87379200 = 0$

$\omega^2 = \frac{36480 \pm \sqrt{36480^2 - 8 \times 87379200}}{4} = \underline{15404}, \underline{2836}$ [2]

$\Rightarrow \omega_2 = 124$ rad/s , $\omega_1 = 53.3$ rad/s

$= \underline{19.8 \text{ Hz}}$ $= \underline{8.5 \text{ Hz}}$

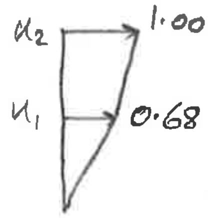
(2) continued

$$\omega_1^2 = 2836 \Rightarrow \begin{bmatrix} 18720 - 5672 & -8880 \\ -8880 & 8880 - 2836 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Rightarrow 13048\, u_1 = 8880\, u_2 \Rightarrow \frac{u_1}{u_2} = \underline{0.68}$$

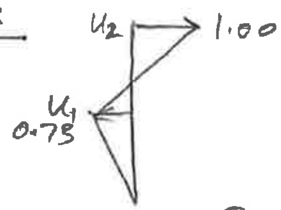
$$\omega_2^2 = 15404 \Rightarrow -12088\, u_1 = 8880\, u_2 \Rightarrow \frac{u_1}{u_2} = \underline{-0.73}$$

So mode shapes are: $f_1 = 8.5\,\text{Hz}$, $\underline{T_1 = 0.12\,\text{s}}$



[2]

$f_2 = 19.8\,\text{Hz}$, $\underline{T_2 = 0.05\,\text{s}}$



b). Find equivalent SDOF models: $M_{eq} = \underline{u}^t M \underline{u}$, $K_{eq} = \underline{u}^t K \underline{u}$, $f_{eq} = F\underline{u}$

Note M_{eq} not required

$$\left[M_{eq}^{(1)} = \begin{bmatrix} 0.68 & 1 \end{bmatrix} \begin{bmatrix} 2000 & \\ & 1000 \end{bmatrix} \begin{bmatrix} 0.68 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.68 & 1 \end{bmatrix} \begin{bmatrix} 1360 \\ 1000 \end{bmatrix} = \underline{1925\,\text{kg}} \right]$$

$$K_{eq}^{(1)} = \begin{bmatrix} 0.68 & 1 \end{bmatrix} \begin{bmatrix} 18.72 & -8.88 \\ -8.88 & 8.88 \end{bmatrix} \begin{bmatrix} 0.68 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.68 & 1 \end{bmatrix} \begin{bmatrix} 3.85 \\ 2.84 \end{bmatrix} = \underline{5.46\,\text{MN/m}}$$ [2]

$F_{eq}^{(1)} = 1.00 \times 60\,\text{kN} = \underline{60\,\text{kN}}$ [2]

$\therefore$ static deflection, $u_{st}^{(1)} = \dfrac{60 \times 10^3}{5.46 \times 10^6} = 0.0110 = 11.0\,\text{mm}$

$\dfrac{t_d}{T_1} = \dfrac{0.1}{0.12} = 0.83 \Rightarrow \underline{\text{DAF} \simeq 1.45}$ $\therefore$ $u_{max}^{(1)} = \underline{1.45 \times 11.0 = 16.0\,\text{mm}}$ [2]

$$\left[M_{eq}^{(2)} = \begin{bmatrix} -0.73 & 1 \end{bmatrix} \begin{bmatrix} 2000 & \\ & 1000 \end{bmatrix} \begin{bmatrix} -0.73 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.73 & 1 \end{bmatrix} \begin{bmatrix} -1460 \\ 1000 \end{bmatrix} = \underline{2066\,\text{kg}} \right]$$

$$K_{eq}^{(2)} = \begin{bmatrix} -0.73 & 1 \end{bmatrix} \begin{bmatrix} 18.72 & -8.88 \\ -8.88 & 8.88 \end{bmatrix} \begin{bmatrix} -0.73 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.73 & 1 \end{bmatrix} \begin{bmatrix} -22.55 \\ 15.36 \end{bmatrix} = \underline{31.82\,\text{MN/m}}$$

$F_{eq}^{(2)} = 1.00 \times 60\,\text{kN} = 60\,\text{kN}$

$u_{st}^{(2)} = \dfrac{60 \times 10^3}{31.82 \times 10^6} = 1.89 \times 10^{-3} = 1.9\,\text{mm}.$

$\dfrac{t_d}{T_2} = \dfrac{0.1}{0.05} = 2.0 \Rightarrow \underline{\text{DAF} \simeq 1.75}$ $\therefore$ $u_{max}^{(2)} = 1.75 \times 1.9 = \underline{3.3\,\text{mm}}$ [3]

(2) Continued.

Hence max. response via SRSS $= \sqrt{16^2 + 3.3^2}$

$= \underline{16.3\,mm}.$ [1]

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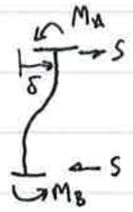
Q3 a), b) Liquefaction + remediation as per lecture notes.

c) $M_{superstruc.} = 20 \times 10^3$ kg

$M_{foundation} = (3 \times 3 \times 0.5 \text{ m}^3) \times (2400 \text{ kg/m}^3) = 10\,800$ kg

$M_{soil} \approx 10 \times M_{found.} = 108{,}000$ kg

Structures Data Book:



$$M_A = \frac{6EI\delta}{L^2} \qquad SL = 2M = \frac{12EI\delta}{L^2}$$

$$S = \frac{12EI\delta}{L^3}$$

Two columns $\Rightarrow$ $S_{TOT} = 2 \times \left(\frac{12EI}{L^3}\right)\delta = \left(2 \times \frac{12 \times 2 \times 10^8 \text{ Nm}^2}{125 \text{ m}^3}\right)\delta$

"$S_{TOT} = K\delta$" $\Rightarrow$ "K" $= 38.4 \times 10^6$ N/m

Fixed base freq $= \frac{1}{2\pi}\sqrt{\frac{K}{M}} = \frac{1}{2\pi}\sqrt{\frac{38.4 \times 10^6}{20 \times 10^3}} = 6.97$ say $\underline{\underline{7 \text{ Hz}}}$

d) $\gamma_{sand} = 20$ kN/m3 $\quad V_{shear} = 100$ m/s $\quad \nu = 0.3$

$$K_{hx} = \frac{Gb}{2-\nu}\left[6.8 + 2.4\right]\left[1 + \left(0.33 + \frac{1.34}{2}\right)\left(\frac{0.5}{1.5}\right)^{0.8}\right] \quad \text{(DataSheet)}$$

$$= G\left(\frac{1.5}{1.7}\right)(9.2)\left(\underbrace{1 + 1 \times \left(\tfrac{1}{3}\right)^{0.8}}_{1.4152}\right) = 11.49\,G$$

$V_s = \sqrt{G/\rho} \Rightarrow G = \rho V_s^2 \Rightarrow G = (2000)(100)^2 = 2 \times 10^7 \text{ N/m}^2$

$K_{hx} = 11.49 \times 20 \times 10^6 = 230 \times 10^6$ N/m.

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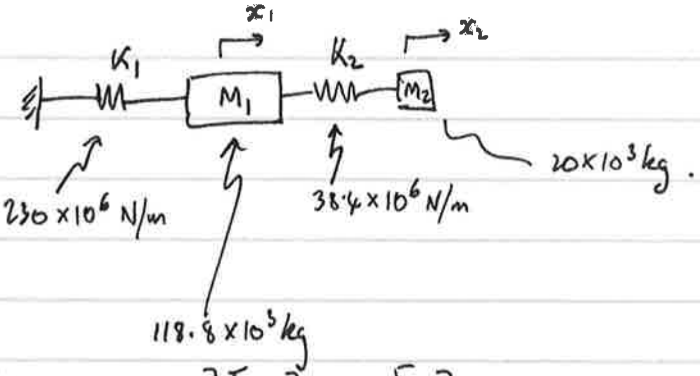

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (K_2+K_1-\omega^2 M_1) & -K_2 \\ -K_2 & K_2-\omega^2 M_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[(K_2+K_1)-\omega^2 M_1\right]\left[K_2-\omega^2 M_2\right]-K_2^2 = 0$$

$$M_1 M_2\,\omega^4 - \left[(K_1+K_2)M_2 + K_2 M_1\right]\omega^2 + \underbrace{(K_1+K_2)K_2 - K_2^2}_{=K_1K_2} = 0$$

$$a\omega^4 + b\omega^2 + c = 0$$

$$a = M_1M_2 = 20\times118.8\times10^6 = \underline{2.376\times10^9}$$
$$b = -\left[(K_1+K_2)M_2 + K_2M_1\right] = -\left[(230+38.4)\times10^6\times20\times10^3 + 38.4\times10^6\times118.8\times10^3\right]$$
$$= -10^9\left[5368 + 4562\right] = \underline{-9.93\times10^{12}}$$
$$c = K_1K_2 = 230\times38.4\times10^{12} = \underline{8.832\times10^{15}}$$

$$\omega^2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{9.93\times10^{12}\pm\sqrt{(9.93\times10^{12})^2-4(2.376\times10^9)(8.832\times10^{15})}}{2(2.376\times10^9)}$$
$$= 10^3\left[\frac{9.93\pm\sqrt{(9.93)^2-4(2.376)(8.832)}}{2(2.376)}\right] = 1284 \quad \text{(lowest root)}$$

$\omega = 35.8$ rads/sec $\qquad f = \frac{\omega}{2\pi} = \underline{5.7\text{ Hz}}$

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Q3 e) $V_s = 15\ m/s$ $G = 2000 \times 15^2 = 450{,}000 = 0.45 \times 10^6\ N/m^2$

$\therefore K_{hx} = 11.49\,G = 11.49 \times 0.45 \times 10^6 = 5.17 \times 10^6\ N/m.$

so K_1 is now 5.17×10^6 rather than 230×10^6 previously.

$a = 2.376 \times 10^9$ still

$b = -10^9\left[(5.17 + 38.4)20 + 4562\right] = -5.43 \times 10^{12}$

$c = 5.17 \times 38.4 \times 10^{12} = 0.199 \times 10^{15}$

$$\omega^2 = 10^3\left[\frac{5.43 - \sqrt{(5.43)^2 - 4(2.376)(0.199)}}{2(2.376)}\right] = 37.2$$

$\omega = 6.1\ rads/s \rightarrow f = 0.97\ Hz.$

much closer to earthquake frequencies.

4D6.
Q4

a) The equations of vertical (heave, h) and torsional (pitch, α) motions of a 2D symmetric section cross-sections are

$$m\ddot{x} + c_x\dot{x} + k_x x = L$$
$$I\ddot{\theta} + c_\theta\dot{\theta} + k_\theta\theta = M$$

These are coupled via the lift L and moment M. For example it is clear that the Lift L will depend on the angle of attack θ. More generally, (but then linearising – i.e. assuming the deck is moving with small harmonic motions) the lift and moment may be written as

$$L = c_1 h + c_2 \dot{h} + c_3\alpha + c_4\dot{\alpha}$$
$$M = c_5\alpha + c_6\dot{\alpha} + c_7 h + c_8\dot{h}$$

where the coefficients c depend on the incident wind speed U and the frequency of the deck's harmonic motion. These coefficients, once suitably non-dimensionalised, are the "flutter derivatives".

In a wind tunnel they can be measured by moving a 2D section model through prescribed harmonic motions at various frequencies. Vertical and torsional motions are usually applied separately. Pressures can be measured during each test, which when suitably integrated, lead to the lift force and moment. Alternatively load cells may measure force and moment directly. For example, if purely vertical motions $h = H_0 \sin(\omega t)$ are applied the resulting lift force L will be harmonic, and its amplitude and phase relative to the input h/t can be separated into the components in-phase with h and in-phase with $\dot{h}$, leading to c_1 and c_2 derivatives at that (non-dimensionalised) frequency.

b) $\left[K - \omega_j^2 M\right]\phi_j = 0$ is basic eigenvalue equation.

$$\therefore \quad K\phi_j = \omega_j^2 M\phi_j$$
$$\therefore \quad \phi_i^T K\phi_j = \omega_j^2 \phi_i^T M\phi_j \qquad (1)$$

By same argument $\phi_j^T K\phi_i = \omega_i^2 \phi_j^T M\phi_i$.
Transpose this last equation

$$\phi_i^T K\phi_j = \omega_i^2 \phi_i^T M\phi_j \qquad (2) \qquad (\text{since } K = K^T \text{ and } M = M^T)$$

Subtract (1) − (2) $\qquad 0 = (\omega_j^2 - \omega_i^2)\,\phi_i^T M\phi_j$

So if $\omega_j \neq \omega_i \Rightarrow \qquad \phi_i^T M\phi_j = 0$

and from (1) $\qquad \phi_i^T K\phi_j = 0$

(i.e. ϕ_i and ϕ_j are orthogonal wrt M and K provided $\omega_i \neq \omega_j$).

4 c). The mechanical admittance $|H_{mech}(\omega)|^2$ relates the power spectrum of an input force $f(t)$ to the ~~out~~ power spectrum of the output displacement of a linear oscillator.

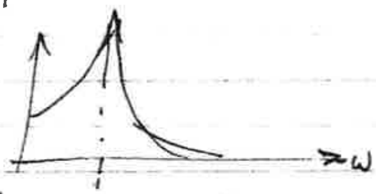
$$\text{ie:} \quad S_{xx}(\omega) = |H_{mech}(\omega)|^2 S_{ff}(\omega)$$

Similarly aerodynamic admittance relates power spectrum of wind velocity to wind force $\quad S_{ff}(\omega) = |H_{aero}(\omega)|^2 S_{uu}(\omega)$

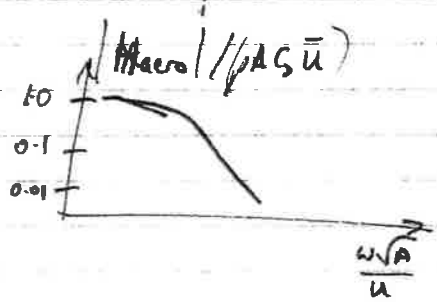
For a sdof oscillator subject to wind excitation, procedure is $u \Rightarrow f \Rightarrow x$

via $\quad S_{xx}(\omega) = |H_{mech}(\omega)|^2 |H_{aero}(\omega)|^2 S_{uu}(\omega)$

The $|H_{mech}(\omega)|^2$ is typically of the form



The $|H_{aero}(\omega)|^2$ is typically of the form



and acts as a sort of "areal reduction factor" since, at any frequency, gusts at one point on the structure will not be correlated ~~with~~ fully with those at other points.

4 d)

1. Increase stand-off – esp. for vehicles, via bollards, barriers etc

2. Robustness – build in redundacy in structure
– detail as if in a seismic area
(ductility, rotation capacity, etc etc)

3. Glazing – ~~eg annealed~~ avoid simple annealed glass

Use toughened glass or polycarbonate – also anti-shatter film and heavy curtains
– but preferably use laminated glass with PVB interlayer
(and design window frames to cope with v. large forces that will result).